

Flow with Slip Between Coaxial Cylinders and in an Equilateral Triangular Pipe

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Abstract: The Navier-Stokes equations are solved exactly, assuming the Navier slip boundary condition, for flow between coaxial circular cylinders (with possibly different slip lengths at the inner and outer walls), and flow in an equilateral triangular pipe. The velocity fields and total flow rates are found analytically. A relatively small slip length can lead to a large increase in the total rate of fluid flow: in the triangular pipe case the flow doubles when the slip length is only about 7.5% of the triangle side.

INTRODUCTION

The idea of slip goes back to Navier [1] in 1823. Later works by Helmholtz and Piotrowski and by Kundt and Warburg were discussed by Maxwell in 1879 [2]. The recent interest is vigorous [3-25]. Briefly, flow through nanometer-scale channels has shown slip of liquid over solid when the interface is hydrophobic. The experimental data [references 3-9, 11-13, 21 and 24] indicate that slip lengths of the order of 10 to 50 nm characterize liquid flow over hydrophobic surfaces. (Slip lengths from 200 nm to 20 μm have been observed in nanogated superhydrophobic microchannels [12, 19].) Thus slip can become important (or even dominant) for very small channels.

Here we shall consider flow with slip in some pipes of simple cross-sections, assuming a uniform slip length s , as defined (implicitly) by Navier,

$$s = \left(\frac{v_{||}}{\partial_{\perp} v_{||}} \right)_{\text{wall}} \quad (1)$$

where $v_{||}$ is the velocity component along the wall. The normal derivative $\partial_{\perp} v_{||}$ is understood to be taken inward into the pipe from the wall or walls.

In general, for incompressible steady flow in the z -direction, the Navier-Stokes equations reduce to $\partial_x p = 0 = \partial_y p$ and

$$\left(\partial_x^2 + \partial_y^2 \right) v_z = \frac{1}{\eta} \partial_z p \quad (2)$$

where η is the fluid viscosity and p is the pressure. For uniform flow v_z does not depend on z , and so the pressure

gradient $\partial_z p$ is constant. Hence the flow problem is solved when the two-dimensional Laplacian of v_z is constant, and (1) is satisfied.

Our new results are for flow with slip between coaxial circular cylinders, and within a pipe of equilateral triangular cross-section. For completeness we shall begin with two simple well-known solutions.

Flow Between Parallel Plates

Let the flow be in the z direction, between plates at $y = -h$ and $y = h$. We consider flow far away from the sides of the pipe at $x = -X$ and $+X$. When slip is absent the fluid speed is parabolic:

$$v_z = u_0 \left[1 - \frac{y^2}{h^2} \right] \quad (3)$$

With slip, the conditions of constant Laplacian and the slip boundary condition are also satisfied by a parabolic velocity profile:

$$v_z = u \left[1 - \frac{y^2}{h^2 + 2hs} \right] \quad (4)$$

The maximum flow speeds $u_0 = h^2(-\partial_z p) / 2\eta$ and $u = (h^2 + 2hs)(-\partial_z p) / 2\eta$ follow from (2).

Circular Pipe

In a circular pipe of radius a the velocity (everywhere parallel to the pipe wall) is given by the usual Hagen-Poiseuille flow profile

$$v_z = u_0 \left[1 - \frac{r^2}{a^2} \right] \quad (5)$$

[the Laplacian $\left(\partial_r^2 + \frac{1}{r} \partial_r \right) v_z$ is constant, and v_z is zero at $r = a$].

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With slip, the flow is again modified in a simple way:

$$v_z = u \left[1 - \frac{r^2}{a^2 + 2as} \right] \quad (6)$$

Then $v_{||} = v_z$ and $\partial_{\perp} v_{||} = -\partial_r v_z$ at the wall $r = a$ are equal to $2aus/[a^2 + 2as]$ and $2au/[a^2 + 2as]$ respectively, with ratio s in accord with (1). The speed u determined by (2) is $u = \frac{1}{4}(a^2 + 2as)(-\partial_z p)/\eta$. The total flow rate in a circular pipe (of radius a , area $A = \pi a^2$) is thus

$$Q = \frac{A^2}{8\pi\eta} (-\partial_z p) \left[1 + \frac{4s}{a} \right] \quad (7)$$

CONCENTRIC CIRCULAR PIPES

Slip-free flow between coaxial circular cylinders of radii a and b (with $b > a$) is given by

$$v_z = u_0 \left[\frac{\ln \frac{r}{a} - \frac{r^2 - a^2}{b^2 - a^2}}{\ln \frac{b}{a}} \right] \quad (8)$$

This clearly satisfies the conditions of constant Laplacian and zero speed at $r = a$ and $r = b$. The speed u_0 determined by (2) is $u_0 = \frac{1}{4}(b^2 - a^2)(-\partial_z p)/\eta$. The total no-slip flow rate is

$$Q_0 = 2\pi \int_a^b dr r v_z(r) = \frac{\pi}{8\eta} (-\partial_z p)(b^2 - a^2) \left[a^2 + b^2 - \frac{b^2 - a^2}{\ln(b/a)} \right] \quad (9)$$

Note that Q_0 is not simply a constant times the square of the area between the pipes, $A^2 = \pi^2 (b^2 - a^2)^2$, in contradistinction to Q_0 for the circular pipe (above) and the triangular pipe (next section).

Next we allow for possibly different slip lengths s_a and s_b at the inner and outer walls, applying (1) with the normal derivative $\partial_{\perp} v_{||}$ taken inwardly into the pipe at both walls.

The constant, $\ln(r)$ and r^2 terms which appear in the slip-free solution (8) all have constant Laplacian (zero for the constant and $\ln(r)$ terms), and can be linearly superposed in the right proportions to give the solution corresponding to slip lengths s_a and s_b at the inner and outer walls:

$$v_z = u \left(\begin{aligned} & (b^2 - a^2) \ln \frac{r}{a} - (r^2 - a^2) \ln \frac{b}{a} + \frac{s_a}{a} \left[b^2 - r^2 - 2a^2 \ln \frac{b}{r} \right] \\ & + \frac{s_b}{b} \left[2b^2 \ln \frac{r}{a} - (r^2 - a^2) \right] \\ & + 2 \frac{s_a s_b}{ab} (b^2 - r^2) \end{aligned} \right) \left/ \left[\left(\ln \frac{b}{a} + \frac{s_a}{a} + \frac{s_b}{b} \right) (b^2 - a^2) \right] \right. \quad (10)$$

The prefactor u equals u_0 (this special result follows because the Laplacian of $\ln r$ is zero). At $r = a$ and $r = b$ the fluid velocity takes the values

$$\begin{aligned} v_a &= u \frac{\frac{s_a}{a} \left[b^2 - a^2 - 2a^2 \ln \frac{b}{a} + \frac{2s_b}{b} (b^2 - a^2) \right]}{4 \left(\ln \frac{b}{a} + \frac{s_a}{a} + \frac{s_b}{b} \right) (b^2 - a^2)} \\ v_b &= u \frac{\frac{s_b}{b} \left[a^2 - b^2 + 2b^2 \ln \frac{b}{a} + \frac{2s_a}{a} (b^2 - a^2) \right]}{4 \left(\ln \frac{b}{a} + \frac{s_a}{a} + \frac{s_b}{b} \right) (b^2 - a^2)} \end{aligned} \quad (11)$$

Fig. (1) shows the velocity profiles of the flow between two concentric circular cylinders, with $b = 2a$ and equal slip lengths $s_a = s_b$ varying from 0 to $0.3a$, in steps of $0.1a$. (For inner and outer radii of 100 nm and 200 nm, the slip lengths would be 10, 20 and 30 nm). The shape of the velocity profile changes very little with slip, the main change in velocity being a constant increase proportional to s , roughly equal to the average of v_a and v_b . The increase also depends on r , but the variation with r is too weak to show in the figure.

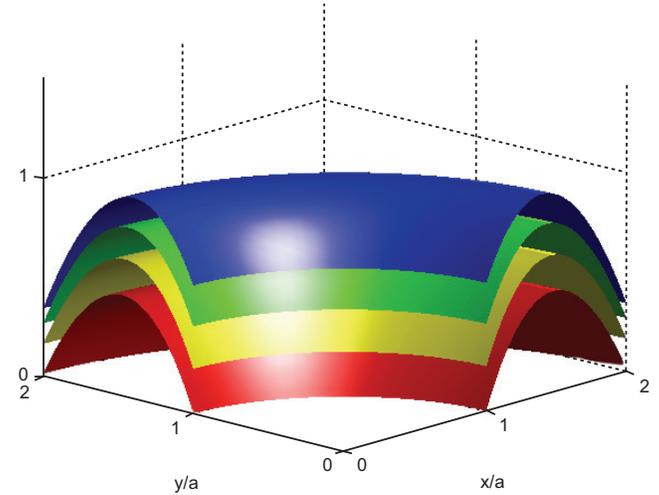


Fig. (1). Flow speed between cylindrical surfaces at $r = a$ and $r = b$, for $b = 2a$ and equal slip lengths $s_a = s_b = s$, with s equal to 0, $0.1a$, $0.2a$, and $0.3a$. The velocity profiles are shown nested, and over only one quarter of the circumference, to reveal the variation with r .

The total flow rate when the slip lengths are s_a and s_b at the inner and outer cylinder walls is given by

$$Q = \frac{\pi}{8\eta} (-\partial_z p) \left(\begin{aligned} & (b^2 - a^2) \left[(a^2 + b^2) \ln \frac{b}{a} - (b^2 - a^2) \right] \\ & + \frac{s_a}{a} \left[4a^4 \ln \frac{b}{a} - (b^2 - a^2)(3a^2 - b^2) \right] \\ & + \frac{s_b}{b} \left[4b^4 \ln \frac{b}{a} - (b^2 - a^2)(3b^2 - a^2) \right] \\ & + 4 \frac{s_a s_b}{ab} (b^2 - a^2)^2 \end{aligned} \right) \left/ \left(\ln \frac{b}{a} + \frac{s_a}{a} + \frac{s_b}{b} \right) \right. \quad (12)$$

To lowest order in s_a and s_b , the ratio of Q to Q_0 is

$$\frac{Q}{Q_0} \approx 1 + \frac{\frac{s_a}{a} \left(b^2 - a^2 - 2a^2 \ln \frac{b}{a} \right)^2 + \frac{s_b}{b} \left(2b^2 \ln \frac{b}{a} + a^2 - b^2 \right)^2}{(b^2 - a^2) \ln \frac{b}{a} \left[(a^2 + b^2) \ln \frac{b}{a} + a^2 - b^2 \right]} \quad (13)$$

The flow rate increases monotonically with both s_a and s_b . When $s_a = s = s_b$ and $b = 2a$ the total flow doubles when $s \approx 0.166a$, so a slip length of about 17 nm is sufficient to double the flow between cylinders of radii 100 nm and 200 nm. The total flow when $s = a$ and $b = 2a$ (corresponding to a slip length of 100 nm for the example above, which is well within the experimental range of slip lengths) is about 7 times the zero-slip flow, for the same pressure gradient.

We note in passing that there is no fundamental reason for the slip length to be substantially smaller than the pipe dimensions. The greatest drag reductions are obtained by modifying the surface with hydrophobic protrusions (see for example [12]), which reduce the effective contact area between the fluid and the surface. The slip lengths obtained in this way exceed 20 μm , for rectangular pipes in which the smallest dimension is as low as 76 μm [12]. The smallest dimension can in principle be reduced further, the physical difficulty being to retain an even gap between the protrusions forming the slipping surface and the opposite wall.

The effect of slip (for given slip lengths) is clearly greatest for small inter-cylinder spacing. For small $b - a$ the leading terms in Q_0 and Q are of different degree in $b - a$:

$$Q_0 = \frac{\pi}{6\eta} (-\partial_z p) \left[a(b-a)^3 + O(b-a)^4 \right] \quad (14)$$

$$Q = \frac{2\pi}{\eta} (-\partial_z p) \left[a(b-a)^2 \frac{s_a s_b}{s_a + s_b} + O(b-a)^3 \right] \quad (15)$$

EQUILATERAL TRIANGULAR PIPE

So far we have considered geometries that are mathematically one-dimensional (for example, dependent only on the radial distance from the axis of symmetry). Two-variable geometries are more interesting, and more complex. One motivation for looking at a triangular geometry is curiosity about slip-flow in the corners. In the absence of slip the flow speed goes to zero linearly with the distance from the walls, except at the corners, where it goes to zero quadratically (with angular dependence). What happens with slip?

Fig. (2) shows an equilateral triangle of side a with no-slip laminar flow, the solution of which is well-known as one of the simplest of an infinity of solvable flows [26].

If $L = 0$, $R = 0$ and $B = 0$ symbolically denote the linear equations for the left, right and basal sides of the triangle (given for a particular coordinate system in (16) below), the solution of (2) satisfying $v_z = 0$ on the sides of the triangle is $v_z \sim LRB$. The value of the constant of proportionality depends on the value of the (constant) right-hand side of (2), and on our choice of L , R and B . Let the base coincide with the line $y = 0$, and let the line $x = 0$ bisect the triangle. Then we can take

$$L = \frac{a}{2} + x - \frac{y}{\sqrt{3}}, \quad R = \frac{a}{2} - x - \frac{y}{\sqrt{3}}, \quad B = \frac{2y}{\sqrt{3}} \quad (16)$$

and the zero-slip velocity profile is given by

$$v_z = \frac{27u_0}{a^3} LRB \quad (17)$$

(The factor $27/a^3$ is chosen to make $v_z = u_0$ at the centroid of the triangle). The Laplacian of v_z is $-36u_0/a^2$, so (2) gives $u_0 = \frac{a^2}{36\eta} (-\partial_z p)$. The total rate of slip-free fluid flow through the pipe (volume per unit time) is

$$Q_0 = 2 \int_0^{a/2} dx \int_0^{\sqrt{3}(\frac{a}{2}-x)} dy v_z(x,y) = \frac{9\sqrt{3}}{80} u_0 a^2 = \frac{A^2}{20\sqrt{3}\eta} (-\partial_z p) \quad (18)$$

where $A = \frac{1}{4}\sqrt{3}a^2$ is the area of the triangle.

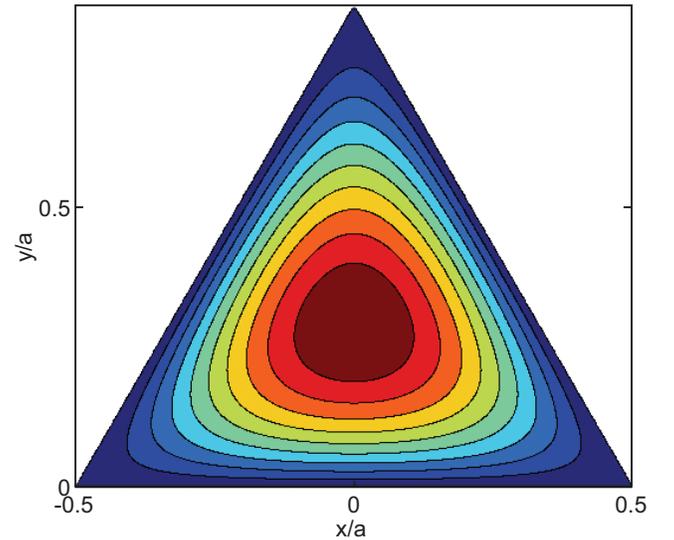


Fig. (2). Velocity distribution in an equilateral triangular pipe, without slip. The contours are drawn at 0.1 to 0.9 in increments of 0.1 of the maximum velocity (which is located at the centroid of the triangle).

Now we consider flow with slip, with constant slip length s . We still need to satisfy Navier-Stokes equation (2), but now the boundary condition is that v_z on the boundary is non-zero, and equal to s times its normal derivative (taken inward from the pipe walls). The symmetry of the problem, and the fact that the Laplacian of v_z must be a constant, suggests adding terms proportional to $LR + RB + BL$ and $L + R + B$ to the zero-slip flow field (17). Note that the use of symmetry implies a particular choice of relative normalization of L , R and B . If these are expressed as $mx + ny + c = 0$, we have chosen $m^2 + n^2 = 4/3$ in writing (16). With this choice we have $L + R + B = a$ and

$$LR + RB + BL = \frac{a^2}{4} + \frac{ay}{\sqrt{3}} - x^2 - y^2 \quad (19)$$

The trial velocity profile is taken to be

$$v_z = \frac{27u}{a^3} LRB + \frac{3v}{a^2} (LR + RB + BL) + w \quad (20)$$

The Laplacian of v_z is a constant, namely $-(36u + 12v)/a^2$, so (2) can be satisfied. To satisfy the boundary condition (1) we calculate v_z and $\partial_{\perp} v_z$ on the walls. For example, on the base $B = 0$ we have

$$v_z \rightarrow \frac{3}{4}v \left[1 - \left(\frac{2x}{a} \right)^2 \right] + w \quad (21)$$

$$\partial_y v_z \rightarrow \frac{\sqrt{3}}{a} \left\{ \frac{9}{2}u \left[1 - \left(\frac{2x}{a} \right)^2 \right] + v \right\} \quad (22)$$

The ratio of (21) to (22) will be a constant (equal to the slip length s) if

$$v = 6\sqrt{3}u \frac{s}{a}, \quad w = 18u \left(\frac{s}{a} \right)^2 \quad (23)$$

The velocity at the centroid $\{x = 0, y = a/2\sqrt{3}\}$ is

$$V_0 = u + v + w = u \left\{ 1 + 6\sqrt{3} \frac{s}{a} + 18 \left(\frac{s}{a} \right)^2 \right\} \quad (24)$$

The velocity on the base $y = 0$ is

$$V_b = \frac{9\sqrt{3}}{2}u \frac{s}{a} \left[1 - \left(\frac{2x}{a} \right)^2 \right] + \frac{4}{\sqrt{3}} \frac{s}{a} \quad (25)$$

At the corners $x = \pm a/2$ this gives a quadratic dependence on the slip length:

$$V_c = 18u \left(\frac{s}{a} \right)^2 \quad (26)$$

The Laplacian of v_z is $-\frac{36u}{a^2} \left(1 + 2\sqrt{3} \frac{s}{a} \right)$, which equals $(\partial_z p) / \eta$ by (2). Thus

$$v_z(x, y) = \frac{(-\partial_z p)}{\eta} \frac{3LRB/a + 2\sqrt{3}(LR + RB + BL)s/a + 2s^2}{1 + 2\sqrt{3} \frac{s}{a}} \quad (27)$$

Fig. (3) shows the velocity profiles for s equal to 0, 0.1a, 0.2a, and 0.3a. Note how the flow is constrained at the vertices of the triangle, even for a substantial s/a ratio.

Finally, the total flow rate is computed from (27), as in (18). We find

$$Q = \frac{A^2}{20\sqrt{3}\eta} (-\partial_z p) \left[1 + 10\sqrt{3} \frac{s}{a} + 40 \left(\frac{s}{a} \right)^2 \right] / \left[1 + 2\sqrt{3} \frac{s}{a} \right] \quad (28)$$

At $s = 0.02a$ the increase due to slip predicted by (28) is 27%, which is remarkable for such a small slip length. The flow rate doubles when s is about 7.5% of a side length; when $s = a$ the flow rate is about 13 times the zero slip value (Alternatively: at constant flow rate the pressure drop decreases by factors of 2 and 13, respectively).

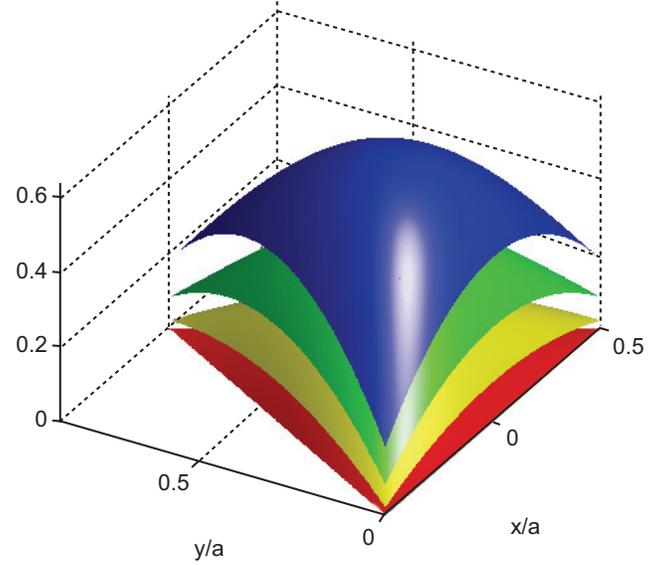


Fig. (3). Flow pattern in an equilateral triangular pipe with the slip length varying from 0 to 0.3a, in steps of 0.1a.

CONCLUSION

The Introduction gave examples of the importance of slip on fluid flow in small channels. Just how important slip is depends on the geometry as well on the relative size of the slip length to the dimensions the channels. For the circular pipe of radius a and slip length s , the rates of flow ratio Q/Q_0 is equal to $1 + 4s/a$. For the equilateral triangular pipe of side a , Q/Q_0 equals $(1 + 10\sqrt{3}s/a + 40s^2/a^2) / (1 + 2\sqrt{3}s/a)$. Between concentric circular cylinders of radii a and b the flow is more complicated (see (10) and (12)). When the slip lengths at the inner and outer walls are equal, the leading term in Q/Q_0 when s is large compared to both a and b is

$$\frac{4s(b-a) \ln \frac{b}{a}}{(a^2 + b^2) \ln \frac{b}{a} + a^2 - b^2} \quad (29)$$

For slip lengths large compared to the other pipe dimensions, the ratio of slipping flow to non-slip flow thus becomes proportional to s for the circular pipe, for the concentric cylinders, and for the triangular pipe. For small slip lengths, the total flows increase linearly with slip: the leading terms are given by (13) for concentric cylinders, and by $1 + 8\sqrt{3}s/a$ for the equilateral triangular pipe. However, the intermediate behaviour is quite different in the two cases.

We conclude that slip can have a dramatic effect on fluid flow when the slip lengths are commensurate with the pipe dimensions, and that the effects are strongly dependent on pipe shape.

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