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RESEARCH ARTICLE

A Novel Arithmetic Technique for Generalized Interval-Valued Triangular Intuitionistic Fuzzy Numbers and Its Application in Decision Making

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Abstract:

Background:

Uncertainty is an integral part of decision-making process which arises due to the lack of knowledge, data or information. Initially fuzzy set theory (FST) was used to handle this type of uncertainty. Later, intuitionistic fuzzy set (IFS) was developed to encounter uncertainty in a more specific manner. However, it is observed that due to the existence of different types of uncertainties, the membership function (MF) of IFS itself is uncertain and consequently, the concept of interval-valued intuitionistic fuzzy sets (IVIFS) came into the picture. But IVIFS is also not capable of handling uncertainty. To overcome the limitations of the existing IVIFS, generalized interval valued intuitionistic fuzzy sets (GIVIFS) have been defined and it has been observed that it has utmost applicability in real world situations as the parameter height characterises the degree of buoyancy of judgment of decision maker in a very specific compartment.

Objective:

An arithmetic operation on GIVTIFNs is always a critical concern and the conventional way of performing arithmetic operations on GIVTIFNs has some shortcomings. This paper attempts to devise a novel technique to effectively resolve the drawbacks of conventional arithmetic operations on GIVTIFNs. Numerical examples are illustrated herewith and to justify the need of a new solution. Furthermore, an application of multi-criteria group decision-making problem was also carried out under this setting.

Method:

For the arithmetic operations on GIVTIFNs, the largest membership function is truncated at the minimum height first and the non-membership function is truncated at the maximum height. Accordingly, arithmetic operations on GIVTIFNs are defined. For this purpose, Decomposition theorems for GIVTrIFNs are discussed first.

Result:

The outputs are obtained as generalized interval-valued trapezoidal intuitionistic fuzzy numbers (GIVTrIFNs). The interesting part of the proposed approach is that it produces GIVTrIFNs. To check the validity and novelty of the approach, a multi criteria decision making was performed which obtained desirable results.

Conclusion:

The arithmetic GIVTIFNs conventional approach produces invariant output in the form of GIVTIFNs for GIVTIFNs of different height. But for the same input GIVTIFNs, the present approach provided different GIVTrIFNs. It was observed that the proposed approach is efficient, simple, logical, technically sound and general enough for implementation. Researchers may apply this approach in any field where GIVTIFNs are involved.

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1. INTRODUCTION

In the presence of different constraints in real life situation and due to highly complex environment, decision makers may provide their opinion under uncertain and imprecise nature. Due to the involvement of uncertainty crisp data are not always adequate to model in many real-life situations whereas FST introduced by L.A. Zadeh [1], is more suitable and realistic to handle such type of situation. After the development of fuzzy set theory, further developments have been made by different researchers. Chen [2] further developed the fuzzy set theory (fuzzy numbers) and named as generalized fuzzy numbers (GFN) and performed all arithmetic operations between GFNs based on function principal. GFNs have been applied in different fields such as reliability analysis, risk analysis, pattern recognition, Rotor Fault Diagnosis, maximal flow problems, series-parallel system etc.

On the other hand, an important generalization of fuzzy set theory is the theory of Intuitionistic Fuzzy Set (IFS), introduced by Atanassov [3] describing a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed to one. It is observed that fuzzy sets are IFSs but the converse is not necessarily correct. Further, Atanassov and Gargov [4] developed the notion of Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) in relation with interval-valued fuzzy sets and IFS. The IVIFSs are characterised by a membership function and a nonmembership function that take interval values rather than the exact number. In human cognitive and decision making processes, it is not absolutely justifiable or technically sound to represent the membership and nonmembership in terms of a single numeric value. Thus, IVIFSs have got more attention due to its ability to handle imprecise and unorganised information in terms of intervals instead of taking a single numeric value [5]. IFS and IVIFS have been successfully applied [6 - 12] in different areas like decision making, pattern recognition, medical diagnosis. Yager, Yuan and Li [13 - 15] studied the cut set characteristic of IVIFS. Following the work of Szmidt and Kacprzyk [16], Xu and Qiansheng [17 - 19] applied IVIFS to pattern recognition. Further Yingjie and Qiansheng [20, 21] studied the interval-valued intuitionistic fuzzy reasoning. Xu and Li [22 - 24], successfully carried IVIFS to decision making problems. The Generalized Intuitionistic Fuzzy Sets (GIFSs) were proposed by Mondal and Samanta [25] under the constraint that the minimum of the two degrees does not exceed half. Shu *et al*. [26], first introduced the concept of Generalized Intuitionistic Fuzzy Numbers (GIFNs) and defined arithmetic operations between them. But later, it was found that there are some errors and misprints in the definition of the four arithmetic operations and those errors were conducted by Li [27]. Zhenhua *et al*. [28] introduced the construction method of the Generalized Interval-Valued Intuitionistic Fuzzy Sets with Parameters (GIVIFSP), and defined complement operation, intersection operation and union operation on GIVIFS. Furthermore, they proved that like IFS and IVIFS, GIVIFS is a closed algebraic system for all these operations. Bhowmik *et al*. [29], Zhi *et al*. [30], and Adak *et al*. [31] also studied different concepts of GIVIFSs. Baloui and Nadarajah [32] extended the IFSs to the concept of GIFSs and introduced some operators on GIFSs. Based on GIFSs, Shabani and Baloui [33] introduced GIFNs. Baloui [34] considered a new GIVIFSs and introduced some operators on GIVIFSs. He studied different basic operations like union, intersection, subset complement *etc*. and also transformed the operations on IVIFSs for the GIVIFSs.

This paper presents a novel efficient approach to perform arithmetic operations on GIVTIFNs using cut method. This approach effectively resolves the shortcomings of the existing approach. Numerical examples are illustrated. Also, to show the proper justification, validity, efficiency and applicability of the proposed approach, a multi-criteria group decision-making problem was carried out. The detail work has been compressed as follows. Section 2 starts with some relevant preliminary definitions. In section 3, decomposition theorems are discussed by using GIVTrIFNs. Section 4 presents the proposed approach of arithmetic operations of GIVTIFNs for different heights and the positivity of the proposed method in comparison to the earlier methods. Numerical examples are shown in section 5. Section 6 discusses the ranking of GIVTrIFNs. A multi-criteria decision-making problem is discussed by using the proposed arithmetic operations in section 7. Finally, a concrete conclusion has been drawn in section 8.

1.1. Drawback of Existing Approach and Motivation

In this section, we perform all the conventional basic arithmetic operations on GIVTIFNs. However, the problem can be seen in the arithmetic on GIVTIFNs due to different heights.

Considering two Generalized Interval-Valued Triangular Intuitionistic Fuzzy Numbers (GIVTIFNs)

$$A=\left\langle\left[\left(a^{lm},b,c^{lm};w^{lm}\right),\left(a^{um},b,c^{um};w^{um}\right)\right],\left[\left(a^{ln},b,c^{ln};\eta^{ln}\right),\left(a^{un},b,c^{un};\eta^{un}\right)\right]\right\rangle$$

and

$$B=\left\langle\left[\left(a_1^{lm},b_1,c_1^{lm};w_1^{lm}\right),\left(a_1^{um},b_1,c_1^{um};w_1^{um}\right)\right],\left[\left(a_1^{ln},b_1,c_1^{ln};\eta_1^{ln}\right),\left(a_1^{un},b_1,c_1^{un};\eta_1^{un}\right)\right]\right\rangle$$

, the arithmetic operations are as follows:

$$A+B=\left\langle\begin{array}{l}\left[\left(a^{l_m}+a_1^{lm},b+b_1,c^{l_m}+c_1^{lm};\min\left(w^{lm},w_1^{lm}\right)\right),\left(a^{u_m}+a_1^{um},b+b_1,c^{u_m}+c_1^{um};\min\left(w^{um},w_1^{um}\right)\right)\right],\\ \left[\left(a^{l_n}+a_1^{ln},b+b_1,c^{l_n}+c_1^{ln};\max(\eta^{ln},\eta_1^{ln})\right),\left(a^{u_n}+a_1^{un},b+b_1,c^{u_n}+c_1^{un};\max(\eta^{un},\eta_1^{un})\right)\right]\end{array}\right\rangle$$

$$A-B=\left\langle\begin{array}{l}\left[\left(a^{l_m}-a_1^{lm},b-b_1,c^{l_m}-c_1^{lm};\min\left(w^{lm},w_1^{lm}\right)\right),\left(a^{u_m}-a_1^{um},b-b_1,c^{u_m}-c_1^{um};\min\left(w^{um},w_1^{um}\right)\right)\right],\\ \left[\left(a^{l_n}-a_1^{ln},b-b_1,c^{l_n}-c_1^{ln};\max(\eta^{ln},\eta_1^{ln})\right),\left(a^{u_n}-a_1^{un},b-b_1,c^{u_n}-c_1^{un};\max(\eta^{un},\eta_1^{un})\right)\right]\end{array}\right\rangle$$

$$AB=\left\langle\begin{array}{l}\left[\left(a^{lm}a_1^{lm},bb_1,c^{lm}c_1^{lm};\min\left(w^{lm},w_1^{lm}\right)\right),\left(a^{um}a_1^{um},bb_1,c^{um}c_1^{um};\min\left(w^{um},w_1^{um}\right)\right)\right],\\ \left[\left(a^{l_n}a_1^{ln},bb_1,c^{l_n}c_1^{ln};\max\left(\eta^{ln},\eta_1^{ln}\right)\right),\left(a^{u_n}a_1^{un},bb_1,c^{u_n}c_1^{un};\max\left(\eta^{un},\eta_1^{un}\right)\right)\right]\end{array}\right\rangle$$

$$A/B=\left\langle\begin{array}{l}\left[\left(a^{l_m}/c_1^{lm},b/b_1,c^{l_m}/a_1^{lm};\min\left(w^{lm},w_1^{lm}\right)\right),\left(a^{u_m}/c_1^{um},b/b_1,c^{u_m}/a_1^{um};\min\left(w^{um},w_1^{um}\right)\right)\right],\\ \left[\left(a^{l_n}/c_1^{ln},b/b_1,c^{l_n}/a_1^{ln};\max\left(\eta^{ln},\eta_1^{ln}\right)\right),\left(a^{u_n}/c_1^{un},b/b_1,c^{u_n}/a_1^{un};\max\left(\eta^{un},\eta_1^{un}\right)\right)\right]\end{array}\right\rangle$$

But this approach has some drawbacks and gives illogical results as during the operation, first it reduces the height of their respective (LMF and UMF) higher MFs to the height of the lower ones (*i.e.*, make it as GIVTFN by reducing the height based on Cheng's [26] function principle) and similarly for the respective (LNMF and UNMF) NMFs, it increases the minimum height to maximum one to make it a generalized interval-valued triangular non-membership function. Therefore, this approach produces GIVTIFN with MF at the minimum height of the given respective (LMF and UMF) MFs of the GIVTIFNs and height of the NMF is maximum of the given respective (LNMF and UNMF) NMFs of GIVTIFNs to perform the arithmetic operations. The major drawback of this approach is that when performing arithmetic operations between a fixed GIVTIFN with different GIVTIFNs with the same support but different heights, the height of MFs of the fixed GIVTIFN is lesser and the height of NMFs is higher than other GIVTIFNs; then it is seen that each time, the resultant GIVTIFN remains invariant, which is illogical. For example consider $A=\langle\ [(2,4,6;0.7),(1,4,7;0.8)],[(1,4,7;0.3),(0.5,4,8.5;0.2)]\ \rangle$ be a fixed GIVTIFN and $B_1=\langle\ [(10,14,18;0.8),(8,14,20;0.9)],[(9,14,19;0.2),(7,14,21;0.1)]\ \rangle$ & $B_2=\langle[(10,14,18;0.9),(8,14,20;0.95)],[(9,14,19;0.1),(7,14,21;0.05)]\rangle$ be two different GIVTIFNs with different heights. Then performing $A*B_i$ (i=1,2), where * is the basic arithmetic operation using the existing approach to provide the same GIVTIFN. That is, the conventional approach gives

$A+B_1=A+B_2=\langle[(12,18,24;0.7),(9,18,27;0.8)],[(10,18,26;0.3),(7.5,18,29.5;0.2)]\rangle$, which is clearly illogical as and are two different GIVTIFNs and the sum of these two GIVTIFNs with the fixed GIVTIFN A is identical. The following Figs. (**1**, **2** and **3**) represent the above example.

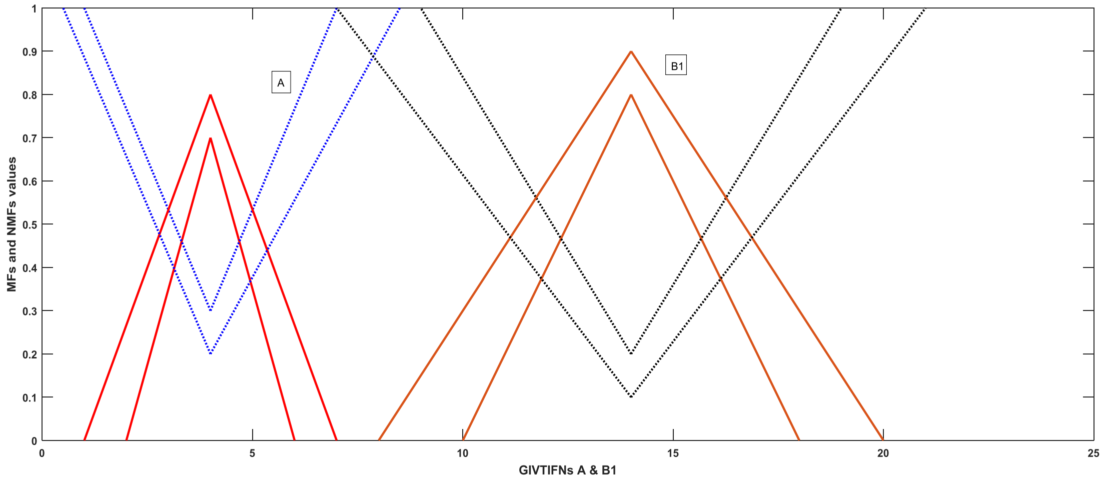

Fig. (1). A and B1.

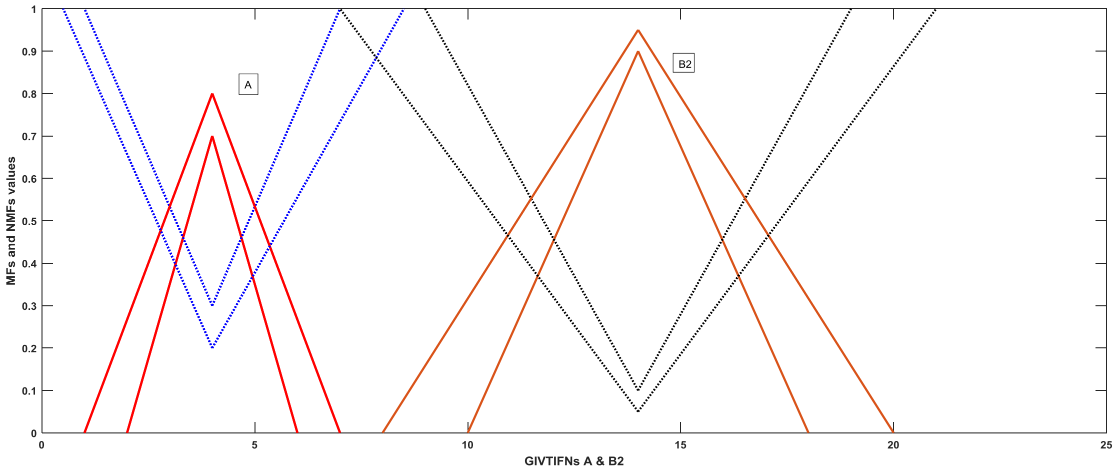

Fig. (2). A and B2.

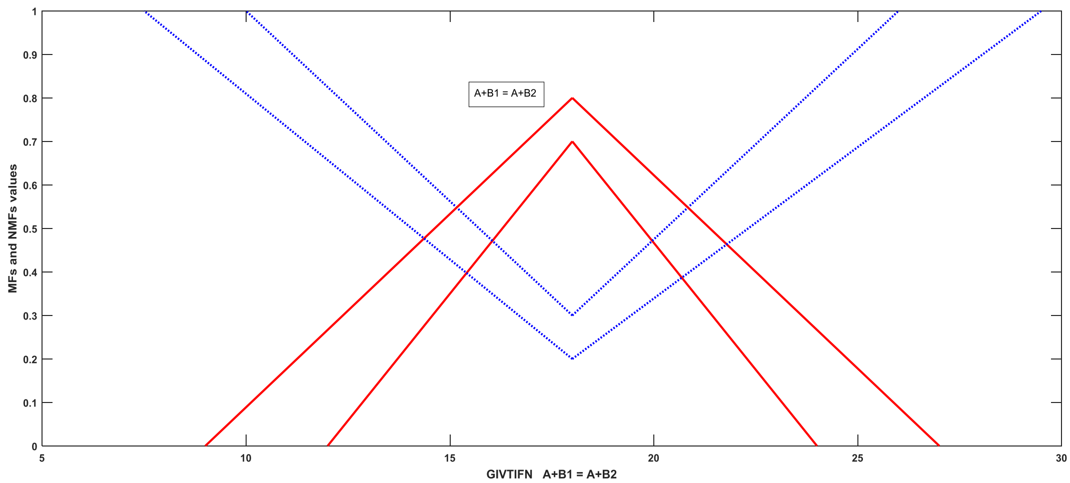

Fig. (3). Sum of GIVTIFNs.

1.2. Motivation

GIVTIFNs play an important role while dealing with uncertainty modeling problems in real life situations as they have the capability to represent imprecision, uncertainty in a proper manner, and are desirable to address such problems.

GIVTIFNs are used as mathematical assessment for different linguistic variables, ratings, and weights in various problems like decision making, medical diagnosis, pattern recognition etc. Proper arithmetic operations on GIVTIFNs are very important for the correct output in different problems. The existing approach produces some illogical results while performing arithmetic operation on GIVTIFNs with different heights. To overcome the shortcomings of the existing approach and for proper evaluation, it is always useful to define novel techniques for arithmetic on GIVTIFNs.

2. PRELIMINARIES

In this section, some basic definitions of FS, IFS and IVIFS have been discussed.

2.1. Definition (Fuzzy Set)

Let X be a universe of discourse; then the fuzzy subset A of X is defined by its membership function

$$\mu_A : X \rightarrow [0,1]$$

which assigns a real number μ_A (x) in the interval [0, 1], to each element $x \in A$, where the value of μ_A (x) at x shows the grade of membership of x in A.

2.2. Definition

Given a fuzzy set A in X and any real number $\boldsymbol{\alpha} \in$ [0, 1]. Then,

(a) (*$\boldsymbol{\alpha}$-cut*) the $\boldsymbol{\alpha}$ -cut fuzzy set A, denoted by ${}^{\alpha}A$ is the crisp set:

$${}^{\alpha}A = \left\{ x \in X : \mu_A(x) \geq \alpha \right\}$$

(b) (*Strong a- cut*) the strong a -cut, denoted by ${}^{\alpha+}A$ is the crisp set:

$${}^{\alpha}A = \left\{ x \in X : \mu_A(x) > \alpha \right\}$$

2.3. Definition (Support)

The support of a fuzzy set A defined on X is a crisp set defined as:

$$Supp(A) = \left\{ x \in X : \mu_A(x) > 0 \right\}$$

2.4. Definition (Height)

[1] The height of a fuzzy set A, denoted by $h(A)$, is the largest membership grade obtained by any element in the set and it is denoted as $h(A) = \sup_{x \in X} \mu_A(x)$.

2.5. Definition (Generalized Fuzzy Numbers (GFN))

The membership function of GFN A= [$a,b,c,d;w$] where $a \leq b \leq c \leq d$, $0 < w \leq 1$ is defined as:

$$\mu_A(x) = \begin{cases} 0, x < a \\ w\dfrac{x-a}{b-a}, a \leq x \leq b \\ w, b \leq x \leq c \\ w\dfrac{d-x}{d-c}, c \leq x \leq d \\ 0, x > d \end{cases}$$

If w = 1, then GFN A is a normal trapezoidal fuzzy number A = [a, b, c, d]. If a = b and c = d, then A is a crisp interval.If b = c then A is a generalized triangular fuzzy number. If a = b = c = d and w = 1 then A is a real number. Compared to normal fuzzy number, the GFN can deal with uncertain information in a more flexible manner because of the parameter w that represents the degree of confidence of opinions of decision maker's.

2.6. Definition (Intuitionistic Fuzzy Set (IFS))

An Intuitionistic fuzzy set A on a universe of discourse X is of the form:

$$A = \{(x, \mu_A(x), \nu_A(x) : x \in X\},$$

Where $\nu_A(x) \in [0,1]$ is called the "degree of membership of x in A", $\nu_A(x) \in [0,1]$ is called the "degree of non-membership of x in A", and where $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \le \mu_A(x) + \nu_A(x) \le 1$$

The amount $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$ is called hesitancy of x which is a reflection of lack of commitment or uncertainty associated with the membership or non-membership or both in A.

2.7. Definition (Generalized triangular intuitionistic fuzzy number (GTIFN))

The membership function of GTIFN $A = \langle [a_1, b, c_1; w], [a_2, b, c_2; \eta] \rangle$, where $a' \le a \le b \le c \le c'$ is defined as:

$$\mu_A(x) = \begin{cases} w\dfrac{x - a_1}{b - a_1}, a_1 \le x \le b \\ w\dfrac{c_1 - x}{c_1 - b}, b \le x \le c_1 \\ 0, \text{ otherwise} \end{cases}$$

and the non-membership function of the GTIFS A is defined as:

$$\nu_A(x) = \begin{cases} \dfrac{(b - x) + (x - a_2)\eta}{b - a_2}, a_2 \le x \le b \\ \dfrac{(x - b) + (c_2 - x)\eta}{c_2 - b}, b \le x \le c_2 \\ 1, \text{ otherwise} \end{cases}$$

For example, consider the GTIFS $A = \langle [4, 6, 8; 0.7], [2, 6, 10; 0.2] \rangle$. The MF and NMF of A are shown in Fig. (**4**).

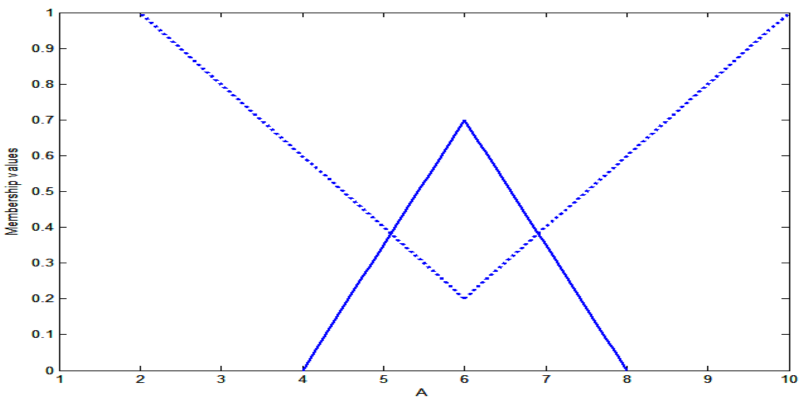

Fig. (4). Membership and non-membership of GTIFS $A = \langle [a_1, b, c, d_1; w], [a_2, b, c, d_2; \eta] \rangle$

2.8. Definition (Generalized trapezoidal intuitionistic fuzzy number (GTrIFN))

The membership function of trapezoidal GTrIFN $A=\langle [a_1,b,c,d_1;w],[a_2,b,c,d_2;\eta] \rangle$, where $a' \le a \le b \le c \le d \le d'$ is defined as:

$$\mu_A(x)=\begin{cases} w\dfrac{x-a_1}{b-a_1}, a_1 \le x \le b \\ w, \quad b \le x \le c \\ w\dfrac{d_1-x}{d_1-c}, c \le x \le d_1 \\ 0, \text{ otherwise} \end{cases}$$

and the non-membership function of the GTrIFN A is defined as:

$$\nu_A(x)=\begin{cases} \dfrac{(b-x)+(x-a_2)\eta}{b-a_2}, a_2 \le x \le b \\ \eta, \quad b \le x \le c \\ \dfrac{(x-c)+(d_2-x)\eta}{d_2-c}, c \le x \le d_2 \\ 1, \text{ otherwise} \end{cases}$$

For example, consider the GTrIFS $A=\langle [4,6,8,10;0.6],[2,4,6,12;0..3] \rangle$. The MF and NMF of A are shown in Fig. (**5**).

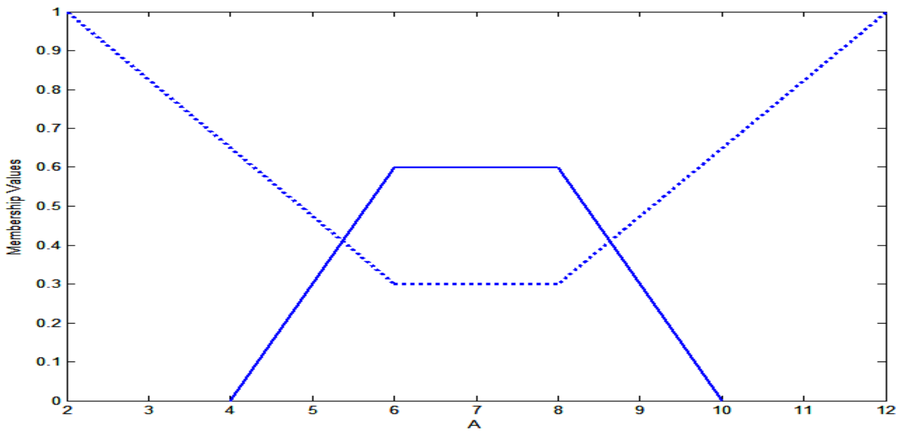

Fig. (5). Membership and non-membership of GTrIFS $A=\langle [4,6,8,10;0.6],[2,4,6,12;0..3] \rangle$.

2.9. Definition (Positive and Negative GIFS)

A GIFS is said to be positive GIFS $a_2 > 0$ if and negative GIFS if $a_2 < 0$.

2.10. Definition (Support of a GIFS)

Let $A=\langle [a_1,b,c_1;w],[a_2,b,c_2;\eta] \rangle$ be a GIFS then support of A is defined as:

$$Supp(A) = \{x \in X : \mu_A(x) > 0 \,\&\, \nu_A(x) < 1\}$$

2.11. Definition (Height of a GFIS)

Let $A = \langle [a_1, b, c_1; w], [a_2, b, c_2; \eta] \rangle$ be a GIFS then height for MF is defined as:

$Hgt(A_+) = Sup\{\mu_A(x)\}$ and height for NMF is defined as $Hgt(A_-) = Inf\{\nu_A(x)\}$.

2.12. Definition (Level Set of GFIS)

Let $A = \langle [a_1, b, c_1; w], [a_2, b, c_2; \eta] \rangle$ be a GIFS then level set for MF is defined as $\Lambda(A_+) = \{\alpha : \mu_A(x) = \alpha, \alpha \in [0, w]\}$

and level set for NMF is defined as $\Lambda(A_-) = \{\alpha : \mu_A(x) = \alpha, \alpha \in [\eta, 1]\}$

2.13. Definition (The α -cut of MF and NMF of the GIFS)

The α -cut of the MF of the GIFS $A = \langle [a_1, b, c_1; w], [a_2, b, c_2; \eta] \rangle$ is defined as:

$$^{\alpha}A_+ = \left[\frac{\alpha}{w}(b - a_1) + a_1, c_1 - \frac{\alpha}{w}(c_1 - b) \right], \alpha \in [0, w]$$

The α -cut of the NMF of the GIFS $A = \langle [a_1, b, c_1; w], [a_2, b, c_2; \eta] \rangle$ is defined as:

$$^{\alpha}A_- = \left[\frac{(b - a_2\eta) - (b - a_2)\alpha}{1 - \eta}, \frac{\alpha(c_2 - b) - (c_2\eta - b)}{1 - \eta} \right], \alpha \in [\eta, 1].$$

2.14. Definition (The α -cut GIFS)

Let $A = \{(x, \mu_A(x), \nu_A(x) : x \in X\}$, be a GIFS and –-cut of MF (μ_A) be $^{\alpha}A_+ = [L_{\overline{x_\alpha}}, R_{\overline{x_\alpha}}]$ and NMF (ν_A) be $^{\alpha}A_- = [L_{\underline{x_\alpha}}, R_{\underline{x_\alpha}}]$ respectively. Then –-cut of GIFS A can be evaluated by the following formula:

$$^{\alpha}A = \begin{cases} ([L_{\overline{x_\alpha}}, R_{\overline{x_\alpha}}], [L_{\underline{x_\alpha}}, R_{\underline{x_\alpha}}]), \alpha \le h(A_+) \;\&\; \alpha \ge h(A_-). \\ (\phi, [L_{\underline{x_\alpha}}, R_{\underline{x_\alpha}}]), \alpha > h(A_+) \;\&\; \alpha \ge h(A_-). \\ ([L_{\overline{x_\alpha}}, R_{\overline{x_\alpha}}], \phi), \alpha \le h(A_+) \;\&\; \alpha < h(A_-). \\ (\phi, \phi), \alpha > h(A_+) \;\&\; \alpha < h(A_-). \end{cases}$$

where $\forall \alpha : L_{\underline{x_\alpha}} \le L_{\overline{x_\alpha}} \le R_{\overline{x_\alpha}} \le R_{\underline{x_\alpha}}$, A_+ & A_- mF and NMF such that $h(A_+) = Sup\,\mu_A(x)$ is the height of MF, $h(A_-) = \inf \nu_A(x)$ is the height of NMF and $\emptyset$ is an empty set.

2.15. Definition (Special IFS)

Let $A = \{(x, \mu_A(x), \nu_A(x) : x \in X\}$, be a GIFS defined on the universe of discourse X, then a special IFS can be defined as $\alpha A = \alpha.{}^{\alpha}A$.

In particular, for MF $\alpha A_+ = \alpha.^{\alpha}A_+$, and for NMF, $\alpha A_- = \alpha.^{\alpha}A_-$.

2.16. Definition (Generalized Interval-Valued Triangular Intuitionistic Fuzzy Number (GIVTIFN))

The membership functions (lower MF (LMF) and upper MF (UMF) of GIVTIFN

$$A=\left\langle\left[\left(a_1^{lm},b,c_1^{lm};w^{lm}\right),\left(a_1^{um},b,c_1^{um};w^{um}\right)\right],\left[\left(a_2^{ln},b,c_2^{ln};\eta^{ln}\right),\left(a_2^{un},b,c_2^{un};\eta^{un}\right)\right]\right\rangle,$$

where $a_2^{u_n} \le a_1^{u_m} \le b \le c_1^{u_m} \le c_2^{u_n}$, $a_2^{l_n} \le a_1^{l_m} \le b \le c_1^{l_m} \le c_2^{l_n}$ is defined as:

$$\mu_A(x)=\begin{cases}\left.\begin{cases} w^{lm}\dfrac{x-a_1^{lm}}{b-a_1^{lm}}, a_1^{lm}\le x\le b\\ w^{lm}\dfrac{c_1^{lm}-x}{c_1^{lm}-b}, b\le x\le c_1^{lm}\\ 0,\ \text{otherwise}\end{cases}\right\}\ \text{LMF}\\ \left.\begin{cases} w^{um}\dfrac{x-a_1^{um}}{b-a_1^{um}}, a_1^{um}\le x\le b\\ w^{um}\dfrac{c_1^{um}-x}{c_1^{um}-b}, b\le x\le c_1^{um}\\ 0,\ \text{otherwise}\end{cases}\right\}\ \text{UMF}\end{cases}$$

and the non-membership function of the GIVTIFN A is defined as:

$$\nu_A(x)=\begin{cases}\left.\begin{cases} \dfrac{(b-x)+(x-a_2^{ln})\eta^{ln}}{b-a_2^{ln}}, a_2^{ln}\le x\le b\\ \dfrac{(x-b)+(c_2^{ln}-x)\eta^{ln}}{c_2^{ln}-b}, b\le x\le c_2^{ln}\\ 1,\ \text{otherwise}\end{cases}\right\}\ \text{LNMF}\\ \left.\begin{cases} \dfrac{(b-x)+(x-a_2^{un})\eta^{un}}{b-a_2^{un}}, a_2^{um}\le x\le b\\ \dfrac{(x-b)+(c_2^{un}-x)\eta^{un}}{c_2^{un}-b}, b\le x\le c_2^{un}\\ 1,\ \text{otherwise}\end{cases}\right\}\ \text{UNMF}\end{cases}$$

For example, consider the GIVTIFN $A = \langle [(2,3,4;0.4),(1,3,5;0.6)],[(1,3,5;0.5),(0,3,5.5;0.3)] \rangle$. The MFs and NMFs of A are shown in Fig. (**6**).

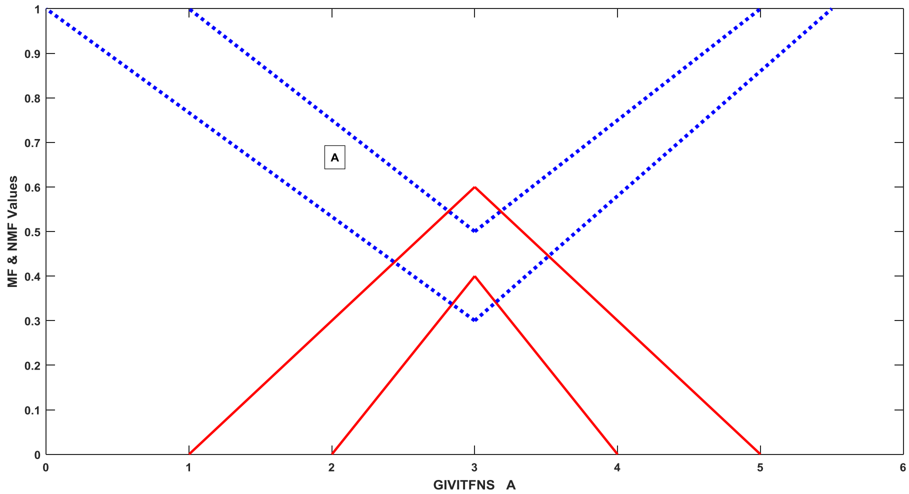

Fig. (6). MFs and NMFs of the GIVTIFNs $A = \langle [(2,3,4;0.4),(1,3,5;0.6)],[(1,3,5;0.5),(0,3,5.5;0.3)] \rangle$.

2.17. Definition (Generalized Interval-Valued Trapezoidal Intuitionistic Fuzzy Number (GIVTrIFN))

The membership functions (LMF and UMF) of GIVTrIFN $A = \left\langle \left[\left(a_1^{lm}, b, c, d_1^{lm}; w^{lm} \right), \left(a_1^{um}, b, c, d_1^{um}; w^{um} \right) \right], \left[\left(a_2^{ln}, b, c, d_2^{ln}; \eta^{ln} \right), \left(a_2^{un}, b, c, d_2^{un}; \eta^{un} \right) \right] \right\rangle$, where $a_2^{u_n} \le a_1^{u_m} \le b \le c \le d_1^{u_m} \le d_2^{u_n}$, $a_2^{l_n} \le a_1^{l_m} \le b \le c \le d_1^{l_m} \le d_2^{l_n}$ are defined as:

$$\mu_A(x) = \begin{cases} \left. \begin{array}{l} w^{lm} \dfrac{x - a_1^{lm}}{b - a_1^{lm}}, a_1^{lm} \le x \le b \\ w^{lm}, \qquad b \le x \le c \\ w^{lm} \dfrac{d_1^{lm} - x}{d_1^{lm} - c}, c \le x \le d_1^{lm} \\ 0, \text{ otherwise} \end{array} \right\} \text{LMF} \\ \left. \begin{array}{l} w^{um} \dfrac{x - a_1^{um}}{b - a_1^{um}}, a_1^{um} \le x \le b \\ w^{um}, \qquad b \le x \le c \\ w^{um} \dfrac{d_1^{um} - x}{d_1^{um} - c}, c \le x \le d_1^{um} \\ 0, \text{ otherwise} \end{array} \right\} \text{UMF} \end{cases}$$

and the non-membership functions (lower NMF (LNMF) and upper NMF (UNMF)) of the GIVTrIFN A are defined as:

$$\nu_{A^l}(x) = \begin{cases} \left.\begin{array}{ll} \dfrac{(b-x)+(x-a_2^{ln})\eta^{ln}}{b-a_2^{ln}}, & a_2^{ln} \le x \le b \\ \eta^{ln}, & b \le x \le c \\ \dfrac{(x-c)+(d_2^{ln}-x)\eta^{ln}}{d_2^{ln}-c}, & c \le x \le d_2^{ln} \\ 1, \text{ otherwise} & \end{array}\right\} \text{LNMF} \\ \left.\begin{array}{ll} \dfrac{(b-x)+(x-a_2^{un})\eta^{un}}{b-a_2^{un}}, & a_2^{un} \le x \le b \\ \eta^{un}, & b \le x \le c \\ \dfrac{(x-c)+(d_2^{un}-x)\eta^{un}}{d_2^{un}-c}, & c \le x \le d_2^{un} \\ 1, \text{ otherwise} & \end{array}\right\} \text{UNMF} \end{cases}$$

For example, consider the GIVTrIFN $A = \left[\left(9, 10.73, 11.27, 13; 0.4\right), \left(7, 10.73, 11.27, 15; 0.6\right), \left(7, 10.73, 11.27, 15; 0.5\right), \left(5, 10.73, 11.27, 16.5; 0.3\right)\right]$. The MFs and NMFs are shown in Fig. (**7**).

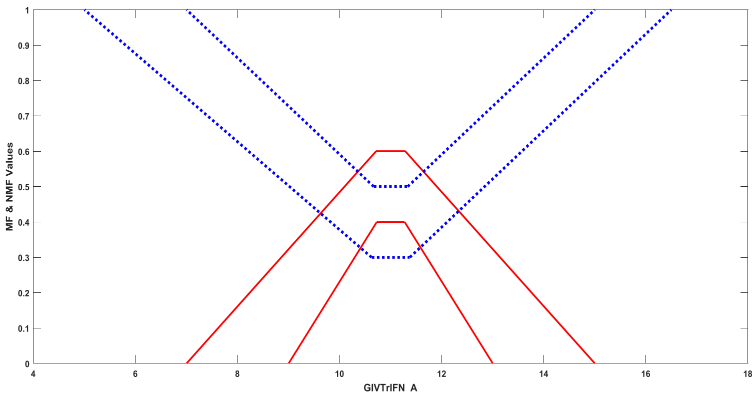

Fig. (7). MFs and NMFs of the GIVTrIFN $A = \left[\left(9, 10.73, 11.27, 13; 0.4\right), \left(7, 10.73, 11.27, 15; 0.6\right), \left(7, 10.73, 11.27, 15; 0.5\right), \left(5, 10.73, 11.27, 16.5; 0.3\right)\right]$.

2.18. Definition (Support of a GIVTIFS)

Let $A = \left\langle \left[\left(a_1^{lm}, b, c_1^{lm}; w^{lm}\right), \left(a_1^{um}, b, c_1^{um}; w^{um}\right)\right], \left[\left(a_2^{ln}, b, c_2^{ln}; \eta^{ln}\right), \left(a_2^{un}, b, c_2^{un}; \eta^{un}\right)\right] \right\rangle$ be a, GIVTIFS then support of A is defined as:

$$Supp(A) = \left\{ x \in X : \mu_A^{lm}(x) > 0,\ \mu_A^{um}(x) > 0 \ \&\ \nu_A^{ln}(x) < 1, \nu_A^{un}(x) < 1 \right\}.$$

2.19. Definition (Height of a GIVTIFS)

Let $A = \left\langle \left[\left(a_1^{lm}, b, c_1^{lm}; w^{lm}\right), \left(a_1^{um}, b, c_1^{um}; w^{um}\right)\right], \left[\left(a_2^{ln}, b, c_2^{ln}; \eta^{ln}\right), \left(a_2^{un}, b, c_2^{un}; \eta^{un}\right)\right] \right\rangle$ be a GIVTIFS, then height for

MFs is defined as:

$Hgt(A_+^{l}) = Sup\{\mu^{lm}{}_A(x)\}$, $Hgt(A_+^{um}) = Sup\{\mu^{um}{}_A(x)\}$ and height for NMFs are defined as $Hgt(A_-^{ln}) = Inf\{\nu^{ln}{}_A(x)\}$, $Hgt(A_-^{un}) = Inf\{\nu^{un}{}_A(x)\}$.

2.20. Definition (Level Set of GIVTIFS)

Let $A = \left\langle \left[\left(a_1^{lm}, b, c_1^{lm}; w^{lm}\right), \left(a_1^{um}, b, c_1^{um}; w^{um}\right)\right], \left[\left(a_2^{ln}, b, c_2^{ln}; \eta^{ln}\right), \left(a_2^{un}, b, c_2^{un}; \eta^{un}\right)\right] \right\rangle$ be a GIVTIFS, then the level set for MF is defined as:

$\Lambda(A_+^{l}) = \left\{\alpha : \mu^{lm}{}_A(x) = \alpha, \alpha \in [0, w^{lm}]\right\}$, $\Lambda(A_+^{um}) = \left\{\alpha : \mu^{um}{}_A(x) = \alpha, \alpha \in [0, w^{um}]\right\}$ and level set for NMFs are defined as

$$\Lambda(A_-^{ln}) = \left\{\alpha : \mu^{ln}{}_A(x) = \alpha, \alpha \in [\eta^{ln}, 1]\right\},\ \Lambda(A_-^{un}) = \left\{\alpha : \mu^{un}{}_A(x) = \alpha, \alpha \in [\eta^{un}, 1]\right\}$$

2.21. Definition (The α -cut of MFs and β -cut of NMFs of the GIVTIFS))

The $\boldsymbol{\alpha}$ -cut of the MFs of the GIVTIFS $A = \left\langle \left[\left(a_1^{lm}, b, c_1^{lm}; w^{lm}\right), \left(a_1^{um}, b, c_1^{um}; w^{um}\right)\right], \left[\left(a_2^{ln}, b, c_2^{ln}; \eta^{ln}\right), \left(a_2^{un}, b, c_2^{un}; \eta^{un}\right)\right] \right\rangle$ is defined as

$$^{\alpha}A^{l}_{+} = \left[\frac{\alpha}{w^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}, c_1^{l_m} - \frac{\alpha}{w^{l_m}}(c_1^{l_m} - b)\right], \alpha \in [0, w^{l_m}]$$

$$^{\alpha}A^{u}_{+} = \left[\frac{\alpha}{w^{\mu_m}}(b - a_1^{u_m}) + a_1^{u_m}, c_1^{u_m} - \frac{\alpha}{w^{\mu_m}}(c_1^{u_m} - b)\right], \alpha \in [0, w^{\mu_m}]$$

The α -cut of the NMFs of the GIVTIFS $A = \left\langle \left[\left(a_1^{lm}, b, c_1^{lm}; w^{lm}\right), \left(a_1^{um}, b, c_1^{um}; w^{um}\right)\right], \left[\left(a_2^{ln}, b, c_2^{ln}; \eta^{ln}\right), \left(a_2^{un}, b, c_2^{un}; \eta^{un}\right)\right] \right\rangle$ is defined as

$$^{\beta}A^{l}_{-} = \left[\frac{(b - a_2^{l_n}\eta^{l_n}) - (b - a_2^{l_n})\beta}{1 - \eta^{l_n}}, \frac{\beta(c_2^{l_n} - b) - (c_2^{l_n}\eta^{l_n} - b)}{1 - \eta^{l_n}}\right], \beta \in [\eta^{l_n}, 1].$$

$$^{\beta}A^{u}_{-} = \left[\frac{(b - a_2^{u_n}\eta^{u_n}) - (b - a_2^{u_n})\beta}{1 - \eta^{u_n}}, \frac{\beta(c_2^{u_n} - b) - (c_2^{u_n}\eta^{u_n} - b)}{1 - \eta^{u_n}}\right], \beta \in [\eta^{u_n}, 1].$$

2.22. Definition (The α, β -cut GIVTIFS)

Let $A = \left\{\left(x, \left[\mu_{A^l}(x), \mu_{A^u}(x)\right], \left[\nu_{A^l}(x), \nu_{A^u}(x)\right] : x \in X\right)\right\}$, be a GIVTIFS and α-cut of MFs $\left(\mu_{A^l}(x), \mu_{A^u}(x)\right)$ be $^{\alpha}A_{+} = \left[{}^{\alpha}A_{+}^{l}, {}^{\alpha}A_{+}^{u}\right]$, where ${}^{\alpha}A_{+}^{l} = \left[L^{l}\overline{x_\alpha}, R^{l}\overline{x_\alpha}\right]$, ${}^{\alpha}A_{+}^{u} = \left[L^{u}\overline{x_\alpha}, R^{u}\overline{x_\alpha}\right]$ and NMFs $\left(\nu_{A^l}(x), \nu_{A^u}(x)\right)$ be $^{\beta}A_{-} = \left[{}^{\beta}A_{-}^{l}, {}^{\beta}A_{-}^{u}\right]$, where ${}^{\beta}A_{-}^{l} = \left[L^{l}_{\underline{x_\beta}}, R^{l}_{\underline{x_\beta}}\right]$, ${}^{\beta}A_{-}^{u} = \left[L^{u}_{\underline{x_\beta}}, R^{u}_{\underline{x_\beta}}\right]$ respectively. Then α,β-cut of GIVTIFS A can be evaluated by the following formula:

$$
{}^{\alpha,\beta}A=\begin{cases}
\left[\left(\left[L^{l}_{\overline{x_\alpha}},R^{l}_{\overline{x_\alpha}}\right],\left[L^{u}_{\overline{x_\alpha}},R^{u}_{\overline{x_\alpha}}\right]\right),\left(\left[L^{l}_{\underline{x_\alpha}},R^{l}_{\underline{x_\alpha}}\right],\left[L^{u}_{\underline{x_\alpha}},R^{u}_{\underline{x_\alpha}}\right]\right)\right], \alpha\le h\left(A^{l}_{+}\right),h\left(A^{u}_{+}\right)\ \&\ \alpha\ge h\left(A^{l}_{-}\right),h\left(A^{u}_{-}\right).\\
\left[\phi,\left(\left[L^{l}_{\underline{x_\beta}},R^{l}_{\underline{x_\beta}}\right],\left[L^{u}_{\underline{x_\beta}},R^{u}_{\underline{x_\beta}}\right]\right)\right], \beta> h\left(A^{l}_{+}\right),h\left(A^{u}_{+}\right)\ \&\ \beta\ge h\left(A^{l}_{-}\right),h\left(A^{u}_{-}\right)\\
\left[\left(\left[L^{l}_{\overline{x_\alpha}},R^{l}_{\overline{x_\alpha}}\right],\left[L^{u}_{\overline{x_\alpha}},R^{u}_{\overline{x_\alpha}}\right]\right),\varphi\right], \alpha\le h\left(A^{l}_{+}\right),h\left(A^{u}_{+}\right)\ \&\ \alpha< h\left(A^{l}_{-}\right),h\left(A^{u}_{-}\right).\\
(\phi,\phi), \beta> h\left(A^{l}_{+}\right),h\left(A^{u}_{+}\right)\ \&\ \beta< h\left(A^{l}_{-}\right),h\left(A^{u}_{-}\right).
\end{cases}
$$

where A^{l}_{+}, A^{u}_{+} & A^{l}_{-}, A^{u}_{-} MFs and NMFs such that $h\left(A^{l}_{+}\right)=Sup\,\mu_{A^{l}}(x)$, $h\left(A^{u}_{+}\right)=Sup\,\mu_{A^{u}}(x)$ are the height of MFs, $h\left(A^{l}_{-}\right)=\inf \nu_{A^{l}}(x)$, $h\left(A^{u}_{-}\right)=\inf \nu_{A^{u}}(x)$ are the height of NMFs and is an empty set.

3. DECOMPOSITION THEOREM FOR GIVTrIFN

In this section, decomposition theorems for GIVTrIFN have been discussed.

3.1. Theorem (First Decomposition Theorem)

Let X be a universe of discourse. For any GIVTrIFN $A=\left\langle\left[\left(a_1^{lm},b,c,d_1^{lm};w^{lm}\right),\left(a_1^{um},b,c,d_1^{um};w^{um}\right)\right],\left[\left(a_2^{ln},b,c,d_2^{ln};\eta^{ln}\right),\left(a_2^{un},b,c,d_2^{un};\eta^{un}\right)\right]\right\rangle$ in X,

$$
A=\begin{cases}
\left.\begin{array}{l}\bigcup_{\alpha\in[0,w^{lm}]}\ \alpha A^{lm}_{+}\\ \bigcup_{\alpha\in[0,w^{um}]}\ \alpha A^{lm}_{+}\end{array}\right\}\text{ for MFs}\\
\left.\begin{array}{l}\bigcap_{\beta\in[\eta^{ln},1]}\ \beta A^{ln}_{-}\\ \bigcap_{\beta\in[\eta^{un},1]}\ \beta A^{un}_{-}\end{array}\right\}\text{ for NMFs}
\end{cases}
$$

where $\cup$ and $\cap$ are standard fuzzy union and intersection, respectively.

Proof

For MF, let for each $x\in X$, $\mu_{A^{ln}}(x)=a$ where $a\in[0,w^{ln}]$ which indicates the degree of belonging in A. Then,

$$
\begin{aligned}
\bigcup_{\alpha\in[0,w^{lm}]}\alpha A^{lm}_{+} &= \underset{\alpha\in[0,w^{lm}]}{Sup}\ \alpha\,{}^{\alpha}\mathrm{A}^{lm}_{+}\\
&=\max\left[\underset{\alpha\in[0,a]}{Sup}\ \alpha\,{}^{\alpha}A^{lm}_{+},\ \underset{\alpha\in(a,w]}{Sup}\ \alpha\,{}^{\alpha}A^{lm}_{+}\right]
\end{aligned}\tag{3.1}
$$

If $\alpha \in [0,a]$ then $\alpha \le a = \mu_{A^{l_m}}(x)$

i.e., $\alpha \in {}^{\alpha}A_{+}^{l_m}$ then $\alpha\, {}^{\alpha}A_{+}^{l_m} = \alpha.$

If $\alpha \in (a, w^{l_m}]$then $\alpha > a = \mu_{A}^{l_m}(x)$

i.e., $\alpha \notin {}^{\alpha}A_{+}^{l_m}$ then $\alpha\, {}^{\alpha}A_{+}^{l_m} = 0.$

Hence from (3.1), we have

$$\underset{\alpha\in[0,w^{l_m}]}{\mathrm{U}} \alpha A_{+}^{l_m} = \max\left[\underset{\alpha\in[0,a]}{Sup} \alpha, 0\right]$$

$$= a$$

$$= \mu_{A^{l_m}}(x)$$

Similarly, $\underset{\alpha\in[0,w^{u_m}]}{\mathrm{U}} \alpha A_{+}^{u_m} = \mu_{A^{u_m}}(x)$

For NMF, let for each $x \in X, \nu_{A^{l_n}}(x) = b$where $b \in [\eta^{l_n}, 1]$ which indicates the degree of non-belonging in *A*.

$$\begin{aligned} \underset{\beta\in[\eta^{l_n},1]}{\mathrm{I}} \beta \mathrm{A}_{-}^{l_n} &= \underset{\beta\in[\eta^{l_n},1]}{Inf} \beta\, {}^{\beta}A_{-}^{l_n} \\ &= \min\left[\underset{\beta\in[\eta^{l_n},b)}{Inf} \beta\, {}^{\beta}A_{-}^{l_n},\ \underset{\beta\in[b,1]}{Inf} \beta.{}^{\beta}A_{-}^{l_n}\right] \end{aligned} \tag{3.2}$$

If $\beta \in [\eta^{l_n}, b)$ then $\beta < b = \nu_{A^{l_n}}(x)$

i.e., $\beta \notin {}^{\beta}A_{-}^{l_n}$ then $\beta\, {}^{\beta}A_{-}^{l_n} = 1.$

If $\beta \in [b,1]$ then $\beta \ge b = \nu_{A^{l_n}}(x)$

i.e., $\beta \in {}^{\beta}A_{-}^{l_n}$ then $\beta\, {}^{\beta}A_{-}^{l_n} = \beta.$

Hence from (3.2), we have

$$\underset{\beta\in[\eta^{l_n},1]}{\mathrm{I}} \beta\, {}^{\beta}A_{-}^{l_n} = \min\left[1,\ \underset{\beta\in[b,1]}{Inf}\beta\right]$$

$$= b$$

$$= \nu_{A^{l_n}}(x)$$

Similarly, $\bigcap_{\beta\in[\eta^{l_n},1]} \beta A_{-}^{l_n} = \nu_{A^{l_n}}(x)$.

3.2. Theorem (Second Decomposition Theorem)

Let X be a universe of discourse. For any GIVTrIFN $A=\left\langle\left[\left(a_1^{l_m},b,c,d_1^{l_m};w^{l_m}\right),\left(a_1^{u_m},b,c,d_1^{u_m};w^{u_m}\right)\right],\left[\left(a_2^{l_n},b,c,d_2^{l_n};\eta^{l_n}\right),\left(a_2^{u_n},b,c,d_2^{u_n};\eta^{u_n}\right)\right]\right\rangle$ in X,

$$A=\begin{cases}\left.\begin{array}{l}\bigcup_{\alpha\in[0,w^{l_m}]} \alpha {+}A_{+}^{l_m}\\ \bigcup_{\alpha\in[0,w^{u_m}]} \alpha {+}A_{+}^{u_m}\end{array}\right\} & \text{for MFs}\\ \left.\begin{array}{l}\bigcap_{\beta\in[\eta^{l_n},1]} \beta {+}A_{-}^{l_n}\\ \bigcap_{\beta\in[\eta^{u_n},1]} \beta {+}A_{-}^{u_n}\end{array}\right\} & \text{for NMFs}\end{cases}$$

where $\cup$ and $\cap$ are standard fuzzy union and intersection, respectively.

Proof

For MF, let for each $x\in X$, $\nu_{A^{l_n}}(x)=a$ where $a\in[0,w^{l_n}]$ which indicates the degree of belonging in A. Then,

$$\begin{aligned}\bigcup_{\alpha\in[0,w^{l_m}]} \alpha + A_{+}^{l_m} &= \operatorname*{Sup}_{\alpha\in[0,w^{l_m}]} \alpha^{\alpha+}A_{+}^{l_m}\\ &= \max\left[\operatorname*{Sup}_{\alpha\in[0,a)} \alpha^{\alpha+}A_{+}^{l_m},\ \operatorname*{Sup}_{\alpha\in[a,w^{l_m}]} \alpha^{\alpha+}A_{+}^{l_m}\right]\end{aligned} \tag{3.3}$$

If $\alpha\in[0,a)$ then $\alpha<a=\mu_{A^{l_m}}(x)$

i.e., $\alpha\in{}^{\alpha+}A^{l_m}$ then $\alpha^{\alpha+}{}_{A^{l_m}}=\alpha$.

If $\alpha\in[a,w^{l_m}]$then $\alpha\geq a=\mu_{A^{l_m}}(x)$

i.e., $\alpha\notin{}^{\alpha+}A_{+}^{l_m}$ then $\alpha^{\alpha+}A_{+}^{l_m}=0$.

Hence from (3.3), we have

$$\begin{aligned}\bigcup_{\alpha\in[0,w^{l_m}]} \alpha + A_{+}^{l_m} &= \max[\operatorname*{Sup}_{\alpha\in[0,a)} \alpha, 0]\\ &= a\\ &= \mu_{A^{l_m}}(x)\end{aligned}$$

Similarly, $\bigcup\limits_{\alpha\in[0,w^{u_m}]} \alpha + A_{+}^{u_m} = \mu_{A^{u_m}}(x)$

For NMF, let for each $x \in X$, $\nu_{A^{l_n}}(x) = b$ where $b \in [\eta^{l_n}, 1]$ which indicates the degree of non-belonging in A.

$$
\begin{aligned}
\bigcap_{\beta\in[\eta^{l_n},1]} \beta \mathrm{A}_{-}^{l_n} &= \inf_{\beta\in[\eta,1]} \beta\, {}^{\beta}\mathrm{A}_{-}^{l_n} \\
&= \min\left[\inf_{\beta\in[\eta^{l_n},b]} \beta\, {}^{\beta}\mathrm{A}_{-}^{l_n},\ \inf_{\beta\in(b,1]} \beta\, {}^{\beta}\mathrm{A}_{-}^{l_n}\right]
\end{aligned}
\tag{3.4}
$$

If $\beta \in [\eta^{l_n}, b]$ then $\beta \le b = \nu_{A^{l_n}}(x)$

i.e., $\beta \notin {}^{\beta}\mathrm{A}_{-}^{l_n}$ then $\beta\, {}^{\beta}\mathrm{A}_{-}^{l_n} = 1.$

If $\beta \in (b, 1]$ then $\beta > b = \nu_{A}^{l_n}(x)$

i.e., $\beta \in {}^{\beta+}\mathrm{A}_{-}^{l_n}$ then $\beta + {}^{\beta+}\mathrm{A}_{-}^{l_n} = \beta.$

Hence from (3.4), we have

$$
\begin{aligned}
\bigcap_{\beta\in[\eta^{l_n},1]} \beta\, {}^{\beta}\mathrm{A}_{-}^{l_n} &= \min\left[1,\ \inf_{\beta\in[b,1]} \beta\right] \\
&= b \\
&= \nu_{A^{l_n}}(x)
\end{aligned}
$$

3.3. Theorem (Third Decomposition Theorem)

Let X be a universe of discourse. For any GIVTrIFN $A = \left\langle \left[\left(a_1^{l_m}, b, c, d_1^{l_m}; w^{l_m}\right), \left(a_1^{u_m}, b, c, d_1^{u_m}; w^{u_m}\right)\right], \left[\left(a_2^{l_n}, b, c, d_2^{l_n}; \eta^{l_n}\right), \left(a_2^{u_n}, b, c, d_2^{u_n}; \eta^{u_n}\right)\right] \right\rangle$ in X,

$$
A = \begin{cases}
\left.\begin{array}{l} \bigcup\limits_{\alpha\in\Lambda(A)} \alpha A_{+}^{l_m} \\ \bigcup\limits_{\alpha\in\Lambda(A)} \alpha A_{+}^{l_n} \end{array}\right\} \text{for MFs} \\
\left.\begin{array}{l} \bigcap\limits_{\beta\in\Lambda(A)} \alpha A_{+}^{u_m} \\ \bigcap\limits_{\beta\in\Lambda(A)} A_{+}^{u_n} \end{array}\right\} \text{for NMFs}
\end{cases}
$$

where $\cup$ and $\cap$ are standard fuzzy union and intersection, respectively, and $\lambda(A)$ is the level set of A.

4. PROPOSED ARITHMETIC TECHNIQUE FOR GIVTIFNS

GIVTIFN is the extended version of GTIFN. Arithmetic on GIVTIFNs is a crucial issue. Let us consider that and are two GIVTIFNs with different heights. Here a novel approach will be discoursed to perform the arithmetic operation between GIVTIFNs A and B. In this approach, the MFs are truncated at the smallest height of their respective (LMF and UMF) MFs. Similarly, the NMFs are truncated at the maximum heights of their respective (LNMF and UNMF) NMFs. The interesting part of this approach is that it produces GIVTrIFNs.

Supposing that the MFs and NMFs of two GIVTIFNs $A=\left\langle\left[\left(a_1^{lm},b,c_1^{lm};w^{lm}\right),\left(a_1^{um},b,c_1^{um};w^{um}\right)\right],\left[\left(a_2^{ln},b,c_2^{ln};\eta^{ln}\right),\left(a_2^{un},b,c_2^{un};\eta^{un}\right)\right]\right\rangle$ and $B=\left\langle\left[\left(a_3^{lm},b,c_3^{lm};w^{lm}\right),\left(a_3^{um},b,c_3^{um};w^{um}\right)\right],\left[\left(a_4^{ln},b,c_4^{ln};\eta^{ln}\right),\left(a_4^{un},b,c_4^{un};\eta^{un}\right)\right]\right\rangle$ are:

$$\mu_A(x)=\begin{cases}\left.\begin{cases} w_A^{lm}\dfrac{x-a_1^{lm}}{b-a_1^{lm}}, a_1^{lm}\le x\le b\\ w_A^{lm}\dfrac{c_1^{lm}-x}{c_1^{lm}-b}, b\le x\le c_1^{lm}\\ 0,\ \text{otherwise}\end{cases}\right\}\text{LMF}\\ \left.\begin{cases} w_A^{um}\dfrac{x-a_1^{um}}{b-a_1^{um}}, a_1^{um}\le x\le b\\ w_A^{um}\dfrac{c_1^{um}-x}{c_1^{um}-b}, b\le x\le c_1^{um}\\ 0,\ \text{otherwise}\end{cases}\right\}\text{UMF}\end{cases}$$

$$\nu_A(x)=\begin{cases}\left.\begin{cases} \dfrac{(b-x)+(x-a_2^{ln})\eta_A^{ln}}{b-a_2^{ln}}, a_2^{ln}\le x\le b\\ \dfrac{(x-b)+(c_2^{ln}-x)\eta_A^{ln}}{c_2^{ln}-b}, b\le x\le c_2^{ln}\\ 1,\ \text{otherwise}\end{cases}\right\}\text{LNMF}\\ \left.\begin{cases} \dfrac{(b-x)+(x-a_2^{un})\eta_A^{un}}{b-a_2^{un}}, a_2^{um}\le x\le b\\ \dfrac{(x-b)+(c_2^{un}-x)\eta_A^{un}}{c_2^{un}-b}, b\le x\le c_2^{un}\\ 1,\ \text{otherwise}\end{cases}\right\}\text{UNMF}\end{cases}$$

and

$$\mu_B(x)=\begin{cases}\left.\begin{cases} w_B^{lm}\dfrac{x-a_3^{lm}}{b_1-a_3^{lm}}, a_3^{lm}\le x\le b_1\\ w_B^{lm}\dfrac{c_3^{lm}-x}{c_3^{lm}-b_1}, b_1\le x\le c_3^{lm}\\ 0,\ \text{otherwise}\end{cases}\right\}\text{LMF}\\ \left.\begin{cases} w_B^{um}\dfrac{x-a_3^{um}}{b_1-a_3^{um}}, a_3^{um}\le x\le b_1\\ w_B^{um}\dfrac{c_3^{um}-x}{c_3^{um}-b_1}, b_1\le x\le c_3^{um}\\ 0,\ \text{otherwise}\end{cases}\right\}\text{UMF}\end{cases}$$

$$\nu_B(x) = \begin{cases} \left.\begin{array}{l} \dfrac{(b_1 - x) + (x - a_4^{l_n})\eta_B^{l_n}}{b_1 - a_4^{l_n}}, a_4^{l_n} \le x \le b_1 \\ \dfrac{(x - b_1) + (c_4^{l_n} - x)\eta_B^{l_n}}{c_4^{l_n} - b_1}, b_1 \le x \le c_4^{l_n} \\ 1, \text{ otherwise} \end{array}\right\} \text{LMF} \\ \left.\begin{array}{l} \dfrac{(b_1 - x) + (x - a_4^{u_n})\eta_B^{u_n}}{b - a_4^{u_n}}, a_4^{u_m} \le x \le b_1 \\ \dfrac{(x - b_1) + (c_4^{u_n} - x)\eta_B^{u_n}}{c_4^{u_n} - b_1}, b_1 \le x \le c_4^{u_n} \\ 1, \text{ otherwise} \end{array}\right\} \text{UMF} \end{cases}$$

respectively.

then, the $\alpha, \beta -$cut of A are ${}^{\alpha}A = \left[{}^{\alpha}A_{+}^{l}\ , {}^{\alpha}A_{+}^{u}\right]$, ${}^{\beta}A = \left[{}^{\beta}A_{-}^{l}\ , {}^{\beta}A_{-}^{u}\right]$ where

$${}^{\alpha}A_{+}^{l} = \left[\frac{\alpha}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}, c_1^{l_m} - \frac{\alpha}{w_A^{l_m}}(c_1^{l_m} - b), \alpha \in \left[0, w_A^{lm}\right]\right]$$

$${}^{\alpha}A_{+}^{u} = \left[\frac{\alpha}{w_A^{u_m}}(b - a_1^{u_m}) + a_1^{u_m}, c_1^{u_m} - \frac{\alpha}{w_A^{u_m}}(c_1^{u_m} - b), \alpha \in \left[0, w_A^{um}\right]\right]$$

$${}^{\beta}A_{-}^{l} = \left[\frac{(b - a_2^{l_n}\eta_A^{l_n}) - (b - a_2^{l_n})\beta}{1 - \eta_A^{l_n}}, \frac{\beta(c_2^{l_n} - b) - (c_2^{l_n}\eta_A^{l_n} - b)}{1 - \eta_A^{l_n}}\right]$$

$${}^{\beta}A_{-}^{u} = \left[\frac{(b - a_2^{u_n}\eta_A^{u_n}) - (b - a_2^{u_n})\beta}{1 - \eta_A^{u_n}}, \frac{\beta(c_2^{u_n} - b) - (c_2^{u_n}\eta_A^{u_n} - b)}{1 - \eta_A^{u_n}}\right]$$

On the other hand, the $\alpha, \beta -$cut of B are ${}^{\alpha}B = \left[{}^{\alpha}B_{+}^{l}\ , {}^{\alpha}B_{+}^{u}\right]$, ${}^{\beta}B = \left[{}^{\beta}B_{-}^{l}\ , {}^{\beta}B_{-}^{u}\right]$ where

$${}^{\alpha}B_{+}^{l} = \left[\frac{\alpha}{w_B^{l_m}}(b_1 - a_3^{l_m}) + a_3^{l_m}, c_3^{l_m} - \frac{\alpha}{w_B^{l_m}}(c_3^{l_m} - b_1), \alpha \in \left[0, w_B^{lm}\right]\right]$$

$${}^{\alpha}B_{+}^{u} = \left[\frac{\alpha}{w_B^{u_m}}(b_1 - a_3^{u_m}) + a_3^{u_m}, c_3^{u_m} - \frac{\alpha}{w_B^{u_m}}(c_3^{u_m} - b_1), \alpha \in \left[0, w_B^{um}\right]\right]$$

$${}^{\beta}B_{-}^{l} = \left[\frac{(b_1 - a_4^{l_n}\eta_B^{l_n}) - (b - a_4^{l_n})\beta}{1 - \eta_B^{l_n}}, \frac{\beta(c_4^{l_n} - b_1) - (c_4^{l_n}\eta_B^{l_n} - b_1)}{1 - \eta_B^{l_n}}\right]$$

$${}^{\beta}B_{-}^{u} = \left[\frac{(b_1 - a_4^{u_n}\eta_A^{u_n}) - (b_1 - a_4^{u_n})\beta}{1 - \eta_B^{u_n}}, \frac{\beta(c_4^{u_n} - b_1) - (c_4^{u_n}\eta_B^{u_n} - b_1)}{1 - \eta_B^{u_n}}\right]$$

4.1. Theorem (Addition of Two GIVTIFNs with Different Heights Produces a GIVTrIFNs)

Proof

To determine the addition of *GIVTIFNs A* and *B,* we first add the *α, β*cuts of GITVIFNs *A* and *B* using interval arithmetic.

For MFs functions

$$ {}^{\alpha}A_{+}^{l} + {}^{\alpha}B_{+}^{l} = \left[\frac{\alpha}{w_A^{l_m}}(b-a_1^{l_m}) + a_1^{l_m} + \frac{\alpha}{w_B^{l_m}}(b_1-a_3^{l_m}) + a_3^{l_m}, c_1^{l_m} - \frac{\alpha}{w_A^{l_m}}(c_1^{l_m}-b) + c_3^{l_m} - \frac{\alpha}{w_B^{l_m}}(c_3^{l_m}-b_1) \right] $$

where $w^l = \min(w_A^{l_m}, w_B^{l_m})$ and $\alpha \in [0, w^l]$

.

$$ = \left[(a_1^{l_m} + a_3^{l_m}) + \alpha \left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\}, (c_1^{l_m} + c_3^{l_m}) - \alpha \left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\} \right] \tag{4.1} $$

To find the LMF $\mu_{A^l+B^l}(x)$ we equate both the first and second component of (4.1) to *x* which gives

$$ x = (a_1^{l_m} + a_3^{l_m}) + \alpha \left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\} \text{ and } x = (c_1^{l_m} + c_3^{l_m}) - \alpha \left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\} $$

Now, expressing α in terms of x

$$ \alpha = \frac{x-(a_1^{l_m} + a_3^{l_m})}{\left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\}} \tag{4.2} $$

$$ \alpha = \frac{(c_1^{l_m} + c_3^{l_m})-x}{\left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\}} \tag{4.3} $$

Setting $\alpha \geq 0$ & $\alpha \leq w^l$ in (4.2) and $\alpha \leq w^l$ & $\alpha \geq 0$ in (4.3), we get the domain of *x*,

$$ x \in \left[(a_1^{l_m} + a_3^{l_m}), (a_1^{l_m} + a_3^{l_m}) + w^l \left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\} \right] \text{and } x \in \left[(c_1^{l_m} + c_3^{l_m}) - w \left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\}, (c_1^{l_m} + c_3^{l_m}) \right] $$

Hence the LMF $\mu_{A^l+B^l}(x)$ of is

$$ \mu_{A^l+B^l}(x) = \begin{cases} \dfrac{x-(a_1^{l_m}+a_3^{l_m})}{\left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\}}, & x \in \left[(a_1^{l_m}+a_3^{l_m}), w^l \left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\} + (a_1 + a_3^{l_m}) \right] \\ w^l, & x \in \left[w^l \left\{ \frac{b-a_1^{l_m}}{w_A^{l_m}} + \frac{b_1-a_3^{l_m}}{w_B^{l_m}} \right\} + (a_1^{l_m}+a_3^{l_m}), (c_1^{l_m}+c_3^{l_m}) - w^l \left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\} \right] \\ \dfrac{(c_1^{l_m}+c_3^{l_m})-x}{\left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\}}, & x \in \left[(c_1^{l_m}+c_3^{l_m}) - w^l \left\{ \frac{c_1^{l_m}-b}{w_A^{l_m}} + \frac{c_3^{l_m}-b_1}{w_B^{l_m}} \right\}, (c_1^{l_m}+c_3^{l_m}) \right] \end{cases} $$

In a similar manner we also have the UMF $\mu_{A^u+B^u}(x)$ *of A + B* as

$$\mu_{A^u+B^u}(x)=\begin{cases} \dfrac{x-(a_1^{u_m}+a_3^{u_m})}{\left\{\dfrac{b-a_1^{u_m}}{w_A^{u_m}}+\dfrac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}}, & x\in\left[(a_1^{u_m}+a_3^{u_m}),\, w^u\left\{\dfrac{b-a_1^{u_m}}{w_A^{u_m}}+\dfrac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}+(a_1+a_3^{u_m})\right] \\ w^u, & x\in\left[w^u\left\{\dfrac{b-a_1^{u_m}}{w_A^{u_m}}+\dfrac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}+(a_1^{u_m}+a_3^{u_m}),\,(c_1^{u_m}+c_3^{u_m})-w^u\left\{\dfrac{c_1^{u_m}-b}{w_A^{u_m}}+\dfrac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\}\right] \\ \dfrac{(c_1^{u_m}+c_3^{u_m})-x}{\left\{\dfrac{c_1^{u_m}-b}{w_A^{u_m}}+\dfrac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\}}, & x\in\left[(c_1^{u_m}+c_3^{u_m})-w^u\left\{\dfrac{c_1^{u_m}-b}{w_A^{u_m}}+\dfrac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\},\,(c_1^{u_m}+c_3^{u_m})\right] \end{cases}$$

To obtain NMFs, we proceed as:

$$\beta_{\underline{A}^l}+\beta_{\underline{B}^l}=\left[\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}},\frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right]+$$

$$\left[\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}},\frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right]$$

$$=\left[\left\{\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}+\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}\right\},\right.$$

$$\left.\left\{\frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}+\frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}\right]$$

$$=\left[\left\{\frac{(b-a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}}+\frac{(b_1-a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\}-\beta\left\{\frac{(b-a_2^{l_n})}{1-\eta_A^{l_n}}+\frac{(b-a_4^{l_n})}{1-\eta_B^{l_n}}\right\},\right.$$

$$\left.\beta\left\{\frac{(c_2^{l_n}-b)}{1-\eta_A^{l_n}}+\frac{(c_4^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}-\left\{\frac{(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}+\frac{(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}\right]$$

Let's equate each component with x, we have

$$x=\left\{\frac{(b-a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}}+\frac{(b_1-a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\}-\beta\left\{\frac{(b-a_2^{l_n})}{1-\eta_A^{l_n}}+\frac{(b-a_4^{l_n})}{1-\eta_B^{l_n}}\right\}\text{ and}$$

$$x=\beta\left\{\frac{(c_2^{l_n}-b)}{1-\eta_A^{l_n}}+\frac{(c_4^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}-\left\{\frac{(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}+\frac{(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}$$

Now, expressing in terms of x, we obtain

$$\beta=\frac{\left[\left\{\dfrac{(b-a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}}+\dfrac{(b_1-a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\}-x\right]}{\left\{\dfrac{(b-a_2^{l_n})}{1-\eta_A^{l_n}}+\dfrac{(b-a_4^{l_n})}{1-\eta_B^{l_n}}\right\}} \tag{4.4}$$

and

$$\beta = \frac{\left[x + \left\{\frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}}\right\}\right]}{\left\{\frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}}\right\}} \quad \textbf{(4.5)}$$

Putting $\beta \le 1$ and $\beta \ge \eta^l$ in (4.4), where $\eta^l = \max\left\{\eta_A^{l_n}, \eta_B^{l_n}\right\}$, we have

$$1 \ge \frac{\left[\left\{\frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\} - x\right]}{\left\{\frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b - a_4^{l_n})}{1-\eta_B^{l_n}}\right\}}$$

Then, $\left\{\frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b - a_4^{l_n})}{1-\eta_B^{l_n}}\right\} \ge \left[\left\{\frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\} - x\right]$

That is, $x \ge \left\{\frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\} - \left\{\frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b - a_4^{l_n})}{1-\eta_B^{l_n}}\right\}$

$$= \frac{a_2^{l_n}(1-\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{a_4^{l_n}(1-\eta_B^{l_n})}{1-\eta_B^{l_n}}$$

i.e., $x \ge a_2^{l_n} + a_4^{l_n}$

Again, taking $\beta \ge \eta^l$ in (4.4), we have

$$\eta^l \le \frac{\left[\left\{\frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\} - x\right]}{\left\{\frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b - a_4^{l_n})}{1-\eta_B^{l_n}}\right\}}$$

$$x \le \left\{\frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}}\right\} - \eta^l \left\{\frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b - a_4^{l_n})}{1-\eta_B^{l_n}}\right\}$$

i.e., $x \le \frac{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})}{1-\eta_B^{l_n}}$

That is, $x \in \left[a_2^{l_n} + a_4^{l_n}, \frac{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})}{1-\eta_B^{l_n}}\right]$ for (4.4)

Next, Putting $\beta \ge \eta^l$ and $\beta \le 1$ in (4.5), we have

$$\eta^l \le \frac{\left[x + \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} \right]}{\left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}}$$

$$x \ge \eta^l \left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} - \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}$$

i.e, $x \ge \frac{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)}{1-\eta_B^{l_n}}$

Again for, $1 = \beta \ge \frac{\left[x + \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} \right]}{\left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}}$

Then, $\left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} \ge x + \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}$

$$x \le \left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} - \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}$$

i.e., $x \le c_2^{l_n} + c_4^{l_n}$.

That is, $x \in \left[\frac{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)}{1-\eta_B^{l_n}},\ c_2^{l_n} + c_4^{l_n} \right]$ for (4.5).

Hence the required LNMF $\nu_{A^l+B^l}(x)$ is

$$\nu_{A^l+B^l}(x) = \begin{cases} \dfrac{\left[\left\{ \frac{(b - a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n}\eta_B^{l_n})}{1-\eta_B^{l_n}} \right\} - x \right]}{\left\{ \frac{(b - a_2^{l_n})}{1-\eta_A^{l_n}} + \frac{(b_1 - a_4^{l_n})}{1-\eta_B^{l_n}} \right\}}, \\ \quad x \in \left[a_2^{l_n} + a_4^{l_n}, \frac{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})}{1-\eta_B^{l_n}} \right] \\ \eta^l,\ x \in \left[\begin{array}{l} \frac{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})}{1-\eta_B^{l_n}}, \\ \frac{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)}{1-\eta_B^{l_n}} \end{array} \right] \\ \dfrac{\left[x + \left\{ \frac{(c_2^{l_n}\eta_A^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n}\eta_B^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\} \right]}{\left\{ \frac{(c_2^{l_n} - b)}{1-\eta_A^{l_n}} + \frac{(c_4^{l_n} - b_1)}{1-\eta_B^{l_n}} \right\}}, \\ \quad x \in \left[\frac{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)}{1-\eta_B^{l_n}},\ c_2^{l_n} + c_4^{l_n} \right] \\ 1, \text{ otherwise.} \end{cases}$$

In a similar fashion, we have the UNMF $\nu_{A^u+B^u}(x)$ of $A+B$

$$\nu_{A^u+B^u}(x)=\begin{cases}\dfrac{\left[\left\{\dfrac{(b-a_2^{u_n}\eta_A^{u_n})}{1-\eta_A^{u_n}}+\dfrac{(b_1-a_4^{u_n}\eta_B^{u_n})}{1-\eta_B^{u_n}}\right\}-x\right]}{\left\{\dfrac{(b-a_2^{u_n})}{1-\eta_A^{u_n}}+\dfrac{(b_1-a_4^{u_n})}{1-\eta_B^{u_n}}\right\}}, \\ \quad x\in\left[a_2^{u_n}+a_4^{u_n}, \dfrac{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})}{1-\eta_A^{u_n}}+\dfrac{b_1(1-\eta^u)+a_4^{u_n}(\eta^u-\eta_B^{u_n})}{1-\eta_B^{u_n}}\right] \\ \eta^u,\ x\in\left[\begin{array}{l}\dfrac{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})}{1-\eta_A^{u_n}}+\dfrac{b_1(1-\eta^u)+a_4^{u_n}(\eta^u-\eta_B^{u_n})}{1-\eta_B^{u_n}}, \\ \dfrac{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)}{1-\eta_A^{u_n}}+\dfrac{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}\end{array}\right] \\ \dfrac{\left[x+\left\{\dfrac{(c_2^{u_n}\eta_A^{u_n}-b)}{1-\eta_A^{u_n}}+\dfrac{(c_4^{u_n}\eta_B^{u_n}-b_1)}{1-\eta_B^{u_n}}\right\}\right]}{\left\{\dfrac{(c_2^{u_n}-b)}{1-\eta_A^{u_n}}+\dfrac{(c_4^{u_n}-b_1)}{1-\eta_B^{u_n}}\right\}}, \\ \quad x\in\left[\dfrac{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)}{1-\eta_A^{u_n}}+\dfrac{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}},\ c_2^{u_n}+c_4^{u_n}\right] \\ 1,\ \text{otherwise.}\end{cases}$$

Thus, we have

$$A+B=\left\langle\left[\left(f_1^l,f_2^l,f_3^l,f_4^l;w^l\right),\left(f_1^u,f_2^u,f_3^u,f_4^u;w^u\right)\right],\left[\left(g_1^l,g_2^l,g_3^l,g_4^l;\eta^l\right),\left(g_1^u,g_2^u,g_3^u,g_4^u;\eta^u\right)\right]\right\rangle$$

where $f_1^l=(a_1^{l_m}+a_3^{l_m})$

$$f_2^l=w^l\left\{\frac{b-a_1^{l_m}}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\}+(a_1^{l_m}+a_3^{l_m})$$

$$f_3^l=(c_1^{l_m}+c_3^{l_m})-w^l\left\{\frac{c_1^{l_m}-b}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}$$

$$f_4^l=(c_1^{l_m}+c_3^{l_m})$$

and $w^l=\min(w_A^{l_m},w_B^{l_m})$.

$$f_1^u=(a_1^{u_m}+a_3^{u_m})$$

$$f_2^u=w^u\left\{\frac{b-a_1^{u_m}}{w_A^{u_m}}+\frac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}+(a_1+a_3^{u_m})$$

$$f_3^u = (c_1^{u_m} + c_3^{u_m}) - w^u \left\{ \frac{c_1^{u_m} - b}{w_A^{u_m}} + \frac{c_3^{u_m} - b_1}{w_B^{u_m}} \right\}$$

$$f_4^u = (c_1^{u_m} + c_3^{u_m})$$

and $w^u = \min(w_A^{u_m}, w_B^{u_m})$

Also $g_1^l = a_2^{l_n} + a_4^{l_n}$

$$g_2^l = \frac{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})}{1-\eta_A^{l_n}} + \frac{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})}{1-\eta_B^{l_n}}$$

$$g_3^l = \frac{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)}{1-\eta_B^{l_n}}$$

$$g_4^l = c_2^{l_n} + c_4^{l_n}$$

and $\eta^l = \max(\eta_A^{l_n}, \eta_B^{l_n})$

$$g_1^u = a_2^{u_n} + a_4^{u_n}$$

$$g_2^u = \frac{b(1-\eta^u) + a_2^{u_n}(\eta^u - \eta_A^{u_n})}{1-\eta_A^{u_n}} + \frac{b_1(1-\eta^u) + a_4^{u_n}(\eta^u - \eta_B^{u_n})}{1-\eta_B^{u_n}}$$

$$g_3^u = \frac{c_2^{u_n}(\eta^u - \eta_A^{u_n}) + b(1-\eta^u)}{1-\eta_A^{u_n}} + \frac{c_4^{u_n}(\eta^u - \eta_B^{u_n}) + b_1(1-\eta^u)}{1-\eta_B^{u_n}}$$

$g_4^u = c_2^{u_n} + c_4^{u_n}$ and $\eta^u = \max(\eta_A^{u_n}, \eta_B^{u_n})$

$A + B$ is clearly a GIVTrIFN with height of the MFs are w^l, w^u and NMFs are η^l, η^u.

Hence the theorem.

4.2. Theorem (Subtraction of Two GIVTIFNs with Different Heights Produces a GIVTrIFNs)

Proof

To perform subtraction operation of *GIVTIFNs A* and *B,* we subtract the *α, β*-cuts of *A* and *B* using interval arithmetic.

For MF,

$$^{\alpha}A_+^l - {}^{\alpha}B_+^l = \left[\frac{\alpha}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m} - \left\{ c_3^{l_m} - \frac{\alpha}{w_B^{l_m}}(c_3^{l_m} - b_1) \right\},\ c_1^{l_m} - \frac{\alpha}{w_A^{l_m}}(c_1^{l_m} - b) - \left\{ \frac{\alpha}{w_B^{l_m}}(b_1 - a_3^{l_m}) + a_3^{l_m} \right\} \right] \quad \textbf{(4.6)}$$

where $w^l = \min(w_A^{l_m}, w_B^{l_m})$ and $\alpha \in [0, w^l]$.

To find the membership function $\mu_{A^l-B^l}(x)$ we equate both the first and second component of (4.6) to x which gives

$$x=\frac{\alpha}{w_A^{l_m}}(b-a_1^{l_m})+a_1^{l_m}-\left\{c_3^{l_m}-\frac{\alpha}{w_B^{l_m}}(c_3^{l_m}-b_1)\right\}\text{and } x=c_1^{l_m}-\frac{\alpha}{w_A^{l_m}}(c_1^{l_m}-b)-\left\{\frac{\alpha}{w_B^{l_m}}(b_1-a_3^{l_m})+a_3^{l_m}\right\}$$

Now, expressing in terms of x

$$\alpha=\frac{x-(a_1^{l_m}-c_3^{l_m})}{\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}} \tag{4.7}$$

$$\alpha=\frac{(c_1^{l_m}-a_3^{l_m})-x}{\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\}} \tag{4.8}$$

Setting $\alpha \geq 0$ & $\alpha \leq w^L$ in (4.7) and $\alpha \leq w^L$ & $\alpha \geq 0$ in (4.8) we get the domain of x

$$x\in\left[(a_1^{l_m}-c_3^{l_m}),(a_1^{l_m}-c_3^{l_m})+w^l\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}\right] \text{ and } x\in\left[(c_1^{l_m}-a_3^{l_m})-w^l\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\},(c_1^{l_m}-a_3^{l_m})\right]$$

The required LMF is $\mu_{A^l-B^l}(x)$

$$\mu_{A^l-B^l}(x)=\begin{cases}\dfrac{x-(a_1^{l_m}-c_3^{l_m})}{\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}}, & x\in\left[(a_1^{l_m}-c_3^{l_m}),(a_1^{l_m}-c_3^{l_m})+w^l\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}\right]\\ w^l, & x\in\left[(a_1^{l_m}-c_3^{l_m})+w^l\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\},(c_1^{l_m}-a_3^{l_m})-w^l\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\}\right]\\ \dfrac{(c_1^{l_m}-a_3^{l_m})-x}{\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\}}, & x\in\left[(c_1^{l_m}-a_3^{l_m})-w^l\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\},(c_1^{l_m}-a_3^{l_m})\right]\end{cases}$$

where $w^l=\min(w_A^{l_m},w_B^{l_m})$ and $\alpha\in[0,w^l]$.

In a similar way,we can have the UMF $\mu_{A^u-B^u}(x)$ as

$$\mu_{A^u-B^u}(x)=\begin{cases}\dfrac{x-(a_1^{u_m}-c_3^{u_m})}{\left\{\frac{(b-a_1^{u_m})}{w_A^{u_m}}+\frac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\}}, & x\in\left[(a_1^{u_m}-c_3^{u_m}),(a_1^{u_m}-c_3^{u_m})+w^u\left\{\frac{(b-a_1^{u_m})}{w_A^{u_m}}+\frac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\}\right]\\ w^u, & x\in\left[(a_1^{u_m}-c_3^{u_m})+w^u\left\{\frac{(b-a_1^{u_m})}{w_A^{u_m}}+\frac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\},(c_1^{u_m}-a_3^{u_m})-w^u\left\{\frac{(c_1^{u_m}-b)}{w_A^{u_m}}+\frac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}\right]\\ \dfrac{(c_1^{u_m}-a_3^{u_m})-x}{\left\{\frac{(c_1^{u_m}-b)}{w_A^{u_m}}+\frac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}}, & x\in\left[(c_1^{u_m}-a_3^{u_m})-w^u\left\{\frac{(c_1^{u_m}-b)}{w_A^{u_m}}+\frac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\},(c_1^{u_m}-a_3^{u_m})\right]\end{cases}$$

For NMFs

$$\beta_{A^l_-} - \beta_{B^l_-} = \left[\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}, \frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right] - \left[\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}, \frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right]$$

$$= \left[\begin{array}{c} \dfrac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}} - \dfrac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}, \\ \dfrac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}} - \dfrac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}} \end{array}\right]$$

$$= \left[\begin{array}{c} \left\{\dfrac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}} + \dfrac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\} - \beta\left\{\dfrac{b-a_2^{l_n}}{1-\eta_A^{l_n}} + \dfrac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}, \\ \beta\left\{\dfrac{c_2^{l_n}-b}{1-\eta_A^{l_n}} + \dfrac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\} - \left\{\dfrac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}} + \dfrac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\} \end{array}\right]$$

Let's equate each component with *x*, we have

$$x = \left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\} - \beta\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}} + \frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}$$

and

$$x = \beta\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}} + \frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\} - \left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}} + \frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}$$

Now, expressing β in terms of x, we obtain

$$\beta = \frac{\left\{\dfrac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}} + \dfrac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\} - x}{\left\{\dfrac{b-a_2^{l_n}}{1-\eta_A^{l_n}} + \dfrac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}} \quad \textbf{(4.9)}$$

$$\beta = \frac{x + \left\{\dfrac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}} + \dfrac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}}{\left\{\dfrac{c_2^{l_n}-b}{1-\eta_A^{l_n}} + \dfrac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}} \quad \textbf{(4.10)}$$

Putting $\beta \le 1$ and $\beta \ge \eta^l$ in (4.9), we have

$$1 \geq \frac{\left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-x}{\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}}$$

Then, $\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\} \geq \left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-x$

Then, $x \geq \left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}$

Then, $x \geq \frac{a_2^{l_n}(1-\eta_A^{l_n})}{1-\eta_A^{l_n}}-\frac{c_4^{l_n}(1-\eta_B^{l_n})}{1-\eta_B^{l_n}}$

That is, $x \geq a_2^{l_n}-c_4^{l_n}$.

Again, taking $\beta \geq \eta^l$ in (4.9), where $\eta^l = \max\left\{\eta_A^{l_n}, \eta_B^{l_n}\right\}$, we have

$$\eta^l \leq \frac{\left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-x}{\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}}$$

$$\eta^l\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\} \leq \left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-x$$

$$x \leq \left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-\eta^l\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}$$

$$x \leq \frac{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})}{1-\eta_A^{l_n}}-\frac{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)}{1-\eta_B^{l_n}}.$$

That is, $x \in \left[a_2^{l_n}-c_4^{l_n}, \frac{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})}{1-\eta_A^{l_n}}-\frac{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)}{1-\eta_B^{l_n}}\right]$ for (4.9).

Next, Putting $\beta \geq \eta^l$ and $\beta \leq$ in (4.10), we have

$$\eta^l \leq \frac{x+\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}}{\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}}$$

which gives

$$x \geq \eta^{l}\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}-\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}$$

$$x \geq \frac{c_2^{l_n}(\eta^{l}-\eta_A^{l_n})+b(1-\eta^{l})}{1-\eta_A^{l_n}}-\frac{a_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}.$$

Again for $\beta \leq 1$, we have

$$1 \geq \frac{x+\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}}{\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}}.$$

$$\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\} \geq x+\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}$$

$$x \leq \left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}-\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}$$

That is, $x \leq c_2^{l_n}-a_4^{l_n}$.

Then, $x \in \left[\frac{c_2^{l_n}(\eta^{l}-\eta_A^{l_n})+b(1-\eta^{l})}{1-\eta_A^{l_n}}-\frac{a_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}, c_2^{l_n}-a_4^{l_n}\right]$ for (4.10).

Hence, the required LNMF $\nu_{A^l-B^l}(x)$ is

$$\nu_{A^l-B^l}(x)=\begin{cases}\dfrac{\left\{\frac{b-a_2^{l_n}\eta_A^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}\eta_B^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}-x}{\left\{\frac{b-a_2^{l_n}}{1-\eta_A^{l_n}}+\frac{c_4^{l_n}-b_1}{1-\eta_B^{l_n}}\right\}}, \\ \quad x \in \left[a_2^{l_n}-c_4^{l_n}, \frac{b(1-\eta^{l})+a_2^{l_n}(\eta^{l}-\eta_A^{l_n})}{1-\eta_A^{l_n}}-\frac{c_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}\right] \\ \eta^{l},\ x \in \left[\begin{array}{l}\frac{b(1-\eta^{l})+a_2^{l_n}(\eta^{l}-\eta_A^{l_n})}{1-\eta_A^{l_n}}-\frac{c_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}, \\ \frac{c_2^{l_n}(\eta^{l}-\eta_A^{l_n})+b(1-\eta^{l})}{1-\eta_A^{l_n}}-\frac{a_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}\end{array}\right] \\ \dfrac{x+\left\{\frac{c_2^{l_n}\eta_A^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}\eta_B^{l_n}}{1-\eta_B^{l_n}}\right\}}{\left\{\frac{c_2^{l_n}-b}{1-\eta_A^{l_n}}+\frac{b_1-a_4^{l_n}}{1-\eta_B^{l_n}}\right\}}, \\ \quad x \in \left[\frac{c_2^{l_n}(\eta^{l}-\eta_A^{l_n})+b(1-\eta^{l})}{1-\eta_A^{l_n}}-\frac{a_4^{l_n}(\eta^{l}-\eta_B^{l_n})+b_1(1-\eta^{l})}{1-\eta_B^{l_n}}, c_2^{l_n}-a_4^{l_n}\right] \\ 1, \text{ otherwise.}\end{cases}$$

Similarly, we can have UNMF $\nu_{A^u-B^u}(x)$ as follows:

$$
\nu_{A^u-B^u}(x)=\begin{cases}
\dfrac{\left\{\dfrac{b-a_2^{u_n}\eta_A^{u_n}}{1-\eta_A^{u_n}}+\dfrac{c_4^{u_n}\eta_B^{u_n}-b_1}{1-\eta_B^{u_n}}\right\}-x}{\left\{\dfrac{b-a_2^{u_n}}{1-\eta_A^{u_n}}+\dfrac{c_4^{u_n}-b_1}{1-\eta_B^{u_n}}\right\}}, \\
\quad x\in\left[a_2^{u_n}-c_4^{u_n},\dfrac{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})}{1-\eta_A^{u_n}}-\dfrac{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}\right] \\
\eta^u,\ x\in\left[\begin{array}{l}\dfrac{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})}{1-\eta_A^{u_n}}-\dfrac{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}, \\ \dfrac{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)}{1-\eta_A^{u_n}}-\dfrac{a_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}\end{array}\right] \\
\dfrac{x+\left\{\dfrac{c_2^{u_n}\eta_A^{u_n}-b}{1-\eta_A^{u_n}}+\dfrac{b_1-a_4^{u_n}\eta_B^{u_n}}{1-\eta_B^{u_n}}\right\}}{\left\{\dfrac{c_2^{u_n}-b}{1-\eta_A^{u_n}}+\dfrac{b_1-a_4^{u_n}}{1-\eta_B^{u_n}}\right\}}, \\
\quad x\in\left[\dfrac{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)}{1-\eta_A^{u_n}}-\dfrac{a_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}},c_2^{u_n}-a_4^{u_n}\right] \\
1,\ \text{otherwise.}
\end{cases}
$$

Thus, we have

$$A-B=\left\langle\left[\left(f_1^l,f_2^l,f_3^l,f_4^l;w^l\right),\left(f_1^u,f_2^u,f_3^u,f_4^u;w^u\right)\right],\left[\left(g_1^l,g_2^l,g_3^l,g_4^l;\eta^l\right),\left(g_1^u,g_2^u,g_3^u,g_4^u;\eta^u\right)\right]\right\rangle$$

where $f_1^l=a_1^{l_m}-c_3^{l_m}$

$$f_2^l=(a_1^{l_m}-c_3^{l_m})+w^l\left\{\frac{(b-a_1^{l_m})}{w_A^{l_m}}+\frac{c_3^{l_m}-b_1}{w_B^{l_m}}\right\}$$

$$f_3^l=(c_1^{l_m}-a_3^{l_m})-w^l\left\{\frac{(c_1^{l_m}-b)}{w_A^{l_m}}+\frac{b_1-a_3^{l_m}}{w_B^{l_m}}\right\}$$

$$f_4^l=c_1^{l_m}-a_3^{l_m}$$

and $w^l=\min(w_A^{l_m},w_B^{l_m})$.

$f_1^u=a_1^{u_m}-c_3^{u_m}$

$$f_2^u=(a_1^{u_m}-c_3^{u_m})+w^u\left\{\frac{(b-a_1^{u_m})}{w_A^{u_m}}+\frac{c_3^{u_m}-b_1}{w_B^{u_m}}\right\}$$

$$f_3^u=(c_1^{u_m}-a_3^{u_m})-w^u\left\{\frac{(c_1^{u_m}-b)}{w_A^{u_m}}+\frac{b_1-a_3^{u_m}}{w_B^{u_m}}\right\}$$

$$f_4^u=c_1^{u_m}-a_3^{u_m}$$

and $w^u=\min(w_A^{u_m},w_B^{u_m})$

Also $g_1^l=a_2^{l_n}-c_4^{l_n}$

$$g_2^l = \frac{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})}{1-\eta_A^{l_n}} - \frac{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)}{1-\eta_B^{l_n}}$$

$$g_3^l = \frac{c_2^{l_n}(\eta^l-\eta_A^{l_n})+b(1-\eta^l)}{1-\eta_A^{l_n}} - \frac{a_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)}{1-\eta_B^{l_n}}$$

$$g_4^l = c_2^{l_n} - a_4^{l_n}$$

and $\eta^l = \max(\eta_A^{l_n}, \eta_B^{l_n})$

$$g_1^u = a_2^{u_n} - c_4^{u_n}$$

$$g_2^u = \frac{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})}{1-\eta_A^{u_n}} - \frac{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}$$

$$g_3^u = \frac{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)}{1-\eta_A^{u_n}} - \frac{a_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)}{1-\eta_B^{u_n}}$$

$$g_4^u = c_2^{u_n} - a_4^{u_n}$$

and $\eta^u = \max(\eta_A^{u_n}, \eta_B^{u_n})$

Thus, *A* - *B* is also a GIVTrIFN, where height of the MFs are w^l, w^u and NMFs are η^l, η^u .

Hence proved the theorem.

4.3. Theorem (Multiplication of Two GIVTIFNs with Different Heights Produces a GIVTrIFN)

Proof

To calculate multiplication of GIVTIFNs *A* and *B,* we first multiply the *α, β*-cuts of generalized fuzzy numbers *A* and *B* using interval arithmetic.

For MF,

$$^{\alpha}A^l.^{\alpha}B^l = \left[\left\{\frac{\alpha}{w_A^{l_m}}(b-a_1^{l_m})+a_1^{l_m}\right\}.\left\{\frac{\alpha}{w_B^{l_m}}(b_1-a_3^{l_m})+a_3^{l_m}\right\},\left\{c_1^{l_m}-\frac{\alpha}{w_A^{l_m}}(c_1^{l_m}-b)\right\}.\left\{c_3^{l_m}-\frac{\alpha}{w_B^{l_m}}(c_3^{l_m}-b_1)\right\}\right] \quad \textbf{(4.11)}$$

where $w^l = \min(w_B^{l_m}, w_A^{l_m})$ and $\alpha \in [0, w^l]$. .

To find the LMF $\mu_{A^l.B^l}(x)$, we equate both the first and second component of (4.11) to *x* which gives

$$x = \left\{\frac{\alpha}{w_A^{l_m}}(b-a_1^{l_m})+a_1^{l_m}\right\}.\left\{\frac{\alpha}{w_B^{l_m}}(b_1-a_3^{l_m})+a_3^{l_m}\right\}$$

$$\Rightarrow x = \frac{\alpha^2}{w_A^{l_m}w_B^{l_m}}(b-a_1^{l_m})(b_1-a_3^{l_m}) + \frac{\alpha}{w_A^{l_m}}w_A^{l_m}(b-a_1^{l_m}) + \frac{\alpha}{w_B^{l_m}}a_1^{l_m}(b_1-a_3^{l_m}) + a_1^{l_m}a_3^{l_m}$$

$$\Rightarrow \alpha^2\frac{(b-a_1^{l_m})(b_1-a_3^{l_m})}{w_A^{l_m}w_B^{l_m}} + \alpha\left\{\frac{a_3^{l_m}(b-a_1^{l_m})}{w_A^{l_m}} + \frac{a_1^{l_m}(b_1-a_3^{l_m})}{w_B^{l_m}}\right\} + (a_1^{l_m}a_3^{l_m} - x) = 0$$

which is a quadratic equation and by solving it we obtain

$$\alpha=\frac{-\{a_3^{l_m}(b-a_1^{l_m})w_B^{l_m}+a_1^{l_m}(b_1-a_3^{l_m})w_A^{l_m}\}+\sqrt{\{a_3^{l_m}(b-a_1^{l_m})w_B^{l_m}+a_1^{l_m}(b_1-a_3^{l_m})w_A^{l_m}\}^2-4w_A^{l_m}w_B^{l_m}(b-a_1^{l_m})(b_1-a_3^{l_m})(a_1^{l_m}a_3^{l_m}-x)}}{2(b-a_1^{l_m})(b_1-a_3^{l_m})} \tag{4.12}$$

Similarly $x=\left\{c_1^{l_m}-\frac{\alpha}{w_A^{l_m}}(c_1^{l_m}-b)\right\}.\left\{c_3^{l_m}-\frac{\alpha}{w_B^{l_m}}(c_3^{l_m}-b_1)\right\}$ gives

$$\alpha=\frac{c_1^{l_m}(c_3^{l_m}-b_1)w_A^{l_m}+c_3^{l_m}(c_1^{l_m}-b)w_A^{l_m}-\sqrt{\left\{c_1^{l_m}(c_3^{l_m}-b_1)w_A^{l_m}+c_3^{l_m}(c_1^{l_m}-b)w_B^{l_m}\right\}^2-4w_A^{l_m}w_B^{l_m}(c_1^{l_m}-b)(c_3^{l_m}-b_1)(c_1^{l_m}c_3^{l_m}-x)}}{2(c_1^{l_m}-b)(c_3^{l_m}-b_1)} \tag{4.13}$$

Now, setting $\alpha\geq 0$ & $\alpha\leq w^L$ and $\alpha\leq w^L$ & $\alpha\geq 0$ in (4.12) and (4.13), we get the LMF of the resulting GIVTrIFN after multiplication of A and B together with the domain of x

$$\mu_{A^l.B^l}(x)=\begin{cases}\dfrac{-\{a_3^{l_m}(b-a_1^{l_m})w_B^{l_m}+a_1^{l_m}(b_1-a_3^{l_m})w_A^{l_m}\}+\sqrt{\{a_3^{l_m}(b-a_1^{l_m})w_B^{l_m}+a_1^{l_m}(b_1-a_3^{l_m})w_A^{l_m}\}^2-4w_A^{l_m}w_B^{l_m}(b-a_1^{l_m})(b_1-a_3^{l_m})(a_1^{l_m}a_3^{l_m}-x)}}{2(b-a_1^{l_m})(b_1-a_3^{l_m})}\\ \text{if } x\in\left[a_1^{l_m}a_3^{l_m},\, a_1^{l_m}a_3^{l_m}+\dfrac{(w^l)^2}{w_A^{l_m}w_B^{l_m}}(b-a_1^{l_m})(b_1-a_3^{l_m})+\dfrac{w^l}{w_A^{l_m}}a_3^{l_m}(b-a_1^{l_m})+\dfrac{w^l}{w_B^{l_m}}a_1^{l_m}(b_1-a_3^{l_m})\right]\\ w^l,\ x\in\left[\begin{array}{c}a_1^{l_m}a_3^{l_m}+\dfrac{(w^l)^2}{w_A^{l_m}w_B^{l_m}}(b-a_1^{l_m})(b_1-a_3^{l_m})+\dfrac{w^l}{w_A^{l_m}}a_3^{l_m}(b-a_1^{l_m})+\dfrac{w^l}{w_B^{l_m}}a_1^{l_m}(b_1-a_3^{l_m}),\\ c_1^{l_m}c_3^{l_m}+\dfrac{(w^l)^2}{w_A^{l_m}w_B^{l_m}}(c_1^{l_m}-b)(c_3^{l_m}-b_1)-\dfrac{w}{w_A^{l_m}}c_3^{l_m}(c_1^{l_m}-b)-\dfrac{w}{w_B^{l_m}}c_1^{l_m}(c_3^{l_m}-b_1)\end{array}\right]\\ \dfrac{c_1^{l_m}(c_3^{l_m}-b_1)w_A^{l_m}+c_3^{l_m}(c_1^{l_m}-b)w_B^{l_m}-\sqrt{\left\{c_1^{l_m}(c_3^{l_m}-b_1)w_A^{l_m}+c_3^{l_m}(c_1^{l_m}-b)w_B^{l_m}\right\}^2-4w_A^{l_m}w_B^{l_m}(c_1^{l_m}-b)(c_3^{l_m}-b_1)(c_1^{l_m}c_3^{l_m}-x)}}{2(c_1^{l_m}-b)(c_3^{l_m}-b_1)}\\ \text{if } x\in\left[c_1^{l_m}c_3^{l_m}+\dfrac{(w^l)^2}{w_A^{l_m}w_B^{l_m}}(c_1^{l_m}-b)(c_3^{l_m}-b_1)-\dfrac{w^l}{w_A^{l_m}}c_3^{l_m}(c_1^{l_m}-b)-\dfrac{w^l}{w_B^{l_m}}c_1^{l_m}(c_3^{l_m}-b_1),\, c_1^{l_m}c_3^{l_m}\right]\end{cases}$$

Also with a similar manner, we have the UMF $\mu_{A^u.B^u}(x)$ as

$$\mu_{A^u.B^u}(x)=\begin{cases}\dfrac{-\{a_3^{u_m}(b-a_1^{u_m})w_B^{u_m}+a_1^{u_m}(b_1-a_3^{u_m})w_A^{u_m}\}+\sqrt{\{a_3^{u_m}(b-a_1^{u_m})w_B^{u_m}+a_1^{u_m}(b_1-a_3^{u_m})w_A^{u_m}\}^2-4w_A^{u_m}w_B^{u_m}(b-a_1^{u_m})(b_1-a_3^{u_m})(a_1^{u_m}a_3^{u_m}-x)}}{2(b-a_1^{u_m})(b_1-a_3^{u_m})}\\ \text{if } x\in\left[a_1^{u_m}a_3^{u_m},\, a_1^{u_m}a_3^{u_m}+\dfrac{w^2}{w_A^{u_m}w_B^{u_m}}(b-a_1^{u_m})(b_1-a_3^{u_m})+\dfrac{w^u}{w_A^{u_m}}a_3^{u_m}(b-a_1^{u_m})+\dfrac{w^u}{w_B^{u_m}}a_1^{u_m}(b_1-a_3^{u_m})\right]\\ w^u,\ x\in\left[\begin{array}{c}a_1^{u_m}a_3^{u_m}+\dfrac{w^2}{w_A^{u_m}w_B^{u_m}}(b-a_1^{u_m})(b_1-a_3^{u_m})+\dfrac{w^u}{w_A^{u_m}}a_3^{u_m}(b-a_1^{u_m})+\dfrac{w^u}{w_B^{u_m}}a_1^{u_m}(b_1-a_3^{u_m}),\\ c_1^{u_m}c_3^{u_m}+\dfrac{w^2}{w_A^{u_m}w_B^{u_m}}(c_1^{u_m}-b)(c_3^{u_m}-b_1)-\dfrac{w}{w_A^{u_m}}c_3^{u_m}(c_1^{u_m}-b)-\dfrac{w}{w_B^{u_m}}c_1^{u_m}(c_3^{u_m}-b_1)\end{array}\right]\\ \dfrac{c_1^{u_m}(c_3^{u_m}-b_1)w_A^{u_m}+c_3^{u_m}(c_1^{u_m}-b)w_A^{u_m}-\sqrt{\left\{c_1^{u_m}(c_3^{u_m}-b_1)w_A^{u_m}+c_3^{u_m}(c_1^{u_m}-b)w_B^{u_m}\right\}^2-4w_A^{u_m}w_B^{u_m}(c_1^{u_m}-b)(c_3^{u_m}-b_1)(c_1^{u_m}c_3^{u_m}-x)}}{2(c_1^{u_m}-b)(c_3^{u_m}-b_1)}\\ \text{if } x\in\left[c_1^{u_m}c_3^{u_m}+\dfrac{w^2}{w_A^{u_m}w_B^{u_m}}(c_1^{u_m}-b)(c_3^{u_m}-b_1)-\dfrac{w}{w_A^{u_m}}c_3^{u_m}(c_1^{u_m}-b)-\dfrac{w}{w_B^{u_m}}c_1^{u_m}(c_3^{u_m}-b_1),\, c_1^{u_m}c_3^{u_m}\right]\end{cases}$$

For NMF

$$\beta_{A^l_-}.\beta_{B^l_-} = \left[\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}, \frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right]\times$$
$$\left[\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}, \frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right]$$

$$=\left[\left\{\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}\times\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}\right\},\right.$$
$$\left.\left\{\frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\times\frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right\}\right]$$

Now, equating both the terms with x, we obtain

$$\beta^2(b-a_2^{l_n})(b_1-a_4^{l_n})-\beta\{(b-a_2^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})+(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n})\}+$$
$$\{(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}=0$$

and

$$\beta^2(c_2^{l_n}-b)(c_4^{l_n}-b_1)-\beta\{(c_2^{l_n}-b)(c_4^{l_n}\eta_B^{l_n}-b_1)+(c_4^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)\}+$$
$$\{(c_4^{l_n}\eta_B^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}=0$$

It can be expressed in terms of x by solving this quadratic equation,

$$\beta=\frac{\{(b-a_2^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})+(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n})\}-\sqrt{\{(b-a_2^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})+(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n})\}^2-4(b-a_2^{l_n})(b_1-a_4^{l_n})\{(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}}}{2(b-a_2^{l_n})(b_1-a_4^{l_n})} \quad \textbf{(4.14)}$$

and,

$$\beta=\frac{\{(c_2^{l_n}-b)(c_4^{l_n}\eta_B^{l_n}-b_1)+(c_4^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)\}-\sqrt{\{(c_2^{l_n}-b)(c_4^{l_n}\eta_B^{l_n}-b_1)+(c_4^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)\}^2-4(c_2^{l_n}-b)(c_4^{l_n}-b_1)\{(c_4^{l_n}\eta_B^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}}}{2(c_2^{l_n}-b)(c_4^{l_n}-b_1)} \quad \textbf{(4.15)}$$

Putting $\beta\le 1$ and $\beta\ge\eta^l$ in (4.14), where $\eta^l=\max\left(\eta_A^{l_n},\eta_B^{l_n}\right)$ we have

$x=a_2^{l_n}a_4^{l_n}$ and

$$x=\frac{\{b_1(1-\eta^l)+a_4^{l_n}(\eta^l-\eta_B^{l_n})\}\{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}$$

Again, $\beta\ge\eta^l$ and $\beta\le 1$ Putting and in (4.15), we have

$$x=\frac{\{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)\}\{c_2^{l_n}(\eta^l-\eta_A^{l_n})+b(1-\eta^l)\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}$$

and $x = c_2^{l_n} c_4^{l_n}$.

Thus, the LNMF $\nu_{A^l B^l}(x)$ is

$$\nu_{A^l . B^l}(x) = \begin{cases} \dfrac{\{(b-a_2^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})+(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n})\} - \sqrt{\{(b-a_2^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})+(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n})\}^2 - 4(b-a_2^{l_n})(b_1-a_4^{l_n})\{(b-a_2^{l_n}\eta_A^{l_n})(b_1-a_4^{l_n}\eta_B^{l_n})-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}}}{2(b-a_2^{l_n})(b_1-a_4^{l_n})} \\ if\ x \in \left[a_2^{l_n} a_4^{l_n}, \dfrac{\{b_1(1-\eta^l)+a_4^{l_n}(\eta^l-\eta_B^{l_n})\}\{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})} \right] \\ \eta^l, x \in \left[\dfrac{\{b_1(1-\eta^l)+a_4^{l_n}(\eta^l-\eta_B^{l_n})\}\{b(1-\eta^l)+a_2^{l_n}(\eta^l-\eta_A^{l_n})\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}, \dfrac{\{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)\}\{c_2^{l_n}(\eta^l-\eta_A^{l_n})+b(1-\eta^l)\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})} \right] \\ \dfrac{\{(c_2^{l_n}-b)(c_4^{l_n}\eta_B^{l_n}-b_1)+(c_4^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)\} - \sqrt{\{(c_2^{l_n}-b)(c_4^{l_n}\eta_B^{l_n}-b_1)+(c_4^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)\}^2 - 4(c_2^{l_n}-b)(c_4^{l_n}-b_1)\{(c_4^{l_n}\eta_B^{l_n}-b_1)(c_2^{l_n}\eta_A^{l_n}-b)-x(1-\eta_A^{l_n})(1-\eta_B^{l_n})\}}}{2(c_2^{l_n}-b)(c_4^{l_n}-b_1)} \\ if\ x \in \left[\dfrac{\{c_4^{l_n}(\eta^l-\eta_B^{l_n})+b_1(1-\eta^l)\}\{c_2^{l_n}(\eta^l-\eta_A^{l_n})+b(1-\eta^l)\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}, c_2^{l_n} c_4^{l_n} \right] \end{cases}$$

In a similar fashion, we can have UNMF $\nu_{A^u . B^u}(x)$ as follows:

$$\nu_{A^u . B^u}(x) = \begin{cases} \dfrac{\{(b-a_2^{u_n})(b_1-a_4^{u_n}\eta_B^{u_n})+(b-a_2^{u_n}\eta_A^{u_n})(b_1-a_4^{u_n})\} - \sqrt{\{(b-a_2^{u_n})(b_1-a_4^{u_n}\eta_B^{u_n})+(b-a_2^{u_n}\eta_A^{u_n})(b_1-a_4^{u_n})\}^2 - 4(b-a_2^{u_n})(b_1-a_4^{u_n})\{(b-a_2^{u_n}\eta_A^{u_n})(b_1-a_4^{u_n}\eta_B^{u_n})-x(1-\eta_A^{u_n})(1-\eta_B^{u_n})\}}}{2(b-a_2^{u_n})(b_1-a_4^{u_n})} \\ if\ x \in \left[a_2^{u_n} a_4^{u_n}, \dfrac{\{b_1(1-\eta^u)+a_4^{u_n}(\eta^u-\eta_B^{u_n})\}\{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})} \right] \\ \eta^u, x \in \left[\dfrac{\{b_1(1-\eta^u)+a_4^{u_n}(\eta^u-\eta_B^{u_n})\}\{b(1-\eta^u)+a_2^{u_n}(\eta^u-\eta_A^{u_n})\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})}, \dfrac{\{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)\}\{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})} \right] \\ \dfrac{\{(c_2^{u_n}-b)(c_4^{u_n}\eta_B^{u_n}-b_1)+(c_4^{u_n}-b_1)(c_2^{u_n}\eta_A^{u_n}-b)\} - \sqrt{\{(c_2^{u_n}-b)(c_4^{u_n}\eta_B^{u_n}-b_1)+(c_4^{u_n}-b_1)(c_2^{u_n}\eta_A^{u_n}-b)\}^2 - 4(c_2^{u_n}-b)(c_4^{u_n}-b_1)\{(c_4^{u_n}\eta_B^{u_n}-b_1)(c_2^{u_n}\eta_A^{u_n}-b)-x(1-\eta_A^{u_n})(1-\eta_B^{u_n})\}}}{2(c_2^{u_n}-b)(c_4^{u_n}-b_1)} \\ if\ x \in \left[\dfrac{\{c_4^{u_n}(\eta^u-\eta_B^{u_n})+b_1(1-\eta^u)\}\{c_2^{u_n}(\eta^u-\eta_A^{u_n})+b(1-\eta^u)\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})}, c_2^{u_n} c_4^{u_n} \right] \end{cases}$$

Thus, we have

$$AB = \left\langle \left[\left(f_1^l, f_2^l, f_3^l, f_4^l; w^l \right), \left(f_1^u, f_2^u, f_3^u, f_4^u; w^u \right) \right], \left[\left(g_1^l, g_2^l, g_3^l, g_4^l; \eta^l \right), \left(g_1^u, g_2^u, g_3^u, g_4^u; \eta^u \right) \right] \right\rangle$$

where $f_1^l = a_1^{l_m} a_3^{l_m}$

$$f_2^l = a_1^{l_m} a_3^{l_m} + \frac{\left(w^l\right)^2}{w_A^{l_m} w_B^{l_m}} (b - a_1^{l_m})(b_1 - a_3^{l_m}) + \frac{w^l}{w_A^{l_m}} a_3^{l_m} (b - a_1^{l_m}) + \frac{w^l}{w_B^{l_m}} a_1^{l_m} (b_1 - a_3^{l_m})$$

$$f_3^l = c_1^{l_m} c_3^{l_m} + \frac{\left(w^l\right)^2}{w_A^{l_m} w_B^{l_m}} (c_1^{l_m} - b)(c_3^{l_m} - b_1) - \frac{w^l}{w_A^{l_m}} c_3^{l_m} (c_1^{l_m} - b) - \frac{w^l}{w_B^{l_m}} c_1^{l_m} (c_3^{l_m} - b_1)$$

$$f_4^l = c_1^{l_m} c_3^{l_m}$$

and $w^l = \min(w_A^{l_m}, w_B^{l_m})$.

$$f_2^u = a_1^{u_m} a_3^{u_m} + \frac{\left(w^u\right)^2}{w_A^{u_m} w_B^{u_m}} (b - a_1^{u_m})(b_1 - a_3^{u_m}) + \frac{w^u}{w_A^{u_m}} a_3^{u_m} (b - a_1^{u_m}) + \frac{w^u}{w_B^{u_m}} a_1^{u_m} (b_1 - a_3^{u_m})$$

$$f_3^u = c_1^{u_m} c_3^{u_m} + \frac{\left(w^u\right)^2}{w_A^{u_m} w_B^{u_m}} (c_1^{u_m} - b)(c_3^{u_m} - b_1) - \frac{w^u}{w_A^{u_m}} c_3^{u_m} (c_1^{u_m} - b) - \frac{w^u}{w_B^{u_m}} c_1^{u_m} (c_3^{u_m} - b_1)$$

$$f_4^u = c_1^{u_m} c_3^{u_m}$$

and $w^u = \min(w_A^{u_m}, w_B^{u_m})$

Also $g_1^l = a_2^{l_n} a_4^{l_n}$

$$g_2^l = \frac{\{b_1(1-\eta^l) + a_4^{l_n}(\eta^l - \eta_B^{l_n})\}\{b(1-\eta^l) + a_2^{l_n}(\eta^l - \eta_A^{l_n})\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}$$

$$g_3^l = \frac{\{c_4^{l_n}(\eta^l - \eta_B^{l_n}) + b_1(1-\eta^l)\}\{c_2^{l_n}(\eta^l - \eta_A^{l_n}) + b(1-\eta^l)\}}{(1-\eta_A^{l_n})(1-\eta_B^{l_n})}$$

$$g_4^l = c_2^{l_n} c_4^{l_n}$$

and $\eta^l = \max(\eta_A^{l_n}, \eta_B^{l_n})$

$$g_2^u = \frac{\{b_1(1-\eta^u) + a_4^{u_n}(\eta^u - \eta_B^{u_n})\}\{b(1-\eta^u) + a_2^{u_n}(\eta^u - \eta_A^{u_n})\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})}$$

$$g_3^u = \frac{\{c_4^{u_n}(\eta^u - \eta_B^{u_n}) + b_1(1-\eta^u)\}\{c_2^{u_n}(\eta^u - \eta_A^{u_n}) + b(1-\eta^u)\}}{(1-\eta_A^{u_n})(1-\eta_B^{u_n})}$$

$$g_4^u = c_2^{u_n} c_4^{u_n}$$

and $\eta^{u} = \max(\eta_A^{u_n}, \eta_B^{u_n})$

Thus, *AB* is clearly a type of GIVTrIFN, where height of the MFs are w^l, w^u and NMFs are η^l, η^u.

Hence proved the theorem.

4.4. Theorem (Division of Two GIVTIFNs with Different Heights Produces a GIVTrIFNs)

Proof

To divide two GIVTIFNs *A* and *B,* we first divide the *α, β*-cuts of *A* and *B* using interval arithmetic.

For MF,

$$ {}^{\alpha}A_{+}^{l} / {}^{\alpha}B_{+}^{l} = \left[\frac{\frac{\alpha}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{\alpha}{w_B^{l_m}}(c_3^{l_m} - b_1)}, \frac{c_1^{l_m} - \frac{\alpha}{w_A^{l_m}}(c_1^{l_m} - b)}{\frac{\alpha}{w_B^{l_m}}(b_1 - a_3^{l_m}) + a_3^{l_m}} \right] \tag{4.16} $$

where $w^{l} = \min(w_A^{l_m}, w_B^{l_m})$ and $\alpha \in [0, w]$.

To find the LMF $\mu_{A^l/B^l}(x)$ we equate both the first and second component of (4.16) to *x,* which gives

$$ x = \frac{\frac{\alpha}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{\alpha}{w_B^{l_m}}(c_3^{l_m} - b_1)} \text{ and } x = \frac{c_1^{l_m} - \frac{\alpha}{w_A^{l_m}}(c_1^{l_m} - b)}{\frac{\alpha}{w_B^{l_m}}(b_1 - a_3^{l_m}) + a_3^{l_m}} $$

Now, expressing *α* in terms of *x* and setting *α* ≥ 0 & *α* ≤ *w* and *α* ≤ *w* & *α* ≥ 0 in the above expressions, we get *α* together with the domain of *x*

$$ \text{i.e.,} \quad \alpha = \frac{c_3^{l_m} x - a_1^{l_m}}{\left\{ \frac{(b - a_1^{l_m})}{w_A^{l_m}} + \frac{(c_3^{l_m} - b_1)x}{w_B^{l_m}} \right\}}, \; x \in \left[a_1^{l_m} / c_3^{l_m}, \frac{\frac{w^l}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{w^l}{w_B^{l_m}}(c_3^{l_m} - b_1)} \right] \text{and} $$

$$ \alpha = \frac{c_1^{l_m} - a_3^{l_m} x}{\left\{ \frac{(c_1^{l_m} - b)}{w_A^{l_m}} + \frac{(b_1 - a_3^{l_m})x}{w_B^{l_m}} \right\}}, \; x \in \left[\frac{c_1^{l_m} - \frac{w^l}{w_A^{l_m}}(c_1^{l_m} - b)}{a_3^{l_m} + \frac{w^l}{w_B^{l_m}}(b_1 - a_3^{l_m})}, c_1^{l_m} / a_3^{l_m} \right] $$

The required LMF $\mu_{A^l/B^l}(x)$ is:

$$ \mu_{A^l/B^l}(x) = \begin{cases} \dfrac{c_3^{l_m} x - a_1^{l_m}}{\left\{ \frac{(b - a_1^{l_m})}{w_A^{l_m}} + \frac{(c_3^{l_m} - b_1)x}{w_B^{l_m}} \right\}}, & x \in \left[a_1^{l_m} / c_3^{l_m}, \dfrac{\frac{w^l}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{w^l}{w_B^{l_m}}(c_3^{l_m} - b_1)} \right] \\ w^l, & x \in \left[\dfrac{\frac{w^l}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{w^l}{w_B^{l_m}}(c_3^{l_m} - b_1)}, \dfrac{c_1^{l_m} - \frac{w^l}{w_A^{l_m}}(c_1^{l_m} - b)}{a_3^{l_m} + \frac{w^l}{w_B^{l_m}}(b_1 - a_3^{l_m})} \right] \\ \dfrac{c_1^{l_m} - a_3^{l_m} x}{\left\{ \frac{(c_1^{l_m} - b)}{w_A^{l_m}} + \frac{(b_1 - a_3^{l_m})x}{w_B^{l_m}} \right\}}, & x \in \left[\dfrac{c_1^{l_m} - \frac{w^l}{w_A^{l_m}}(c_1^{l_m} - b)}{a_3^{l_m} + \frac{w^l}{w_B^{l_m}}(b_1 - a_3^{l_m})}, c_1^{l_m} / a_3^{l_m} \right] \end{cases} $$

In a similar way, we can have the UMF $\mu_{A^u/B^u}(x)$ as:

$$
\mu_{A^u/B^u}(x)=\begin{cases}
\dfrac{c_3^{u_m}x-a_1^{u_m}}{\left\{\dfrac{(b-a_1^{u_m})}{w_A^{u_m}}+\dfrac{(c_3^{u_m}-b_1)x}{w_B^{u_m}}\right\}}, & x\in\left[a_1^{u_m}/c_3^{u_m},\ \dfrac{\dfrac{w^u}{w_A^{u_m}}(b-a_1^{u_m})+a_1^{u_m}}{c_3^{u_m}-\dfrac{w^u}{w_B^{u_m}}(c_3^{u_m}-b_1)}\right]\\[3ex]
w^u, & x\in\left[\dfrac{\dfrac{w^u}{w_A^{u_m}}(b-a_1^{u_m})+a_1^{u_m}}{c_3^{u_m}-\dfrac{w^u}{w_B^{u_m}}(c_3^{u_m}-b_1)},\ \dfrac{c_1^{u_m}-\dfrac{w^u}{w_A^{u_m}}(c_1^{u_m}-b)}{a_3^{u_m}+\dfrac{w^u}{w_B^{u_m}}(b_1-a_3^{u_m})}\right]\\[3ex]
\dfrac{c_1^{u_m}-a_3^{u_m}x}{\left\{\dfrac{(c_1^{u_m}-b)}{w_A^{u_m}}+\dfrac{(b_1-a_3^{u_m})x}{w_B^{u_m}}\right\}}, & x\in\left[\dfrac{c_1^{u_m}-\dfrac{w^u}{w_A^{u_m}}(c_1^{u_m}-b)}{a_3^{u_m}+\dfrac{w^u}{w_B^{u_m}}(b_1-a_3^{u_m})},\ c_1^{u_m}/a_3^{u_m}\right]
\end{cases}
$$

Where $w^u=\min\left(w_A^{u_m},w_B^{u_m}\right)$.

For NMF

$$
\beta_{A^l_-}/\beta_{B^l_-}=\left[\frac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}},\frac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right]/
$$

$$
\left[\frac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}},\frac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}\right]
$$

$$
=\left[\frac{\left(\dfrac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}\right)}{\dfrac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}},\frac{\left(\dfrac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right)}{\left(\dfrac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}\right)}\right]
$$

Equating each component with *x*, we have

$$
x=\frac{\left(\dfrac{(b-a_2^{l_n}\eta_A^{l_n})-(b-a_2^{l_n})\beta}{1-\eta_A^{l_n}}\right)}{\dfrac{\beta(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}}
$$

Now, expressing β in terms of x, we obtain

$$
\beta\left(\frac{(c_4^{l_n}-b_1)}{1-\eta_B^{l_n}}x+\frac{(b-a_2^{l_n})}{1-\eta_A^{l_n}}\right)=\left(\frac{(b-a_2^{l_n}\eta_A^{l_n})}{1-\eta_A^{l_n}}+\frac{(c_4^{l_n}\eta_B^{l_n}-b_1)}{1-\eta_B^{l_n}}x\right)
$$

$$
\beta=\left(\frac{(b-a_2^{l_n}\eta_A^{l_n})+\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(c_4^{l_n}\eta_B^{l_n}-b_1)x}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(c_4^{l_n}-b_1)x+(b-a_2^{l_n})}\right) \tag{4.17}
$$

Again, expressing $x=\dfrac{\left(\dfrac{\beta(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{1-\eta_A^{l_n}}\right)}{\left(\dfrac{(b_1-a_4^{l_n}\eta_B^{l_n})-(b-a_4^{l_n})\beta}{1-\eta_B^{l_n}}\right)}$ in terms of α we have

$$\beta=\frac{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(b_1-a_4^{l_n}\eta_B^{l_n})x+(c_2^{l_n}\eta_A^{l_n}-b)}{\left[(c_2^{l_n}-b)+\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(b-a_4^{l_n})x\right]}\qquad\textbf{(4.18)}$$

Putting $\beta\le 1$ & $\beta\ge\eta^l$ in (4.17) as well as $\beta\ge\eta^l$ & $\beta\le 1$ in (4.18), where $\eta^l=\max\left(\eta_A^{l_n},\,\eta_B^{l_n}\right)$ we have

$$x\in\left[\frac{a_2^{l_n}}{c_4^{l_n}},\frac{(b-a_2^{l_n}\eta_A^{l_n})-\eta^l(b-a_2^{l_n})}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[\eta^l(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)\right]}\right]\text{ and } x\in\left[\frac{\eta^l(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[(b_1-a_4^{l_n}\eta_B^{l_n})-\eta^l(b-a_4^{l_n})\right]},\frac{c_2^{l_n}}{a_4^{l_n}}\right]$$

Thus, the LNMF $\nu_{A^l/B^l}(x)$ is

$$\nu_{A^l/B^l}(x)=\begin{cases}\left(\dfrac{(b-a_2^{l_n}\eta_A^{l_n})+\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(c_4^{l_n}\eta_B^{l_n}-b_1)x}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(c_4^{l_n}-b_1)x+(b-a_2^{l_n})}\right), & x\in\left[\dfrac{a_2^{l_n}}{c_4^{l_n}},\dfrac{(b-a_2^{l_n}\eta_A^{l_n})-\eta^l(b-a_2^{l_n})}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[\eta^l(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)\right]}\right]\\ \eta^l, & x\in\left[\dfrac{(b-a_2^{l_n}\eta_A^{l_n})-\eta^l(b-a_2^{l_n})}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[\eta^l(c_4^{l_n}-b_1)-(c_4^{l_n}\eta_B^{l_n}-b_1)\right]},\dfrac{\eta^l(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[(b_1-a_4^{l_n}\eta_B^{l_n})-\eta^l(b-a_4^{l_n})\right]}\right]\\ \dfrac{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(b_1-a_4^{l_n}\eta_B^{l_n})x+(c_2^{l_n}\eta_A^{l_n}-b)}{\left[(c_2^{l_n}-b)+\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)(b_1-a_4^{l_n})x\right]}, & x\in\left[\dfrac{\eta^l(c_2^{l_n}-b)-(c_2^{l_n}\eta_A^{l_n}-b)}{\left(\dfrac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[(b_1-a_4^{l_n}\eta_B^{l_n})-\eta^l(b-a_4^{l_n})\right]},\dfrac{c_2^{l_n}}{a_4^{l_n}}\right]\end{cases}$$

Following the same procedure, we can have the UNMF $\nu_{A^u/B^u}(x)$ as:

$$\nu_{A^u/B^u}(x)=\begin{cases}\left(\dfrac{(b-a_2^{u_n}\eta_A^{u_n})+\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)(c_4^{u_n}\eta_B^{u_n}-b_1)x}{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)(c_4^{u_n}-b_1)x+(b-a_2^{u_n})}\right), & x\in\left[\dfrac{a_2^{u_n}}{c_4^{u_n}},\dfrac{(b-a_2^{u_n}\eta_A^{u_n})-\eta^u(b-a_2^{u_n})}{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[\eta^u(c_4^{u_n}-b_1)-(c_4^{u_n}\eta_B^{u_n}-b_1)\right]}\right]\\ \eta^u, & x\in\left[\dfrac{(b-a_2^{u_n}\eta_A^{u_n})-\eta^u(b-a_2^{u_n})}{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[\eta^u(c_4^{u_n}-b_1)-(c_4^{u_n}\eta_B^{u_n}-b_1)\right]},\dfrac{\eta^u(c_2^{u_n}-b)-(c_2^{u_n}\eta_A^{u_n}-b)}{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[(b_1-a_4^{u_n}\eta_B^{u_n})-\eta^u(b-a_4^{u_n})\right]}\right]\\ \dfrac{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)(b_1-a_4^{u_n}\eta_B^{u_n})x+(c_2^{u_n}\eta_A^{u_n}-b)}{\left[(c_2^{u_n}-b)+\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)(b_1-a_4^{u_n})x\right]}, & x\in\left[\dfrac{\eta^u(c_2^{u_n}-b)-(c_2^{u_n}\eta_A^{u_n}-b)}{\left(\dfrac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[(b_1-a_4^{u_n}\eta_B^{u_n})-\eta^u(b-a_4^{u_n})\right]},\dfrac{c_2^{u_n}}{a_4^{u_n}}\right]\end{cases}$$

Thus,

$$A/B=\left\langle\left[\left(f_1^l,f_2^l,f_3^l,f_4^l;w^l\right),\left(f_1^u,f_2^u,f_3^u,f_4^u;w^u\right)\right],\left[\left(g_1^l,g_2^l,g_3^l,g_4^l;\eta^l\right),\left(g_1^u,g_2^u,g_3^u,g_4^u;\eta^u\right)\right]\right\rangle$$

where $f_1^l = a_1^{l_m} / c_3^{l_m}$

$$f_2^l = \frac{\frac{w^l}{w_A^{l_m}}(b - a_1^{l_m}) + a_1^{l_m}}{c_3^{l_m} - \frac{w^l}{w_B^{l_m}}(c_3^{l_m} - b_1)}$$

$$f_3^l = \frac{c_1^{l_m} - \frac{w^l}{w_A^{l_m}}(c_1^{l_m} - b)}{a_3^{l_m} + \frac{w^l}{w_B^{l_m}}(b_1 - a_3^{l_m})}$$

$$f_4^l = c_1^{l_m} / a_3^{l_m}$$

and $w^l = \min(w_A^{l_m}, w_B^{l_m})$.

$$f_1^u = a_1^{u_m} / c_3^{u_m}$$

$$f_2^u = \frac{\frac{w^u}{w_A^{u_m}}(b - a_1^{u_m}) + a_1^{u_m}}{c_3^{u_m} - \frac{w^u}{w_B^{u_m}}(c_3^{u_m} - b_1)}$$

$$f_3^u = \frac{c_1^{u_m} - \frac{w^u}{w_A^{u_m}}(c_1^{u_m} - b)}{a_3^{u_m} + \frac{w^u}{w_B^{u_m}}(b_1 - a_3^{u_m})}$$

$$f_4^u = c_1^{u_m} / a_3^{u_m}$$

and $w^u = \min(w_A^{u_m}, w_B^{u_m})$.

Also $g_1^l = \frac{a_2^{l_n}}{c_4^{l_n}}$

$$g_2^l = \frac{(b - a_2^{l_n}\eta_A^{l_n}) - \eta^l(b - a_2^{l_n})}{\left(\frac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[\eta^l(c_4^{l_n} - b_1) - (c_4^{l_n}\eta_B^{l_n} - b_1)\right]}$$

$$g_3^l = \frac{\eta^l(c_2^{l_n} - b) - (c_2^{l_n}\eta_A^{l_n} - b)}{\left(\frac{1-\eta_A^{l_n}}{1-\eta_B^{l_n}}\right)\left[(b_1 - a_4^{l_n}\eta_B^{l_n}) - \eta^l(b_1 - a_4^{l_n})\right]}$$

$$g_4^l = \frac{c_2^{l_n}}{a_4^{l_n}}$$

and $\eta^{l} = \max(\eta_A^{l_n}, \eta_B^{l_n})$

$$g_1^u = \frac{a_2^{u_n}}{c_4^{u_n}}$$

$$g_2^u = \frac{(b - a_2^{u_n}\eta_A^{u_n}) - \eta^u (b - a_2^{u_n})}{\left(\frac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[\eta^u (c_4^{u_n} - b_1) - (c_4^{u_n}\eta_B^{u_n} - b_1)\right]}$$

$$g_3^u = \frac{\eta^u (c_2^{u_n} - b) - (c_2^{u_n}\eta_A^{u_n} - b)}{\left(\frac{1-\eta_A^{u_n}}{1-\eta_B^{u_n}}\right)\left[(b_1 - a_4^{u_n}\eta_B^{u_n}) - \eta^u (b_1 - a_4^{u_n})\right]}$$

$$g_4^u = \frac{c_2^{u_n}}{a_4^{u_n}}$$

and $\eta^{u} = \max(\eta_A^{u_n}, \eta_B^{u_n})$

Thus, *A/B* is also a GIVTrIFN, where height of the MFs are w^l, w^u and NMFs are η^l, η^u.

Hence proved the theorem .

4.5. Graphical Representation of the Proposed Approach

Let us consider the same example discussed in section-1 where,

$A = \langle [(2,4,6;0.7), (1,4,7;0.8)], [(1,4,7;0.3), (0.5,4,8.5;0.2)] \rangle$ is a fixed GIVTIFN while

$B_1 = \langle [(10,14,18;0.8), (8,14,20;0.9)], [(9,14,19;0.2), (7,14,21;0.1)] \rangle$ and

$B_2 = \langle [(10,14,18;0.9), (8,14,20;0.95)], [(9,14,19;0.1), (7,14,21;0.05)] \rangle$ be two different GIVTIFNs with different heights. In section 1, we have seen that the existing approach [26] produces identical GIVTIFN while carrying out the addition operation between A and B_1 & A and B_2 which are depicted in Figs. (**8** and **9**) respectively. On the other hand, while performing the addition operation for the same pairs of GIVTIFNs by using the proposed approach, it produces two distinct GIVTrIFNs. Thus, the GIVTrIFNs are given as:

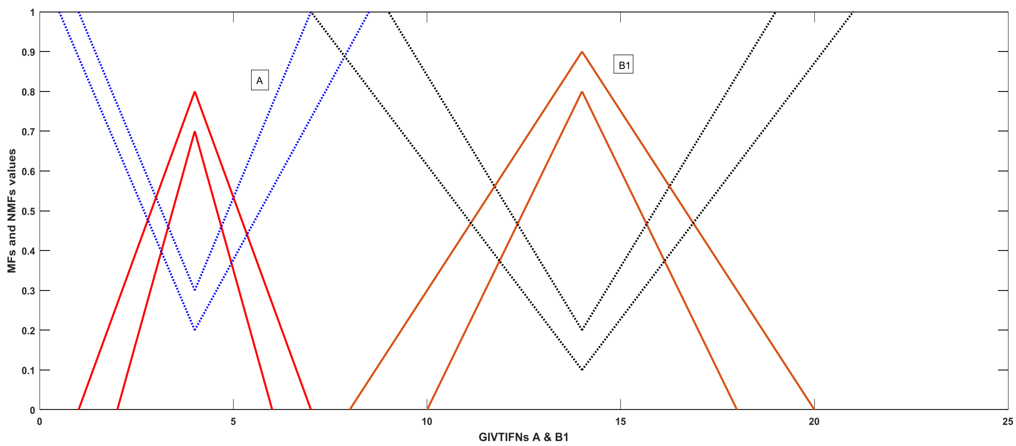

Fig. (8). MFs and NMFs of GIVTIFNs A and B_1

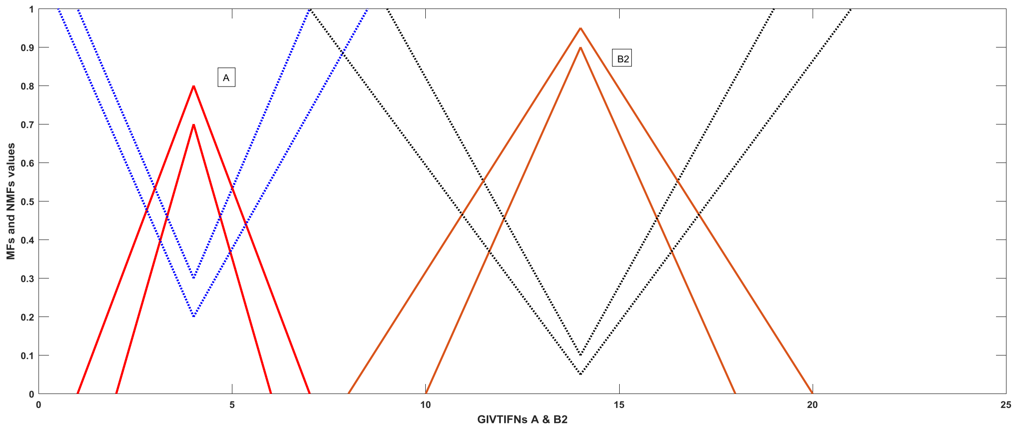

Fig. (9). MFs and NMFs of GIVTIFNs A and B_2

$$A+B_1=\langle[(12,17.5,18.5,24;0.7),(9,17.33,18.67,27;0.8)],[(10,17.375,18.625,26;0.3),(7.5,17.22,18.78,29.5;0.2)]\rangle$$

and

$$A+B_2=\langle[(12,17.11,18.89,24;0.7),(9,17.05,18.94,27;0.8)],[(10,16.88,19.11,26;0.3),(7.5,16.89,19.105,29.5;0.2)]\rangle.$$

The following Figs. (**10** and **11**) are the graphical representation of the GIVTrIFNs $A + B_1$ and $A + B_2$.

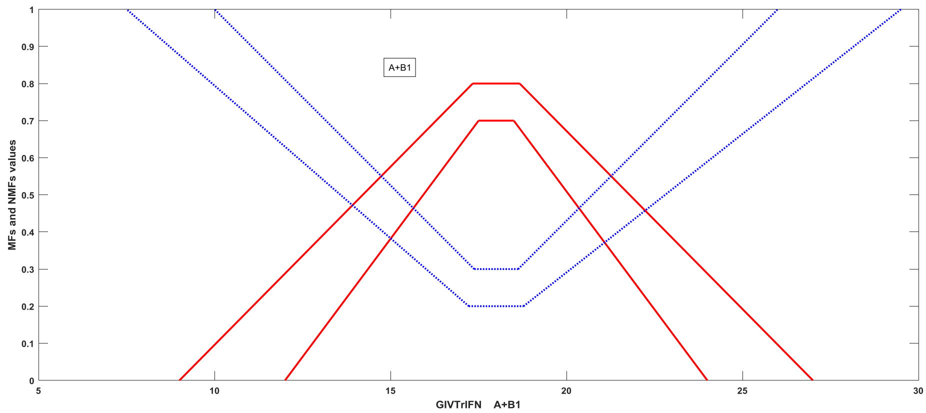

Fig. (10). : MFs and NMFs of GIVTrIFNs A+B_1

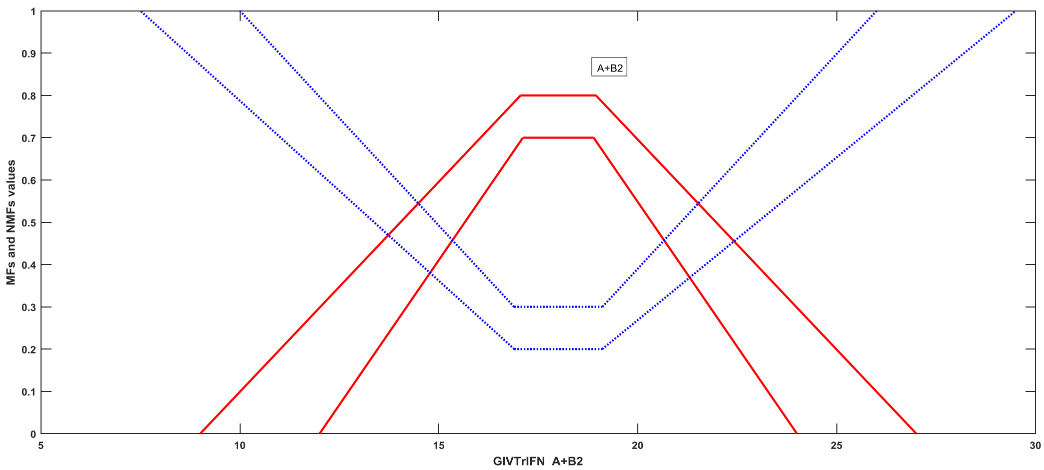

Fig. (11). MFs and NMFs of GIVTrIFNs A+B_2

Similarly, different outputs will be obtained other arithmetic operations which is logical and correct while the existing approach leads to illogical output.

5. NUMERICAL EXAMPLES

Let $A = \langle [10,15,20;0.6];[8,15,22;0.7],[7,15,22;0.25][6,15,24;0.2]\rangle$ and

$B = \langle [4,6,8;0.5];[3,6,10;0.65],[3,6,9;0.4][1,6,13;0.25]\rangle$ be two triangular GIFNs.

Then, using the proposed approach we have

$$A+B = \langle [14,20.17,21.83,28;0.5];[11,20.5,21.5,32;0.65],[10,19.4,22.4,29;0.4];[7,20.44,21.56,37;0.25]\rangle$$

whose MFs and NMFs are:

$$\mu_{A+B}(x) = \begin{cases} \left.\begin{array}{l} 0.5\frac{x-14}{6.17},\ x\in[14,20.17] \\ 0.5, x\in[20.17,21.83] \\ 0.5\frac{28-x}{6.17},\ x\in[21.83,28] \end{array}\right\} \text{LMF} \\ \left.\begin{array}{l} 0.65\frac{x-11}{9.1},\ x\in[11,20.5] \\ 0.65, x\in[20.5,21.5] \\ 0.65\frac{28-x}{10.50},\ x\in[21.5,32] \end{array}\right\} \text{UMF} \end{cases} \quad \text{and} \quad \nu_{A+B}(x) = \begin{cases} \left.\begin{array}{l} 0.4\frac{25.67-x}{6.268},\ x\in[10,19.4] \\ 0.4, x\in[19.4,22.4] \\ 0.4\frac{x-16.67}{5.732},\ x\in[22.4,29] \end{array}\right\} \text{LNMF} \\ \left.\begin{array}{l} 0.25\frac{24.92-x}{7.48},\ x\in[7,20.44] \\ 0.4, x\in[20.44,21.56] \\ 0.4\frac{x-16.42}{5.145},\ x\in[21.56,37] \end{array}\right\} \text{UNMF} \end{cases}$$

respectively.

Then, $A-B = \langle [2,8.165,9.835,16;0.5];[-2,8.5,9.5,19;0.65],[-2,7.4,14.5,19;0.4];[-7,8.437,9.563,23;0.25]\rangle$ whose MFs and NMFs are:

$$\mu_{A-B}(x) = \begin{cases} \left.\begin{array}{l} 0.5\frac{x-2}{6.165},\ x\in[2,8.165] \\ 0.5, x\in[8.165,9.835] \\ 0.5\frac{16-x}{6.165},\ x\in[9.835,16] \end{array}\right\} \text{LMF} \\ \left.\begin{array}{l} 0.65\frac{x+2}{10.5},\ x\in[-2,8.5] \\ 0.65, x\in[8.5,9.5] \\ 0.65\frac{19-x}{9.5},\ x\in[9.5,19] \end{array}\right\} \text{UMF} \end{cases} \quad \text{and} \quad \nu_{A-B}(x) = \begin{cases} \left.\begin{array}{l} 0.4\frac{13.67-x}{6.268},\ x\in[10,19.4] \\ 0.4, x\in[19.4,22.4] \\ 0.4\frac{x-4.67}{13.2},\ x\in[22.4,29] \end{array}\right\} \text{LNMF} \\ \left.\begin{array}{l} 0.25\frac{13.58-x}{5.146},\ x\in[7,20.44] \\ 0.25, x\in[20.44,21.56] \\ 0.25\frac{x-5.083}{4.479},\ x\in[21.56,37] \end{array}\right\} \text{UNMF} \end{cases}$$

respectively.

Also $AB = \langle [40,84.99,95.01,160;0.5];[24,87,93,220;0.5],[21,80.4,98.4,198;0.4],[6,86.625,93.375,312;0.25]\rangle$ whose MFs and NMFs are:

$$\mu_{AB}(x) = \begin{cases} \left.\begin{array}{l} \frac{-11+\sqrt{1+3x}}{10},\ x\in[20,56.67] \\ 0.5, x\in[56.67,63.33] \\ \frac{39-\sqrt{81+12x}}{20},\ x\in[63.33,120] \end{array}\right\} \text{LMF} \\ \left.\begin{array}{l} \frac{-11+\sqrt{1+3x}}{10},\ x\in[24,87] \\ 0.5, x\in[87,93] \\ \frac{24-\sqrt{96+3x}}{10},\ x\in[93,220] \end{array}\right\} \text{UMF} \end{cases}, \nu_{AB}(x) = \begin{cases} \left.\begin{array}{l} \frac{78.15-\sqrt{1.8225+14.4x}}{48},\ x\in[6,86.625] \\ 0.4, x\in[86.625,93.375] \\ \frac{-45.3+\sqrt{136.89+37.8x}}{42},\ x\in[93.375,312] \end{array}\right\} \text{LMF} \\ \left.\begin{array}{l} \frac{78.15-\sqrt{1.8225+14.4x}}{48},\ x\in[6,86.625] \\ 0.4, x\in[86.625,93.375] \\ \frac{-45.3+\sqrt{136.89+37.8x}}{42},\ x\in[93.375,312] \end{array}\right\} \text{UMF} \end{cases},$$

And

$$A/B=\langle[0.462,2.027,3,24;0.2];[0.8,2.416,2.583,7.33;0.65],[0.78,2.23,2.73,7.33;0.4][0.462,2.406,2.594,24;0.25]\rangle$$

whose MFs and NMFs are

$$\mu_{A/B}(x)=\begin{cases}\left.\begin{array}{l}\frac{8x-10}{8.33+4x},\ x\in[0.462,2.027]\\ 0.5,x\in[2.027,3]\\ \frac{20-4x}{8.33+4x},\ x\in[3,24]\end{array}\right\}\text{LMF}\\ \left.\begin{array}{l}\frac{10x-8}{10+6.154x},\ x\in[0.8,2.416]\\ 0.65,x\in[2.416,2.583]\\ \frac{22-3x}{10+4.615x},\ x\in[2.583,7.33]\end{array}\right\}\text{UMF}\end{cases}\quad\text{and}\quad \nu_{A/B}(x)=\begin{cases}\left.\begin{array}{l}\frac{13.25-3x}{8+3.75x},\ x\in[0.78,2.23]\\ 0.4,x\in[2.23,2.73]\\ \frac{6x-9.5}{7+3.75x},\ x\in[2.73,7.33]\end{array}\right\}\text{LNMF}\\ \left.\begin{array}{l}\frac{13.8-2.93x}{9+7.47x},\ x\in[0.462,2.406]\\ 0.25,x\in[2.406,2.594]\\ \frac{6.13x-10.2}{9+5.33x},\ x\in[2.594,24]\end{array}\right\}\text{UNMF}\end{cases}$$

6. RANKING OF GIVTRIFNS BASED ON VALUE INDEX

Deng Feng Li [35] introduced the concept of value and ambiguity of GTIFN and the same concept has been put forward for GTrIFN by De and Das [36]. In this section, we will extend the concept of the value of GTrIFN to GIVTrIFNs.

Definition: Let $A=\left\langle\left[\left(a_1^{lm},b,c,d_1^{lm};w^{lm}\right),\left(a_1^{um},b,c,d_1^{um};w^{um}\right)\right],\left[\left(a_2^{ln},b_1,c_1,d_2^{ln};\eta^{ln}\right),\left(a_2^{un},b_1,c_1,d_2^{un};\eta^{un}\right)\right]\right\rangle$ be a GIVTrIFN and $[L^{l_m}_{\overline{x_\alpha}},R^{l_m}_{\overline{x_\alpha}}]$, $[L^{u_m}_{\overline{x_\alpha}},R^{u_m}_{\overline{x_\alpha}}]$ & $[L^{l_n}_{\underline{x_\beta}},R^{l_n}_{\underline{x_\beta}}]$, $[L^{u_n}_{\underline{x_\beta}},R^{u_n}_{\underline{x_\beta}}]$ be α, β-cuts of the MFs and NMFs of A, respectively. Then the value of MFs and NMFs of A is defined as:

$$V^{lm}{}_{\mu}(A)=\int_0^{w^{lm}}\frac{L^{lm}_{\overline{x_\alpha}}+R^{lm}_{\overline{x_\alpha}}}{2}f(\alpha)d\alpha,\quad V^{um}{}_{\mu}(A)=\int_0^{w^{lu}}\frac{L^{lu}_{\overline{x_\alpha}}+R^{lu}_{\overline{x_\alpha}}}{2}f(\alpha)d\alpha \text{ and}$$

$$V^{ln}{}_{\nu}(A)=\int_{\eta^{ln}}^{1}\frac{L^{ln}{}_{\underline{x_\beta}}+R^{ln}{}_{\underline{x_\beta}}}{2}g(\beta)d\beta,\quad V^{un}{}_{\nu}(A)=\int_{\eta^{un}}^{1}\frac{L^{un}{}_{\underline{x_\beta}}+R^{un}{}_{\underline{x_\beta}}}{2}g(\beta)d\beta \text{ respectively.}$$

Where $f(\alpha)$ is a non-negative and non-decreasing function on $[0,w^{l_m}]$ with $f(0)=0$ and $\int_0^{w^{lm}} f(\alpha)d\alpha=w^{l_m}$ and on $[0,w^{u_m}]$ with $f(0)=0$ & $\int_0^{w^{um}} f(\alpha)d\alpha=w^{u_m}$. The function $f(\beta)$ is a non-negative and non-increasing function on $[\eta^{l_n},1]$ with $g(1)=0$ and $\int_{\eta^{ln}}^{1} g(\beta)d\beta=1-\eta^{l_n}$ also $f(\beta)$ is a non-negative and non-increasing function on $[\eta^{u_n},1]$ with $g(1)=0$ & $\int_{\eta^{un}}^{1} g(\beta)d\beta=1-\eta^{u_n}$

Like [35] and [36], we also choose,

$f(\alpha)=\dfrac{2\alpha}{w^{l_m}}$, $\alpha\in[0,w^{l_m}]$ & $f(\alpha)=\dfrac{2\alpha}{w^{u_m}}$, $\alpha\in[0,w^{u_m}]$ and $g(\beta)=\dfrac{2(1-\beta)}{1-\eta^{l_n}}$, $\alpha\in[\eta^{l_n},1]$ &

$g(\beta)=\dfrac{2(1-\beta)}{1-\eta^{u_n}}, \alpha\in[\eta^{u_n},1]$.

Thus the value of the MFs and NMFs of A are evaluated as:

$$V^{l_m}{}_{\mu}(A)=\frac{a_1^{l_m}+d_1^{l_m}+2(b+c)}{6}w^{l_m} \ \& \ V^{u_m}{}_{\mu}(A)=\frac{a_1^{u_m}+d_1^{u_m}+2(b+c)}{6}w^{u_m} \text{ and}$$

$$V^{l_n}{}_{\nu}(A)=\frac{a_2^{l_n}+d_2^{l_n}+2(b_1+c_1)}{6}(1-\eta^{l_n}) \ \& \ V^{u_n}{}_{\nu}(A)=\frac{a_2^{u_n}+d_2^{u_n}+2(b_1+c_1)}{6}(1-\eta^{u_n}) \text{ respectively.}$$

Zeng *et al*. [37], devised value-index to rank trapezoidal IFS and we extend it for GIVTrIFNs.

That is, for a GIVTrIFS

$A=\left\langle\left[\left(a_1^{l_m},b,c,d_1^{l_m};w^{l_m}\right),\left(a_1^{u_m},b,c,d_1^{u_m};w^{u_m}\right)\right],\left[\left(a_2^{l_n},b_1,c_1,d_2^{l_n};\eta^{l_n}\right),\left(a_2^{u_n},b_1,c_1,d_2^{u_n};\eta^{u_n}\right)\right]\right\rangle$ the value-index of A is defined as:

$V_{\lambda}(A)=\lambda\left(V^{l_m}{}_{\mu}(A)+V^{u_m}{}_{\mu}(A)\right)+(1-\lambda)\left(V^{l_n}{}_{\nu}(A)+V^{u_n}{}_{\nu}(A)\right)$, where $\lambda\in[0,1]$ is a weight which represents the decision maker's preference information.

7. MULTI-CRITERIA GROUP DECISION-MAKING USING ARITHMETIC OPERATIONS ON GIVTIFNs

In general, multi-criteria group decision-making problems include uncertain imprecise data and information. For validity and justification of the approach and to show the application in real-world problem of the proposed approach, a multi-criteria group decision-making model has been carried out to rank the best alternative among the available alternatives based on GIVTIFNs.

7.1. Methodology

Let us suppose that a committee of K expert decision makers D_1, D_2...D_K will choose the best alternative among n alternatives $A_1, A_2,..., A_n$ based on m criteria where $C_1,C_2,...,C_m$ are for each alternative respectively.

The procedure for the decision process is given below:

Step-I: Decision makers choose linguistic weighting variables with respect to the importance weight of criteria and the linguistic ratings variable to evaluate the ratings of alternatives with respect to each criterion which are expressed in terms of positive GIVTIFNs.
Step-II: Decision makers evaluate the importance weight of each criterion using linguistic weighting variables.
Step-III: The weights of criteria are aggregated using.

$$\widetilde{w}_{\mathrm{j}}=\frac{1}{K}\left[\widetilde{w}_{\mathrm{j}}^{1}(+)\widetilde{w}_{\mathrm{j}}^{2}(+)\ldots\ldots(+)\widetilde{w}_{\mathrm{j}}^{K}\right]$$

to get the aggregated fuzzy weight $\widetilde{w}_j$ of the criterion C_j.

The new weight vector can be written as:

$$\widetilde{w}_j=\left[\widetilde{w}_1 \quad \widetilde{w}_2 \quad \ldots\ldots \quad \widetilde{w}_n\right]$$

where each $\tilde{w}_j$ is GIVTIFNs.

Step-IV: Decision makers give their opinion to get the aggregated fuzzy ratings $\tilde{x}_{ij}$ of alternative A_i under criterion C_j. That is,

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots\cdots & C_m \end{array} \\ \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots\cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots\cdots & \tilde{x}_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots\cdots & \tilde{x}_{mn} \end{bmatrix} \end{array}$$

where each $\tilde{x}_{ij}$ is GIVTIFNs.

Step-V: If all the weights and ratings are in the interval [0, 1] (*i.e.*, W and R are normalized) the next step is followed and

if not, they can be normalized by:

$$W^* = \frac{W}{\left\lceil w_j^* / 10 \right\rceil \times 10} \text{ and } R^* = \frac{R}{\left\lceil c^* / 10 \right\rceil \times 10}$$

Where $\lceil \cdot \rceil$ is the ceiling function, $w_j^* = \max(w_{j3})$ and $c^* = \max(c_{ij})$.

Step-VI: Construct the weighted normalized fuzzy decision matrix

$\tilde{D} = [\tilde{d}_{ij}]_{m \times n}$ $i = 1, 2, \ldots\ldots m,\ j = 1,2, \ldots\ldots n$

Where $\tilde{d}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j$ using our proposed arithmetic operations which are normalized positive GIVTIFNs.

Step-VII: Decision makers evaluate $\tilde{d}_i = \sum_{j=1}^{n} \tilde{d}_{ij}$ using our proposed arithmetic operations.

Step-VIII: Based on maximum value-index of $\tilde{d}_i$, decision-makers will choose the suitable alternative A_i.

7.2. Hypothetical Case Study

Let us suppose a committee of three expert decision makers, D1, D2 and D3 which has been formed to conduct the interview for the post of the professor to select the most suitable candidate among the three eligible candidates, namely A1, A2 and A3. Five benefit criteria are considered:

C1: Research Publications,

C2: Teaching skills,

C3: Subject Knowledge,

C4: Experiences in teaching,

C5: Teaching discipline.

7.2.1. Computational Procedure is Discussed in Detail Below:

Step-I: Decision makers choose the linguistic weighting variable (Table **1**) for the importance weight of criteria and the linguistic ratings variable Table **2** to evaluate the ratings of alternatives with respect to each criterion.

Table 1. Linguistic Variable for the Importance Weight of each Criterion.

Very Low (VL)	$<[0,\ 0.0,\ 0.08;0.6][0,\ 0.0,\ 0.1;0.7],[0,\ 0.0,\ 0.15;0.3][0,0,0.2;0.2]>$
Low (L)	$<[0.05,\ 0.1,\ 0.2;0.75][0,\ 0.1,\ 0.25;0.8],[0.02,0.1,0.22;0.2][0,0.1,0.3,0.1]>$
Medium Low (ML)	$<[0.17,\ 0.3,0.35;0.65][0.15,0.3,0.45;0.7],[0.1,0.3,0.4;0.25][0.1,0.3,0.5;0.15]>$
Medium (M)	$<[0.4,\ 0.5,\ 0.6;0.8][0.35,\ 0.5,\ 0.65;0.9],[0.3,\ 0.5,\ 0.63;0.1][0.3,\ 0.5,\ 0.7;0.05]>$
Medium High (MH)	$<[0.6,\ 0.7,\ 0.8;0.7][0.55,\ 0.7,\ 0.85;0.75],[0.58,\ 0.7,\ 0.82;0.25][0.5,\ 0.7,\ 0.9;0.2]>$
High (H)	$<[0.85,\ 0.9,\ 0.95;0.65][0.8,\ 0.9,\ 1.0;0.7],[0.8,0.9,0.97;0.35][0.75,0.9,1;0.3]>$
Very High (VH)	$\langle[0.95,\ 1,\ 1;0.8][0.9,\ 1.0,\ 1.0;0.85],[0.92,\ 1,\ 1;0.15][0.85,1.0,1.0;0.1]\rangle$

Table 2. Linguistic Variables for the Ratings.

Very Poor (VP)	$<[0,\ 0,\ 0.08;0.8][0.0,\ 0.0,\ 0.1;0.9],[0,0,0.12;0.1][0.0,\ 0.0,\ 0.15;0.05]>$
Poor (P)	$<[0.5,0.1,0.15;0.7][0.0,\ 0.1,\ 0.2;0.8],[0.25,0.1,0.2;0.2][0,0.1,0.25.0.1]>$
Medium Poor (MP)	$<[0.25,0.3,0.35;0.75][0.2,\ 0.3,\ 0.5;0.8],[0.15,0.3,0.4;0.25][0.1,0.3,0.55;0.2]>$
Fair (F)	$<[0.4,0.5,0.6;0.7][0.35,0.5,0.65;0.75],[0.33,0.5,0.65;0.25][0.3,0.5,0.7,0.2]>$
Medium Good (MG)	$<[0.55,0.7,0.8;0.8][0.5,\ 0.7,\ 0.9;0.85],[0.5,0.7,0.85;0.15][0.45,0.7,0.9;0.1]>$
Good (G)	$<[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9],[0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]>$
Very Good (VG)	$<[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8],[0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]>$

Step-II: To assess the importance of the criteria (Table **3**) linguistic weighting variables are used Table **1**

Table 3. The importance weight of each criterion given by Decision Makers.

Decision makers **Criterion**	**D1**	**D2**	**D3**
C1	H	VH	MH
C2	VH	VH	VH
C3	VH	H	H
C4	VH	VH	VH
C5	M	MH	MH

Step-III: The weights of criteria are aggregated using *equation* (1) to get the aggregated fuzzy weight $\widetilde{w}_j$ of the criterion C_j and decision makers give their opinion (Table **4**) to get the aggregated fuzzy ratings $\tilde{x}_{ij}$ of alternative A_i under criterion C_j.

$$\widetilde{w}_{\mathrm{j}} = \frac{1}{K}\left[\widetilde{w}_{\mathrm{j}}^{1}(+)\widetilde{w}_{\mathrm{j}}^{2}(+)\ \dots\dots\ (+)\widetilde{w}_{\mathrm{j}}^{K}\right]$$

$$\tilde{w}_1 = \frac{1}{3}\left\{\begin{array}{l} <[0.85,\ 0.9,\ 0.95;0.65][0.8,\ 0.9,\ 1;0.7],[0.8,0.9,0.97;0.35][0.75,0.9,1;0.3]> \\ +\langle[0.95,\ 1,\ 1;0.8][0.9,\ 1,\ 1;0.85],[0.92,\ 1,\ 1;0.15][0.85,1,1;0.1]\rangle \\ +<[0.6,\ 0.7,\ 0.8;0.7][0.55,\ 0.7,\ 0.85;0.75],[0.58,\ 0.7,\ 0.82;0.25][0.5,\ 0.7,\ 0.9;0.2]> \end{array}\right\}$$

$$= <[0.8,0.86,0.87,0.97;0.65][0.75,\ 0.86,0.87,0.95;0.7],[0.76,0.85,0.87,0.93;0.35][0.7,0.81,0.91,0.97;0.3]>$$

Similarly, we have

$$\tilde{w}_2 = <[0.95,1,1,1;0.8];[0.9,1,1,1;0.85],[0.92,1,1,1;0.15][0.85,1,1,1;0.1]>$$

$$\tilde{w}_3 = <[0.833,0.93,0.933,0.96;0.65];[0.83,0.93,0.933,1;0.7],[0.84,0.93,0.933,0.98;0.35][0.783,0.923,0.933,1;0.3]>$$

$$\tilde{w}_4 = <[0.95,1,1,1;0.8];[0.9,1,1,1;0.85],[0.92,1,1,1;0.15][0.85,1,1,1;0.1]>$$

$$\tilde{w}_5 = <[0.533,\ 0.629,0.638,0.733;0.7];[0.483,\ 0.625,0.642,0.783;0.75],$$
$$[0.487,0.622,0.64,0.757;0.25];[0.433,0.623,0.644,0.833;0.2]>$$

Table 4. The Final Aggregate Result Obtained From Ratings Given By Decision Makers.

Criterion	Alternative	Linguistic variable
C1	A1	MG
	A2	G
	A3	VG
C2	A1	G
	A2	VG
	A3	MG
C3	A1	F
	A2	VG
	A3	G
C4	A1	VG
	A2	VG
	A3	G
C5	A1	F
	A2	VG
	A3	G

Step-IV: The fuzzy decision matrix R is constructed as follows using Table **4**.

$$\begin{array}{cc} & \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ R = \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{ccccc} \begin{array}{c} <[0.55,0.7,0.8;0.8][0.5,\ 0.7,\ 0.9;0.85], \\ [0.5,0.7,0.85;0.15][0.45,0.7,0.9;0.1]> \end{array} & \begin{array}{c} <[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9], \\ [0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]> \end{array} & \begin{array}{c} <[0.4,0.5,0.6;0.7][0.35,0.5,0.65;0.75], \\ [0.33,0.5,0.65;0.25][0.3,0.5,0.7,0.2]> \end{array} & \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} & \begin{array}{c} <[0.4,0.5,0.6;0.7][0.35,0.5,0.65;0.75], \\ [0.33,0.5,0.65;0.25][0.3,0.5,0.7,0.2]> \end{array} \\ \begin{array}{c} <[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9], \\ [0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]> \end{array} & \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} & \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} & \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} & \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} \\ \begin{array}{c} <[0.95,1,1;0.75][0.9,\ 1.0,\ 1.0;0.8], \\ [0.93,1,1;0.2][0.85,\ 1.0,\ 1.0;0.15]> \end{array} & \begin{array}{c} <[0.55,0.7,0.8;0.8][0.5,\ 0.7,\ 0.9;0.85], \\ [0.5,0.7,0.85;0.15][0.45,0.7,0.9;0.1]> \end{array} & \begin{array}{c} <[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9], \\ [0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]> \end{array} & \begin{array}{c} <[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9], \\ [0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]> \end{array} & \begin{array}{c} <[0.7,0.8,0.85;0.8][0.65,\ 0.8,\ 0.95;0.9], \\ [0.65,0.8,0.9;0.2][0.6,0.8,0.95;0.1]> \end{array} \end{array}\right] \end{array}$$

Since all the weights and ratings are in the interval [0, 1], so the matrix R is the normalized fuzzy decision matrix.

1. The weighted normalized fuzzy decision matrix is now constructed by using equation (2).

2. To evaluate $\tilde{d}_i = \sum_{j=1}^{n} \tilde{d}_{ij}$ using our proposed arithmetic operations, we have

$$\tilde{d}_1 = \langle [2.554, 3.103, 3.226, 3.512; 0.65], [2.23, 2.703, 3.002, 3.964; 0.7], [2.272, 3.018, 3.286, 3.82; 0.35], [1.912, 2.93, 3.354, 4.106; 0.3] \rangle$$

$$\tilde{d}_2 = \langle [3.661, 4.183, 4.282, 4.518; 0.65], [3.245, 4.122, 4.309, 4.686; 0.7], [3.44, 4.109, 4.301, 4.574; 0.35], [2.9, 3.988, 4.359, 4.755; 0.3] \rangle$$

$$\tilde{d}_3 = \langle [2.949, 3.51, 3.738, 4.209; 0.65], [2.563, 3.446, 3.759, 4.494; 0.7], [2.628, 3.429, 3.726, 4.243; 0.35], [2.283, 3.23, 3.824, 4.561; 0.3] \rangle$$

Table 5. Value-Index and rank.

$\tilde{d}_i$	λ_i	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$
Value-index $V_\lambda(A)$	λ=0.1	4.182994	**5.539223**	4.750128
Value-index $V_\lambda(A)$	λ=0.2	4.171798	**5.547381**	4.761256
Value-index $V_\lambda(A)$	λ=0.3	4.160602,	**5.5556**	4.772384
Value-index $V_\lambda(A)$	λ=0.4	4.149406	**5.56369**	4.783512
Value-index $V_\lambda(A)$	λ=0.5	4.13821	**5.571853**	4.79464
Value-index $V_\lambda(A)$	λ=0.6	4.127014	**5.58001**	4.805768
Value-index $V_\lambda(A)$	λ=0.7	4.115818	**5.588168**	4.816896
Value-index $V_\lambda(A)$	λ=0.8	4.104622	**5.59632**	4.828024
Value-index $V_\lambda(A)$	λ=0.9	4.093426	**5.604483**	4.839152
Rank		3rd	1st	2nd

Step-V: It is clear from the Table **5** that the calculated value-index $V_\lambda(A)$ for d_2 is maximum for any given weight $\lambda \in [0, 1]$. Thus, the ordering of $d_i's (i = 1, 2, 3)$ for any $\lambda \in [0, 1]$ is $\tilde{d}_2 > \tilde{d}_3 > \tilde{d}_1$. Higher the value index of $d_i's$ will indicate the best selection of the alternative $A_i (i = 1, 2, 3)$. Hence, the ranking order of the four alternatives is $A_1 > A_2 > A_3$and the best selection of the alternatives is.

CONCLUSION

The basic idea of IFS is the direct generalization of FST. Later, different developments have been extended, such as IVIFNs, GIVIFNs. Evaluation of arithmetic operation between GIVIFNs is a crucial issue. Arithmetic of conventional approaches produces counterintuitive results. This paper presented a novel technique to perform arithmetic operations on GIVTIFNs which efficiently overcame the shortcomings of conventional approach. The interesting part of the proposed approach is that it produces GIVTrIFN. Numerical illustrations also corroborate the same notion. The applicability and validation of the proposed approach have been shown by solving a multi-criteria group decision-making problem. It is observed that the proposed approach is efficient, simple, logical, technically sound and general enough for implementation. Researchers may apply this approach in any field where uncertainty/ imprecision can be handled using GIVTIFNs. Also, it is seen that both the conventional approach and present approach will be identical only when height of the input GIVTIFNs is same.

CONSENT FOR PUBLICATION

Not applicable.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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