Eigenfrequencies and Critical Speeds on a Beam due to Travelling Waves

T.G. Konstantakopoulos, I.G. Raftoyiannis and G.T. Michaltsos*

Department of Civil Engineering, National Technical University of Athens, Athens 15780, Greece

Abstract: A dynamic load, suddenly applied at a point of a beam, produces a local disturbance that propagates or diffuses to the rest of the beam. This propagation takes place with a speed depending on the material and geometrical characteristics of the beam. It has been demonstrated that an impulsive disturbance involving shear and moment will result in two wave types, one that propagates with the shear wave velocity and a second that propagates with a moment-wave velocity. It is observed that tampering with the cross-section of the beam may result to equal shear wave and moment-wave velocities and the two types of disturbances will travel together along with additionally interfering shear waves from beam's ends reflections. In this paper, the effect of the traveling waves on the dynamic characteristics of a beam is studied. A complete beam model is presented, which motion is governed by the Timoshenko equation. Two main cases are examined, namely a simply supported beam, and a beam resting on a Winkler-type elastic foundation. Analytical results are presented in graphical form showing the influence of the traveling waves on the eigenfrequencies and critical speeds of such a beam and useful conclusions are drawn.

INTRODUCTION

A dynamic load, suddenly applied at a point of a beam, produces a local disturbance that propagates or diffuses to the rest of the beam. This propagation takes place with a speed depending on the beams characteristics as well as the material characteristics from which the beam is made from.

The above simple fact is the basis for the study of the subject known as wave propagation.

This phenomenon is familiar to everyone in various forms such as the transmission of sound in air, or the spreading of ripples on water surface, the transmission of an earthquake waves, or the radio waves.

The physical basis of the propagation of a disturbance is finally caused by the interaction of the discrete atoms within a solid.

In solid and fluid mechanics, the medium is assumed to be continuous and thus, its physical characteristics such as density or modulus of elasticity are also considered to be continuous functions representing average values.

A disturbance to a mass particle is transmitted to the next particle by an imaginary intervening spring. Through this way, the disturbance is transmitted to a remote mass particle. The material characteristics such as density and elastic constants affect the speed of propagation. Increasing the elastic constant (or in other words the spring constant), the speed of propagation increases and vice-versa.

In a solid, two different actions are possible for wave propagation.

In the first case, the solid will transmit stresses (tensile or compressive), and motion of the particles will take place along the direction of the wave motion. The behavior is analogous to that of fluids, and solids of this type do not have any resistance to bending. A characteristic case of this behavior is the flexible string.

A solid though may also transmit shear stresses, and the motion of particles is transverse to the propagation direction. Such a behavior does not exist in fluids.

During their motion, the waves encounter the boundaries of the solid body and, evidently, an interaction between waves and boundaries will be inevitable. The behavior of waves in a solid differs from that in a fluid.

In structural mechanics, the motion of rods, beams, and plates can be described without needing to consider the propagation and reflection of waves within the structure. The so-called “strength of materials” theories may be derived on the basis of various assumptions regarding deformation.

Wave propagation in structures has been studied over a considerable period of time by a significant number of researchers.

Many solution techniques have been reported [1-4] for structures with specific geometrical characteristics and finite, periodic, or semi-infinite boundary conditions.

Among many frequency domain methods, the spectral element method [5] has been proved suitable for analysis of wave’s propagation in real engineering structures. The spectral element method uses the exact solution of the differential equations which govern the problem.

Flügge demonstrated [6,7] that an impulsive disturbance on a beam involving both shear and moment will result in two wave trains, i.e., one that propagates with the shear wave velocity $\upsilon_Q = \sqrt{k'G/\rho}$ and another that propagates with a

*Address correspondence to this author at the Department of Civil Engineering, National Technical University of Athens, Athens 15780, Greece; E-mail: michalts@central.ntua.gr

1874-1584/09 2009 Bentham Open

moment-wave velocity $\upsilon_M = \sqrt{E/\rho}$ along the beam (ρ being the mass per unit volume).

It has been observed that if the cross-section of the beam is such as to have $k'G = E$, then the velocities υ_Q and υ_M will be equal and the two types of disturbances will travel together. In general, these two types of disturbance will travel with different velocities. Moreover, since the reflections of moment waves from the beam's boundaries will result in additional shear waves trains, the moment and shear interfering waves will soon create a very complicated situation.

For studying the aforementioned very interesting problems, numerous mechanical models of beams have been presented. In Fig. (**1**), three mechanical models of beams are presented, in which the main parameters are the lumped masses. These models were first proposed and studied by Schirmer [8]. As the lumping of the masses becomes smaller and approaches in size the uniformly distributed mass and restoration elements of the beam, the wave travel velocities in the models will approach the limiting values υ_Q and υ_M.

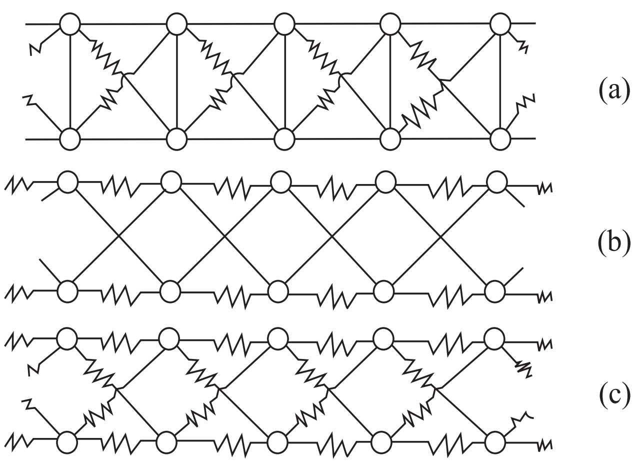


Fig. (1). Lumped mass and spring models with hinged joints for the wave travel properties: (**a**) Shear stiffness, and translation inertia only, (**b**) Flexural stiffness, and translation inertia only, and (**c**) Flexural and shear stiffness as well as translation and rotatory inertias.

In this paper, the effect of the traveling waves on the dynamic characteristics of a beam is studied.

For this study, the complete model (see Fig. **1c**) is used, the motion of which is governed by the Timoshenko equation.

After expressing the Timoshenko equation in its complete form and evaluating the contribution of each term in the equation (and subsequently, the necessity of keeping or ignoring each term), two main cases are examined, namely: the single-span simply supported beam, and the finite beam on a Winkler-type elastic foundation.

Graphical results showing the influence of the traveling waves on the eigenfrequencies and critical speeds of a beam are presented and useful conclusions are gathered.

MATHEMATICAL FORMULATION

Let us consider the one-span simply supported beam of Fig. (**2a**), subjected to the distributed load p(x,t).

After its deformation, the infinitesimal part dx of the beam takes firstly the position abcd (caused by the bending of the beam) and finally the position ABCD because of the shear influence.

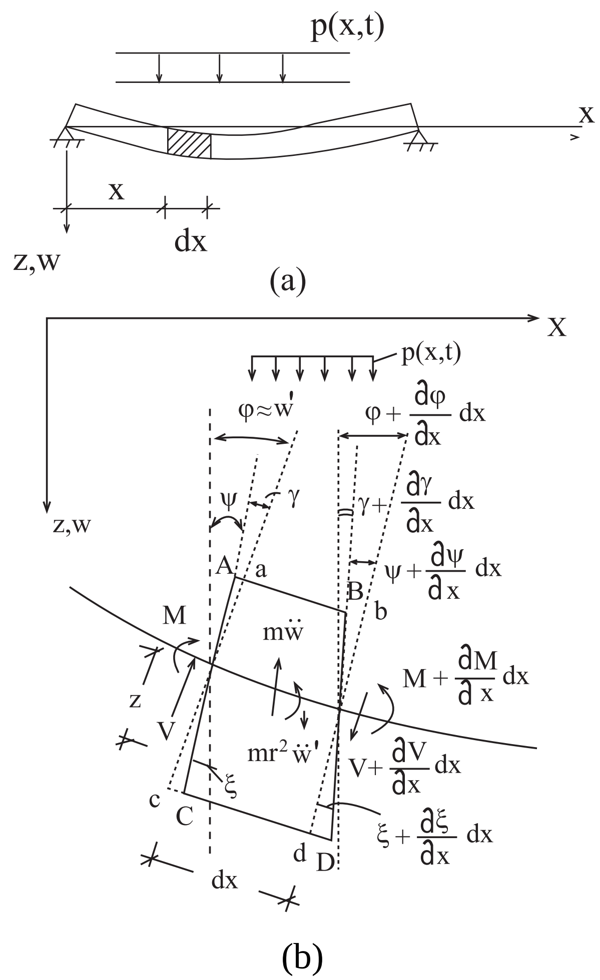


Fig. (2). The deformed state of the infinitesimal part dx.

The final deformation ξ of a fiber in a distance z from the central axis is:

$$\xi(x,t) = -\psi(x,t) \cdot z \tag{1}$$

because, as the coordinate x increases the angle ψ decreases.

On the other hand we have:

$$\varepsilon_x = \frac{d\xi}{dx} = -\frac{\partial\psi(x,t)}{\partial x} \cdot z \tag{2}$$

and finally:

$$\sigma_x = E \cdot \varepsilon_x = -E \cdot \frac{\partial\psi(x,t)}{\partial x} \cdot z \tag{3}$$

Equilibrium of moments gives:

$$\frac{\partial M}{\partial x} = V - \frac{dM_\alpha}{dx} \tag{4}$$

where M_α is the inertia moment produced by the rotation of the infinitesimal part dx.

Equilibrium of forces gives:

$$\frac{\partial V}{\partial x} = m \cdot \ddot{w} - p(x,t) \tag{5}$$

where, $m \cdot \ddot{w}$ is the inertia forces because of the movement of dx in parallel to Oz axis.

1. For the moment M we find the following expression:

$$M = \int_A \sigma_x z\, dA = -\int_A E \cdot \frac{\partial \psi}{\partial x} \cdot z^2 dA = -E \cdot J_y \frac{\partial \psi(x,t)}{\partial x} \tag{6}$$

2. For the shear force V, we have:

$$V = \tau \cdot A = k' \cdot G \cdot \gamma \cdot A \tag{7}$$

where k' is the corrective factor of Timoshenko, expressing the non-uniform distribution of shear stresses along the height of the cross-section.

From Fig. (**2b**) we have the relation:

$$\frac{\partial w}{\partial x} = \psi + \gamma \tag{8}$$

and Eq. (7) becomes:

$$V = k' \cdot G \cdot A \cdot \left(\frac{\partial w}{\partial x} - \psi \right) \tag{9}$$

3. The inertia moment M_α is produced by the inertia forces of the fibers dx in parallel to Ox axis in a distance z from the neutral axis:

$$F_\alpha = -\ddot{\xi} \cdot \rho \cdot dA \cdot dx = \rho \cdot z \cdot \ddot{\psi} \cdot dA \cdot dx$$

where ρ is the mass per unit volume. Therefore the produced moment will be:

$$\begin{aligned} dM_\alpha &= \int_A \rho \cdot z^2 \cdot \ddot{\psi} \cdot dA \cdot dx = \\ &= \rho \cdot \ddot{\psi} \cdot dx \cdot \int_A z^2 dA = \rho \cdot \ddot{\psi} \cdot J_y dx \end{aligned} \tag{10}$$

So, Eq. (4) becomes:

$$\frac{\partial M}{\partial x} = V - \rho \cdot J_y \cdot \ddot{\psi} \tag{11}$$

Introducing M and V from Eqs. (6) and (9) respectively into Eqs. (11) and (5) we obtain the following differential system:

$$\left.\begin{aligned} & E J_y \frac{\partial^2 \psi}{\partial x^2} + k' A G \left(\frac{\partial w}{\partial x} - \psi \right) - \rho J_y \frac{\partial^2 \psi}{\partial t^2} = 0 \\ & k' A G \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) - m \ddot{w} + p = 0 \end{aligned}\right\} \tag{12}$$

From Eq. (12b), we determine $\frac{\partial \psi}{\partial x}, \frac{\partial^2 \psi}{\partial x^2}, \frac{\partial^3 \psi}{\partial x^3}, \frac{\partial^3 \psi}{\partial x \partial t^2}$ that we introduce into the equation getting after differentiation of Eq. (12a). So, we come to the following equation:

$$\begin{aligned} & E J_y \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - \left(\frac{E J_y m}{k' A G} + \rho J_y \right) \frac{\partial^4 w}{\partial^2 x \partial^2 t} + \\ & \qquad + \frac{\rho J_y m}{k' A G} \frac{\partial^4 w}{\partial t^4} = p + \frac{\rho J_y}{k' A G} \ddot{p} - \frac{E J_y}{k' A G} p'' \end{aligned} \tag{13}$$

The equation of the free vibrating beam is:

$$\begin{aligned} & \frac{\partial^4 w}{\partial x^4} + \frac{m}{E J_y} \frac{\partial^2 w}{\partial t^2} - \left(\frac{m}{k' A G} + \frac{\rho}{E} \right) \frac{\partial^4 w}{\partial^2 x \partial^2 t} + \\ & \qquad + \frac{\rho m}{k' A G E} \frac{\partial^4 w}{\partial t^4} = 0 \end{aligned} \tag{14}$$

Assuming a solution of the form:

$$w(x,t) = X(x) \cdot e^{i \omega t} \tag{15}$$

we get the following equation:

$$X'''' + \omega^2 \left(\frac{m}{k' A G} + \frac{\rho}{E} \right) X'' + \left(\frac{\rho m \omega^4}{k' A G E} - \frac{m \omega^2}{E J_y} \right) X = 0$$

and since it is m=ρA, the above equation becomes:

$$X'''' + \omega^2 \left(\frac{\rho}{k' G} + \frac{\rho}{E} \right) X'' + \left(\frac{\rho^2 \omega^4}{k' E G} - \frac{m \omega^2}{E J_y} \right) X = 0 \tag{16}$$

The characteristic equation of the above differential one is:

$$\mu^4 + \omega^2 \left(\frac{\rho}{k' G} + \frac{\rho}{E} \right) \mu^2 + \omega^2 \left(\frac{\rho^2 \omega^2}{k' E G} - \frac{m}{E J_y} \right) = 0$$

The roots of the above equation are:

$$\mu_{1,2} = \pm \lambda \quad \text{and} \quad \mu_{3,4} = \pm i \bar{\lambda} \tag{17}$$

where:

$$\lambda^2 = \frac{\omega}{2} \left[-\left(\frac{\rho}{k' G} + \frac{\rho}{E} \right) \omega + \sqrt{\left(\frac{\rho}{k' G} + \frac{\rho}{E} \right)^2 \omega^2 + 4 \left(\frac{\rho^2 \omega^2}{k' E G} + \frac{m}{E J_y} \right)} \right]$$

$$\bar{\lambda}^2 = \frac{\omega}{2} \left[\left(\frac{\rho}{k' G} + \frac{\rho}{E} \right) \omega + \sqrt{\left(\frac{\rho}{k' G} + \frac{\rho}{E} \right)^2 \omega^2 + 4 \left(\frac{\rho^2 \omega^2}{k' E G} + \frac{m}{E J_y} \right)} \right]$$

Hence, the solution of Eq. (16) is given by the following relation:

$$X(x) = c_1 \sin \bar{\lambda} x + c_2 \cos \bar{\lambda} x + c_3 \sinh \lambda x + c_4 \cosh \lambda x \tag{18}$$

For the case of a simply supported one-span beam, the following boundary conditions are valid:

$$X(0) = X(L) = X''(0) = X''(L) = 0 \tag{19}$$

Using the above conditions we arrive at the system:

$$\left.\begin{aligned} & c_2 + c_4 = 0 \\ & -c_2 \bar{\lambda}^2 + c_4 \lambda^2 = 0 \\ & c_1 \sin \bar{\lambda} L + c_2 \cos \bar{\lambda} L + c_3 \sinh \lambda L + c_4 \cosh \lambda L = 0 \\ & -c_1 \bar{\lambda}^2 \sin \bar{\lambda} L - c_2 \bar{\lambda}^2 \cos \bar{\lambda} L + c_3 \lambda^2 \sinh \lambda L + c_4 \lambda^2 \cosh \lambda L = 0 \end{aligned}\right\}$$

In order for the system to have not only trivial solutions, the following condition must be fulfilled:

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -\bar{\lambda}^2 & 0 & \lambda^2 \\ \sin\bar{\lambda}L & \cos\bar{\lambda}L & \sinh\bar{\lambda}L & \cosh\bar{\lambda}L \\ -\bar{\lambda}^2\sin\bar{\lambda}L & -\bar{\lambda}^2\cos\bar{\lambda}L & \bar{\lambda}^2\sinh\bar{\lambda}L & \bar{\lambda}^2\cosh\bar{\lambda}L \end{vmatrix} = 0$$

which simplifies to the following eigenfrequencies equation:

$$\left(\bar{\lambda}^2+\lambda^2\right)^2\cdot\sin\bar{\lambda}L\cdot\sinh\lambda L = 0$$

and since it is $\left(\bar{\lambda}^2+\lambda^2\right)^2\cdot\sinh\lambda L \neq 0$, it will be:

$$\sin\bar{\lambda}L = 0 \tag{20}$$

The above equation has the solution:

$\bar{\lambda}L = n\pi,\quad n = 1,2,3,\cdots$ or $\bar{\lambda} = \dfrac{n\pi}{L}$ and taking into account Eq. (17), we have:

$$\bar{\lambda}^2 = \frac{n^2\pi^2}{L^2} = \frac{\omega_n^2}{2}\left(\frac{\rho}{k'G}+\frac{\rho}{E}\right) + \frac{\omega_n}{2}\sqrt{\left(\frac{\rho}{k'G}+\frac{\rho}{E}\right)^2\omega_n^2 + 4\left(\frac{\rho^2\omega_n^2}{k'EG}+\frac{m}{EJ_y}\right)}$$

or finally:

$$\frac{\rho^2}{k'GE}\omega_n^4 + \left[\frac{n^2\pi^2}{L^2}\left(\frac{\rho}{k'G}+\frac{\rho}{E}\right)+\frac{m}{EJ_y}\right]\omega_n^2 - \frac{n^4\pi^4}{L^4} = 0 \tag{21}$$

Flügge [6], demonstrated in 1942, that an impulsive disturbance involving shear and moment will result in two wave trains, one that propagates with the moment-wave velocity:

$$\upsilon_M = \sqrt{\frac{E}{\rho}} \tag{22a}$$

and another that propagates with the shear-wave velocity:

$$\upsilon_Q = \sqrt{\frac{k'G}{\rho}} \tag{22b}$$

On the other hand we have:

$$\frac{m}{EJ_y} = \frac{\rho A}{EJ_y} = \frac{1}{\upsilon_M^2}\cdot\frac{1}{r_y^2} = \frac{1}{L^2}\cdot\frac{\lambda^2}{\upsilon_M^2} \tag{22c}$$

where r_y is the radius of gyration in parallel of Oy axis and λ the slenderness of the beam.

Introducing Eqs. (22a,b,c) into Eq. (21), we obtain:

$$\frac{1}{\upsilon_M^2\upsilon_Q^2}\omega_n^4 + \frac{1}{L^2}\left[n^2\pi^2\left(\frac{1}{\upsilon_M^2}+\frac{1}{\upsilon_Q^2}\right)+\frac{\lambda^2}{\upsilon_M^2}\right]\omega_n^2 - \frac{n^4\pi^4}{L^4} = 0 \tag{23}$$

and after some manipulations:

$$\omega_n^4 + \frac{(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2}{L^2}\omega_n^2 - \frac{n^4\pi^4\upsilon_M^2\upsilon_Q^2}{L^4} = 0 \tag{24}$$

The above equation has the solution:

$$\omega_n^2 = \frac{1}{2L^2}\left\{-[(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2] + \sqrt{[(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2]^2 + 4n^4\pi^4\upsilon_M^2\upsilon_Q^2}\right\} \tag{25}$$

Neglecting the term $(\rho^4\omega_n^4/k'EG)$ as being very small compared to the other terms, we arrive at the following approximate relation:

$$\omega_n^2 = \frac{n^4\pi^4}{L^2}\cdot\frac{\upsilon_M^2\upsilon_Q^2}{(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2} \tag{26}$$

THE EIGENFREQUENCIES' CHANGE

The expression giving the eigenfrequencies spectrum by the classical theory is:

$$\omega_{on}^2 = \frac{n^4\pi^4 EJ_y}{mL^4} = \frac{n^4\pi^4}{L^2}\cdot\frac{EJ_y}{\rho A}\cdot\frac{1}{L^2} = \frac{n^4\pi^4}{L^2}\cdot\upsilon_m^2\cdot\frac{r_y^2}{L^2}$$

or finally:

$$\omega_{on}^2 = \frac{n^4\pi^4}{L^2}\cdot\frac{\upsilon_M^2}{\lambda^2} \tag{27}$$

The following ratio is of great interest:

$$\left(\frac{\omega_n}{\omega_{on}}\right)^2 = \frac{\lambda^2\upsilon_Q^2}{(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2} \tag{28}$$

THE CRITICAL SPEEDS' CHANGE

A factor of great interest for the bridges, is the so called "critical speeds". As critical speed, we define the speed that a vehicle needs for crossing a beam in time equal to the corresponding half-period of the beam. Thus for the n^{th} period the critical speed will be:

$$\upsilon_{ncr} = \frac{2L}{T_n} = \frac{L\omega_n}{\pi} \tag{29}$$

Introducing ω_n from Eq. (26) into Eq. (29) we get:

$$\upsilon_{ncr} = \frac{n^2\pi\upsilon_M\upsilon_Q}{\sqrt{(n^2\pi^2+\lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2}} \tag{30}$$

Calling υ_{oncr} the critical speed found by the classical theory, we finally obtain:

$$\frac{\upsilon_{ncr}}{\upsilon_{oncr}} = \frac{\omega_n}{\omega_{on}} \tag{31}$$

From the spectrum of the above critical speeds given by Eq. (30), the first one υ_{1cr} is the most interesting, which from now on we shall call υ_{cr}.

FINITE BEAM ON ELASTIC FOUNDATION

Let us consider now the beam AB in Fig. (**3**), which is based on a Winkler-type elastic foundation.

According to the classical theory, the force per unit length of the beam reacting to the external loading is:

$$P = k \cdot w \tag{32}$$

where k is the so-called Winkler factor.

We shall proceed determining this factor.

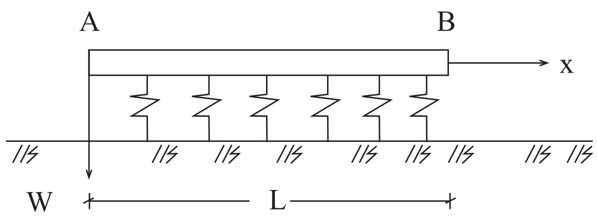


Fig. (3). Finite beam on elastic foundation.

The reduced deformations of an earthen infinitesimal part $dx \cdot dy \cdot dz$ which is under three-axial loading are:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_s}[\sigma_x - \nu_s(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E_s}[\sigma_y - \nu_s(\sigma_z + \sigma_x)] \\ \varepsilon_z &= \frac{1}{E_s}[\sigma_z - \nu_s(\sigma_x + \sigma_y)] \end{aligned}\right\} \tag{33}$$

where E_s is the soil modulus of elasticity and ν_s the Poisson's ratio (with values from 0.2 to 0.4).

If the lateral inflation is restrained (i.e., $\varepsilon_x = \varepsilon_y = 0$), it will be:

$$\sigma_x = \sigma_y = \frac{\nu_s \sigma_z}{1-\nu_s} \tag{34}$$

and introducing σ_x, σ_y, into the third of Eqs. (33):

$$\varepsilon_z = \frac{\sigma_z}{E_s} \cdot \frac{(1+\nu_s)(1-2\nu_s)}{(1-\nu_s)} \tag{35}$$

It is valid that: $\varepsilon_z = \frac{\Delta dz}{dz}$ and thus will be: $\Delta dz = \varepsilon_z dz = \frac{\sigma_z}{E_s} \cdot \frac{(1+\nu_s)(1-2\nu_s)}{(1-\nu_s)} dz$.

Assuming additionally that E_s and σ_z remain constant along the depth H (see Fig. **4**) we will have:

$$w = \int_0^H \Delta dz = \frac{\sigma_z}{E_s} \cdot \frac{(1+\nu_s)(1-2\nu_s)}{(1-\nu_s)}$$

and the force reacting per unit length of the beam will be:

$$P = \sigma_z b = \frac{E_s}{H} \cdot \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \cdot b \cdot w \tag{36}$$

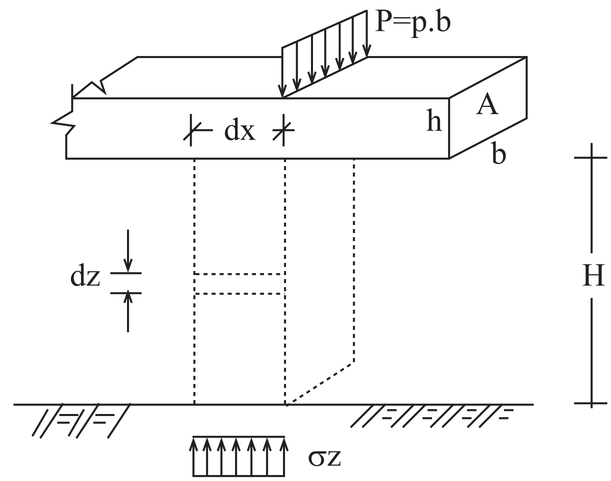


Fig. (4). Stress distribution under a beam of width b.

We next set:

$$\left.\begin{aligned} \rho_s &= \zeta_1 \rho \\ h &= \zeta_2 L \\ H &= \zeta_3 L \\ \xi &= \frac{\zeta_1}{\zeta_2 \zeta_3} \cdot \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \end{aligned}\right\} \tag{37}$$

where ρ_s is the mass per unit volume of the earth, and h is the height of the equivalent orthogonal cross-section that has the same area with the one of the beam AB.

Thus Eq. (36) becomes:

$$P = \frac{E_s}{\rho_s} \cdot \rho_s \cdot \frac{A}{h \cdot H} \cdot \frac{(1-\nu_s)}{(1+\nu_s)(1-2\nu_s)} \cdot w$$

or finally:

$$P = \xi \cdot \frac{E_s}{\rho_s} \cdot \frac{\rho \cdot A}{L^2} \cdot w$$

and thus:

$$k = \xi \cdot \frac{E_s}{\rho_s} \cdot \frac{\rho \cdot A}{L^2} \tag{38}$$

Introducing the reaction of the foundation into Eq. (13) we get:

$$\begin{aligned} E J_y \frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(\frac{EJ_y m}{k'AG} + \rho J_y\right)\frac{\partial^4 w}{\partial^2 x \partial^2 t} + \\ + \xi \frac{E_s}{\rho_s} \cdot \frac{\rho A}{L^2 EJ_y} w = p \end{aligned} \tag{39}$$

where the higher order terms have been neglected.

The equation of the free vibrating beam is:

$$\frac{\partial^4 w}{\partial x^4} + \frac{m}{EJ_y}\frac{\partial^2 w}{\partial t^2} - \left(\frac{m}{k'AG} + \frac{\rho}{E}\right)\frac{\partial^4 w}{\partial^2 x \partial^2 t} + \xi \frac{E_s}{\rho_s} \cdot \frac{\rho A}{L^2 EJ_y} w = 0 \tag{40}$$

Following a similar process, like the one for Eq. (14), we finally obtain the equation:

$$\frac{n^2\pi^2}{L^2} = \frac{\omega_n^2}{2}\left(\frac{\rho}{k'G} + \frac{\rho}{E}\right) + \\ + \frac{\omega_n}{2}\sqrt{\left(\frac{\rho}{k'G} + \frac{\rho}{E}\right)^2 \omega_n^2 + 4\left(\frac{m}{EJ_y} - \xi\frac{E_s}{\rho_s}\cdot\frac{\rho A}{L^2 EJ_y \omega_n^2}\right)}$$

or

$$\left[\frac{n^2\pi^2}{L^2}\left(\frac{1}{\upsilon_Q^2} + \frac{1}{\upsilon_M^2}\right) + \frac{1}{L^2}\cdot\frac{\lambda^2}{\upsilon_M^2}\right]\omega_n^2 = \frac{n^4\pi^4}{L^4} + \frac{\xi}{L^4}\cdot\frac{\upsilon_s^2}{\upsilon_M^2}\cdot\lambda^2 \text{ and}$$

finally:

$$\omega_n^2 = \frac{(n^4\pi^4\upsilon_M^2 + \xi\lambda^2\upsilon_s^2)\cdot\upsilon_Q^2}{L^2[(n^2\pi^2 + \lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2]} \quad (41a)$$

and

$$\upsilon_s = \sqrt{\frac{E_s}{\rho_s}} \quad (41b)$$

where υ_s is the speed of the sound in the earth (depended on the characteristics of the ground), and υ_M, υ_Q are given by Eqs. (22a,b).

Thus the critical speed will be:

$$\upsilon_{scr} = \frac{2L}{T_1} = \frac{L\omega_1}{\pi} = \frac{1}{\pi}\cdot\sqrt{\frac{(n^4\pi^4\upsilon_M^2 + \xi\lambda^2\upsilon_s^2)\cdot\upsilon_Q^2}{(n^2\pi^2 + \lambda^2)\upsilon_Q^2 + n^2\pi^2\upsilon_M^2}} \quad (42)$$

From Eqs. (30) and (42), arises the following interesting ratio between the υ_{cr} of the one-span simply supported beam and the υ_{scr} of the same beam but laid on an elastic foundation:

$$\frac{\upsilon_{bcr}}{\upsilon_{scr}} = \frac{n^2\pi^2\upsilon_M}{\sqrt{n^4\pi^4\upsilon_M^2 + \xi\lambda^2\upsilon_s^2}} \quad (43)$$

NUMERICAL RESULTS AND DISCUSSION

The materials used in engineering structures have modulus of elasticity waving from $E=0.4\text{x}10^{10}$ dN/m^2 to $2.1\text{x}10^{10}$ dN/m^2 , while their mass per unit volume starts from $\rho=200$ kg/m^3 to 800 kg/m^3.

Assuming that the Timoshenko coefficient $k'=0.2$ to 1.00 we get the following values for the speed of the traveling waves:

a. Speed of the moment waves: 3.5 km/sec to 5.5 km/sec, and

b. Speed of the shear waves: 0.4 km/sec to 3.2 km/sec

For different soil qualities we have:

a. Modulus of elasticity: $E_s=0.15\text{x}10^6$ dN/m^2 to $15\text{x}10^6$ dN/m^2

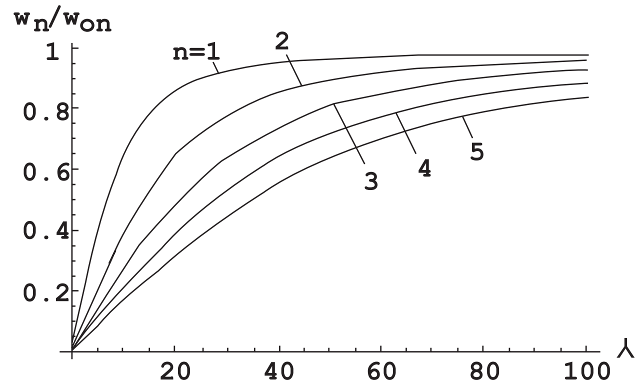
b. Mass per unit volume: $\rho_s=140$ kg/m^3 to 300 kg/m^3, and thus we can determine the waves speed: $\upsilon_s = 0.025\,\text{km/sec}$ to $0.350\,\text{km/sec}$.

The One-Span Simply Supported Beam

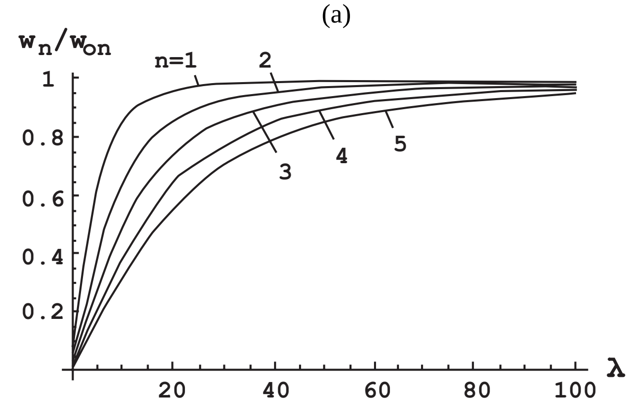
Let us consider firstly a beam of steel in which the speed of moment waves is $\upsilon_M = 5.172\,\text{km/sec}$. For values of Timoshenko's factor $k' = 0.2$ to 1.0 we get speeds of shear waves from υ_Q =1.42 to 3.192 km/sec respectively.

In the diagram of Fig. (**5a**) it is plotted the ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 5.172$ km/sec, $\upsilon_Q = 1.42$ km/sec , and n=1, 2, 3, 4, 5.

In the diagram of Fig. (**5b**) it is plotted the ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 5.172$ km/sec, $\upsilon_Q = 3.192$ km/sec , and n=1, 2, 3, 4, 5.



(a)



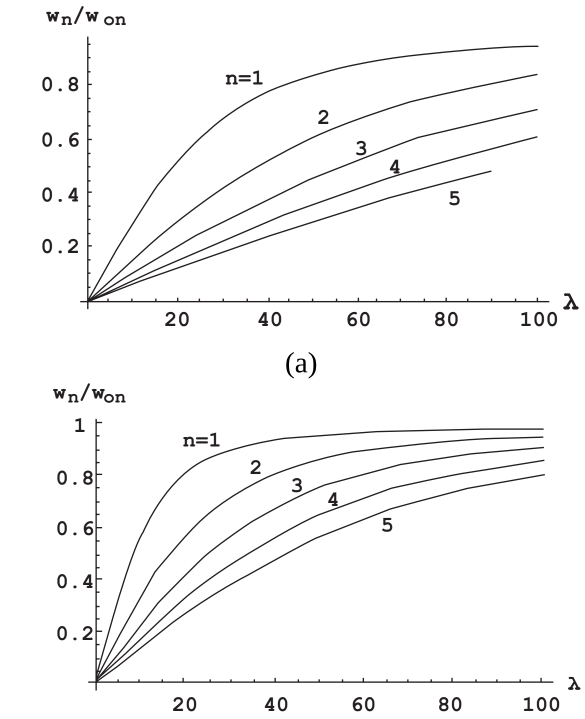
(b)

Fig. (5). a. Ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 5.172\,\text{km/sec}$ and $\upsilon_Q = 1.42\,\text{km/sec}$.

b. Ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 5.172\,\text{km/sec}$ and $\upsilon_Q = 3.19\,\text{km/sec}$.

We next consider a beam made from concrete, in which the speed of moment waves is $\upsilon_M = 4.082\,\text{km/sec}$. For values of Timoshenko's factor from $k' = 0.2$ to 1.0 the speeds of shear waves vary from υ_Q =0,04 to 0,913 km/sec respectively.

In Fig. (**6**), we see the same as above diagrams corresponding to a concrete beam.

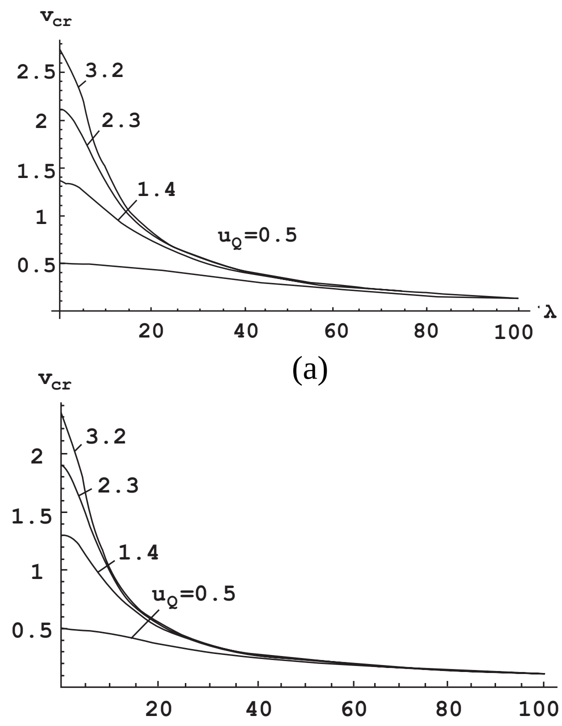


(a)

(b)

Fig. (6). (**a**) The ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 4.082\,\text{km/sec}$ and $\upsilon_Q = 0.40\,\text{km/sec}$.

(**b**) The ratio ω_n/ω_{on} versus λ, for $\upsilon_M = 4.082\,\text{km/sec}$ and $\upsilon_Q = 0.913\,\text{km/sec}$



(a)

(b)

Fig. (7). (**a**) υ_{cr} versus λ for υ_M=5,5 km/sec and various values of υ_Q.

(**b**) υ_{cr} versus λ for υ_M=3,5 km/sec and various values of υ_Q.

In Fig. (**7a**), the diagrams of the critical speed υ_{cr} of the beam versus the slenderness λ for $\upsilon_M = 5.5\,\text{km/sec}$ and various values of υ_Q are drawn.

In Fig. (**7b**), the same as in Fig. (**7a**) diagrams are drawn for $\upsilon_M = 3.5\,\text{km/sec}$ and various values of υ_Q.

Finite Beam on Elastic Foundation

The commonly used material for such a beam is concrete.

For concrete, it is υ_M=4,082 km/sec while the values of υ_Q vary from 0,4 km/sec up to 0,913 km/sec for k'=0,2 to 1,00, respectively. For soil, the corresponding values for the speed of waves are υ_s=0.025 km/sec up to 0.35 km/sec.

Considering a beam with length L=20 m, b=2.00 m and h=0.2 up to 1.0 m, and H varying from 10 to 30 m, the coefficients ζ_1, ζ_2, and ζ_3 take the following values:

$$\left.\begin{array}{l}\zeta_1 = 0.55 \quad \text{to} \quad 1.25\\ \zeta_2 = 0.01 \quad \text{to} \quad 0.05\\ \zeta_3 = 0.20 \quad \text{to} \quad 0.40\end{array}\right\}$$

and ξ waves from ~4 to ~940.

In Fig. (**8**), on can see the diagrams $\upsilon_{bcr}/\upsilon_{scr}$ versus υ_s for different λ, and for ξ=4, 100, 300 and 800.

CONCLUSIONS

From the results obtained and the diagrams shown in the preceding analysis, we can draw the following conclusions:

Single-Span Simply Supported Beam

The use of the exact theory gives eigenfrequencies, which are significantly smaller than the ones obtained using the classical simplified theory.

a. For steel beams and for speed of the shear waves $\upsilon_Q = 1.42\,\text{km/sec}$, this effect is significant and amounts: for λ=20, and n=5 to 80%, for λ=20, and n=1 to 18%, for λ=40, and n=5 to 55%, and for λ=40, and n=1 to 7%. For λ>40, the effect is still significant for the eigenfrequencies and takes its minima (~1%) for λ=100 and n=1.

For speed of the shear waves $\upsilon_Q = 3.19\,\text{km/sec}$, the above values decrease as follows: for λ=20, and n=5 the influence is 55%, for λ=20, and n=1 it is 5%, while for λ=100 it is 5% (for n=5) and 0.5% (for n=1), respectively.

For beams made from concrete, the effect is more significant, i.e., for $\upsilon_Q = 0.40\,\text{km/sec}$, it is 55% for λ =20, and n=1,90% for λ =20, and n=5,20% for λ=40, and n=1,75% for λ =40, and n=5,10% for λ=80, and n=1,55% for λ =80, and n=5,5% for λ=100, and n=1,45% for λ =100, and n=5, while for $\upsilon_Q = 0.913\,\text{km/sec}$, it is 20% for λ=20, and n=1,85% for λ=20, and n=5,10% for λ=40, and n=1), 55% for λ=40, and n=5,5% for λ=80, and n=1, 25% for λ=80, and n=5,2% for λ=100, and n=1, 15% for λ=100, and n=5.

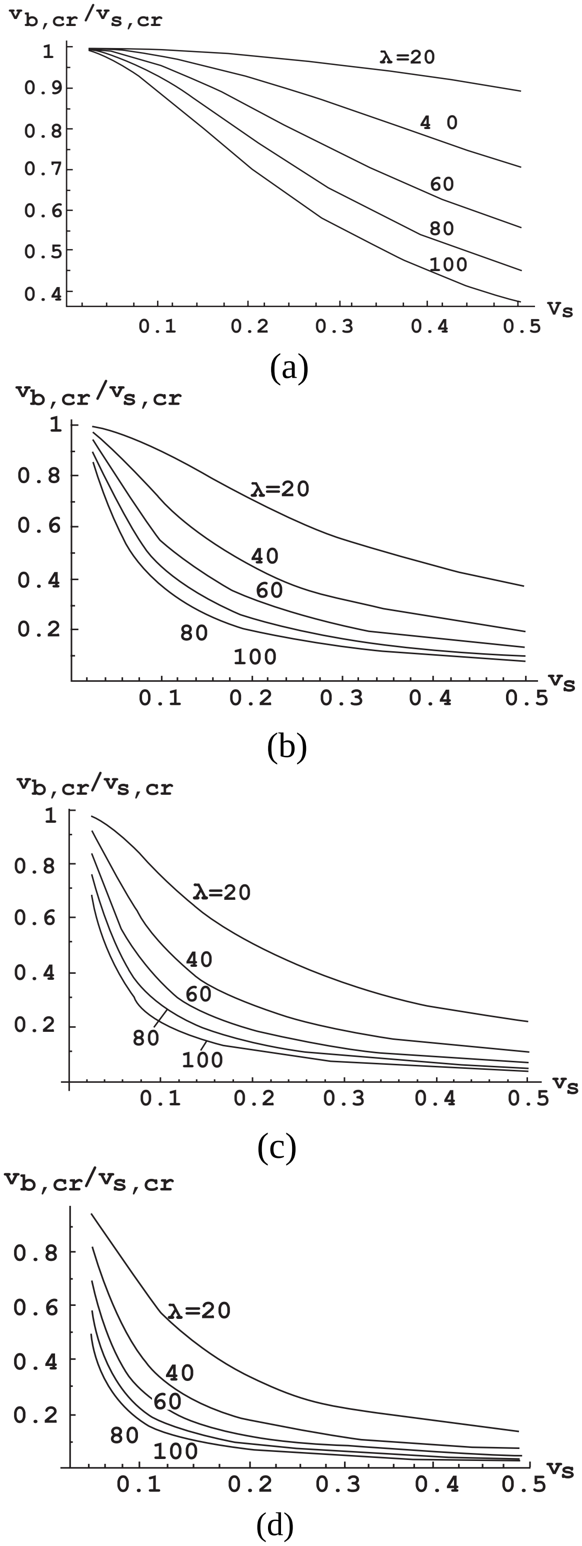


Fig. (8). Velocity ratio $\upsilon_{bcr} / \upsilon_{scr}$ vs soil speed υ_s .

The critical speeds are especially affected by the speed of the shear wave υ_Q for slenderness λ<40. The speed υ_Q (strongly depended on the cross-sectional shape) is possible to result (for the same slenderness) differences amounted to about 30% (for λ=40) to 100% (for λ=30), etc.

Finite Beam on Elastic Foundation

The problem of how the traveling waves affect the critical speeds is very complicated in this case.

The characteristics of the soil that, according to Eqs. (42) and (43), affect the critical speed are two: the factor ξ and the speed of sound in the soil υ_s. Higher values of ξ correspond to a more coherent soil.

The critical speeds of a beam resting on an elastic foundation, compared to the ones of the corresponding one-span simply supported beam with the same characteristics, are significantly higher.

For different values of the slenderness λ, we observe that the primary critical speed of a finite beam on elastic foundation is much higher than the one of the corresponding simply supported beam. Only for stiff beams and especially soft soils (see Fig. **8a**, for ξ=4 and λ=20) there is an insignificant difference.

For λ>20 and usual soil types, the primary critical speed is higher from 2 times (Fig. **8a**) up to 30 times (Fig. **8d**).

REFERENCES

[1] Bathe KJ. Finite element procedures in engineering analysis. Prentice-Hall: Englewood Cliffs, NJ 1982.
[2] Redwood M. Mechanical wave guides. Pergamon Press: New York, NY 1960.
[3] Chenny YK. Finite strip method in structural analysis. Pergamon Press: New York, NY 1997.
[4] Graff FK. Wave motion in elastic solids. New York, NY; Dover Publication 1991.
[5] Doyle JF. Aspectrally formulation finite element for longitudinal wave propagation. Int J Anal Exp Modal Anal 1988; 3: 1-5.
[6] Flügge W. Die ausbreitung von biegungswellen in stäben. Z Angew Math Mech 1942; 22: 312-18.
[7] Flügge W. Viscoelasticity. 2nd ed. Springer Verlag: Berlin 1975.
[8] Shirmer H. Über biegewellen in stäben. Archiv Appl Mech 1952; 20: 247-257.

Received: March 31, 2008 Revised: June 15, 2008 Accepted: July 22, 2008