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Numerical Calibration of Turbulent Compressible Models Using Rapid Distortion Theory

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Abstract: The study of compressible homogeneous shear flows has been largely analyzed in literature. Predictions for these flows require an improvement of existing turbulence models. Rapid-distortion-theory (RDT) has largely showed its relevance and its utility to identify compressibility effects which allow us to analyze performances of compressible turbulence models from the existing literature. This paper describes an evaluation of two models of homogeneous compressible turbulence by comparison with RDT results in compressible regime of flow. This model evaluation stays in the field of RDT validity. Precisely, the manuscript deals with two pressure-strain correlation models developed by Hamba and Marzougui *et al*. Linear parts of these models do not correctly predict compressible turbulence. A numerical calibration of coefficients included in these models provides broadly a net improvement with RDT results.

Keywords: Compressible homogeneous turbulence, rapid-distortion analysis, models.

1. INTRODUCTION

This paper is mainly in the area of a well known rapid-distortion-theory (RDT). It's important to insist on the essential advantage of this approach. Indeed, if homogeneous turbulence, initially isotropic, is submitted to mean velocity gradient for small values of time (small values of non-dimensional time *St* in the shear case) linear effects are dominant at first and non-linear effects take place after wards. Constitutive equations of RDT are the ''exact'' linearized equations of turbulent motion and there is no need to any model to close this set of equations. For small values of time, the numerical solutions obtained in the frame work of RDT are suitable to assess performances of linear contribution models of turbulence and the RDT is an efficient and accurate method for testing linear turbulence models.

The prediction of compressible turbulent flows by rapid-distortion-theory (RDT) provides useful results which can be used to test turbulence models from the existing literature. In this work, RDT code developed by authors [1] solves linearized equations for compressible homogeneous shear flows. It has been validated [1] for various values of gradient Mach number M_g which is the major compressibility parameter. Validation has been performed by comparing RDT results with direct numerical simulation (DNS) and RDT of Simone [2] related to the turbulent kinetic energy, the non-dimensional production term and the turbulent kinetic energy growth rate. It was found that this code is valid for small values of non-dimensional time *St* (until 2.5), where *S* is the shear rate; for high values of *St*, the non-linear effects are dominant so that the RDT is no longer relevant. In this paper, the RDT code is used to critically evaluate two compressible pressure-strain models Π_{ij} developed by Hamba [3] and Marzougui *et al*. [4]. Hamba [3] shows that the model constants values are not yet good enough to predict the pressure-strain accurately. Marzougui *et al.* [4] extended the incompressible Launder, Reece and Rodi (LRR) [5] model to compressible homogeneous shear flow for the pressure-strain correlation. Indeed, the standard LRR model in conjunction with dilatational terms proposed by Sarkar ([6, 7]) yields poor predictions for compressible homogeneous shear flow. The dilatational terms are much smaller to reflect the correct physics of compressibility. Thus, the concept of the growth rate of turbulent kinetic energy [8] allows Marzougui *et al*. [4] to construct a compressible correction to the LRR model. This correction is essentially related to the C_1, C_3 and C_4 coefficients which become, in compressible turbulence, function of the turbulent Mach number defined was earlier. The C_1 coefficient is associated with the non linear part of this model.

The paper is organized as follows. In Section 2, governing equations and numerical method to obtain RDT solutions are given. Turbulence models to be tested are described in Section 3. Initial parameters are investigated in Section 4. The numerical procedure and discussion are introduced in Section 5. Conclusion is given in Section 6.

2. RAPID-DISTORTION ANALYSIS FOR COMPRESSIBLE HOMOGENEOUS TURBULENCE

We focus on the formulation of the RDT analysis and the numerical method used to obtain RDT solutions. We retain the same RDT equations as adopted by Simone, Coleman & Cambon [2].

The linearized equations of continuity, momentum and entropy controlling the fluctuating of velocity u_i and pressure p lead to general RDT equations:

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$$\left(\frac{\dot{p}}{\gamma \overline{P}}\right) = -u_{i,i}, \tag{1}$$

$$\dot{u}_i + u_j \frac{\partial \overline{U_i}}{\partial x_j} = -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial x_i} + \frac{\nu}{3}\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \frac{\nu}{3}\frac{\partial^2 u_i}{\partial {x_j}^2}, \tag{2}$$

where $\overline{\rho}$ is the mean density and ν is the kinematic viscosity.

The dot superscript denotes a substantial derivative along the mean flow trajectories related to mean field velocity $\overline{U_i}$.

The Fourier transform (denoted here by the symbol "^") of the various terms in equations (1) and (2) leads to the following equations expressed in the spectral space, in the case of turbulent shear flow:

$$\frac{\dot{\hat{p}}}{\gamma \overline{P}} = -\hat{u}_{i,i}, \tag{3}$$

$$\dot{\hat{u}}_i + \lambda_{ij}\,\hat{u}_j + \frac{\nu}{3}k_i\,k_j\,\hat{u}_j + \frac{\nu}{3}k^2\,\hat{u}_i = -Ik_i\frac{\hat{p}}{\overline{\rho}}, \tag{4}$$

where λ_{ij} is the mean velocity gradient defined by:

$$\lambda_{ij} = S\delta_{i1}\delta_{j2}. \tag{5}$$

In Fourier space, velocity field is split into solenoidal and dilatational contributions (Helmholtz decomposition of the velocity field) by adopting an orthonormal local frame of reference (Cambon *et al.* [9]).

Using Green's functions, resolution of the differential system involving these functions is carried out using a fourth-order Runge-Kutta numerical integration scheme (Riahi *et al.* [1]).

3. TURBULENCE MODELS

The modeling of the pressure-strain correlation Π_{ij} is the subject of significant literature. For compressible turbulence, the modeling of this correlation articulates on simple extension of incompressible models as mentioned in Hamba [3] and Marzougui *et al.* [4] papers. Linear parts of models proposed by these authors are summarized in Table **1**.

In the Hamba model, the non-dimensional parameter $\chi_p = \dfrac{\overline{p^2}}{2\overline{\rho}^2 c^2 q^2}$ is a normalized pressure variance that corresponds to the ratio of potential to kinetic energy for weak fluctuations. c is the mean speed of sound and $q^2/2$ the turbulent kinetic energy. Constants model are set to C_{ps2}=0.6, C_{ps3}, 0.15, C_{ps4}=0.15 and C_{ps5}=4. S_{ij} is the mean rate of strain tensor ($S_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})$). $P_{ij} = -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}$ and $d_{ij} = -\overline{u_i u_k}\frac{\partial U_k}{\partial x_j} - \overline{u_j u_k}\frac{\partial U_k}{\partial x_i}$ are two production terms. In the pressure-dilatation Π_d model which is a function of the turbulent Mach number M_t, $\gamma = \frac{c_p}{c_v}$ is the ratio of specific heats. C_{pd1} and C_{pd3} are constants taking values 1.2 and 6, respectively.

The pressure-strain model suggested by Marzougui *et al.* takes into account the turbulent Mach number M_t and the anisotropy tensor $b_{\mathrm{ij}} = \frac{\overline{u_i u_j}}{\overline{q^2}} - \frac{1}{3}\delta_{\mathrm{ij}}$. Coefficients C_i^I come from the incompressible LRR pressure-strain model ($C_2^I = 0.8, C_3^I = 1.75$ and $C_4^I = 1.31$).

4. INITIAL PARAMETERS

Intrinsic parameters that characterize the flow include the initial turbulent Mach number $M_{t0} = \frac{q_0}{c}$ (recall that $\frac{1}{2}q_0^2$ is the initial turbulent kinetic energy), the initial gradient Mach number $M_{g0} = M_{t0}S\frac{q_0^2}{\varepsilon_0}$ (ε_0 is the initial total (solenoidal

Table 1. Hamba and Marzougui *et al.* Models

Models	Linear Parts
Hamba	$\Pi_{\mathrm{ij}} = \frac{2}{3}\Pi_d\delta_{ij} - (1 - C_{ps5}\chi_p)\overline{\rho}\Big[C_{ps2}(P_{ij} - \frac{1}{3}P_{kk}\delta_{ij}) + C_{ps3}q^2 S_{ij} + C_{ps4}(d_{ij} - \frac{1}{3}d_{kk}\delta_{ij})\Big]$ $\Pi_{\mathrm{d}} = -(1 - C_{pd3}\chi_p)\Big[C_{pd1}M_t^2\frac{d}{dt}(\overline{\rho}q^2) + C_{pd2}\gamma\overline{\rho}M_t^2 q^2\frac{\partial U_i}{\partial x_i}\Big]$
Marzougui *et al.*	$\Pi_{\mathrm{ij}} = C_2^I\overline{\rho}q^2 S_{ij} + C_3^I(1 - 1.5M_t^2)\overline{\rho}q^2\Big[b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}\Big] + C_4^I(1 - 0.5M_t)\overline{\rho}q^2\big[b_{ik}\omega_{jk} + b_{jk}\omega_{ik}\big]$; $\omega_{\mathrm{ij}} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i})$

and dilatational) rate of turbulent kinetic energy dissipation) and the initial turbulent Reynolds number $R_{et0} = \frac{q_0^4}{\nu\varepsilon_0}$. The ratio of the gradient Mach number to the turbulent Mach number $r = S\frac{q_0^2}{\varepsilon_0}$ characterizes the rapidity of the shear.

In order to evaluate models of Hamba [3] and Marzougui *et al.* [4], the initial gradient Mach number M_{g0} was set to 48 which characterizes the compressible regime. Table **2** lists initial values of parameters used to evaluate these models.

Table 2. Initial Parameters. Quantity M_{g0} Based on Large-Eddy Lengthscale $\frac{q^3}{\varepsilon}$

M_{t0}	M_{g0}	r_0	R_{et0}
0.4	48	120	296

5. NUMERICAL PROCEDURE AND DISCUSSION

We recall that RDT code developed by authors has been validated [1] for various values of gradient Mach number M_g with reference to RDT and DNS results of Simone [2]. In a preliminary study, we never observe any anomaly in physics parameters (no problem of realizability), nor any instability in the calculation course for $\Delta(St)<5.10^{-3}$. The grid spacing parameters have been chosen in order to perform convergence, accuracy and reasonable execution time.

We now focus on the numerical procedure adopted to calibrate coefficients included in linear parts of Hamba [3] and Marzougui *et al.* [4] models for the pressure-strain correlation. Calibration carried out to minimize differences between RDT and models results in the compressible regime (M_{g0}=48). To this end, we first recall the expressions of C_3 and C_4 coefficients corrected by Marzougui *et al.* [4]:

$$C_3 = C_3^I(1 - aM_t - bM_t^2) \text{ and}$$
$$C_4 = C_4^I(1 - aM_t - bM_t^2), \tag{6}$$

where a and b are constants.

We have to make again a numerical calibration of a and b using RDT results. In the same way, the C_2^I coefficient will be calibrated. Differences between RDT and models results for Π_{ij} can be represented as follows (for 24 time values):

$$e_{ij} = \sum_{k=1}^{24}(\Pi_{ij}(k) - \Pi_{ij}(k)^{RDT})^2. \tag{7}$$

Table **3** lists the values of different coefficients C_2^I, a and b in Marzougui *et al.* [4] model before and after calibration. We use the same systematic procedure to calibrate Hamba model [3]. It consists of chosen values of C_{ps2}, C_{ps3}, C_{ps4}, C_{ps5}, C_{pd1} and C_{pd3} which minimize the difference with RDT results. Table **4** summarizes values of these constants before and after calibration.

In Marzougui *et al*. model [4], values of M_t, b_{11}, b_{12} and b_{22} are found from RDT analysis. Results presented on the different figures correspond, in fact, to non-dimensional parameters while bringing them to $\overline{Sq^2}/2$.

Evolution of pressure-strain correlations Π_{11}, Π_{12} and Π_{22} are represented in Fig. (**1**). Solid lines denote RDT results whereas dashed and dashed-dot lines denote respectively results of Marzougui *et al.* and Hamba models before calibration. Curves with triangles and circles correspond respectively to Hamba's and Marzougui *et al.* results after calibration. These results clearly reveal that these models do not take into account the specific effects of compressibility and lead to appreciable discrepancies with RDT. Hamba [3] has confirmed that values of constants chosen in his model do not give satisfaction.

In order to improve performances of Marzougui *et al*. and Hamba models in the compressible regime (M_{g0}=48), a numerical calibration of coefficients included in these models has been carried out. It will be shown from Fig. (**1**) that after the numerical procedure of calibration, compressibility effects are effectively taken into account in Marzougui *et al*. model for Π_{12} and Π_{22} with a good agreement for Π_{12}.

Table 3. Values of C_2^I, *a* and *b* Coefficients in Marzougui *et al*. Model Before and After Calibration

Coefficients	C_2^I	$C_3 = C_3^I(1-aM_t-bM_t^2)$		$C_4 = C_4^I(1-aM_t-bM_t^2)$	
		a	***b***	***a***	***b***
Before calibration	0.8	0	1.5	0.5	0
After calibration	0.02	0.1	1.2	1.45	0.1

Table 4. Values of C_{ps2}, C_{ps3}, C_{ps4}, C_{ps5}, C_{pd1} and C_{pd3} Coefficients in Hamba Model Before and After Calibration

Coefficients	C_{ps2}	C_{ps3}	C_{ps4}	C_{ps5}	C_{pd1}	C_{pd3}
Before calibration	0.6	0.15	0.15	4	6	1.2
After calibration	0.37	1	0.4	2.01	6	0.2

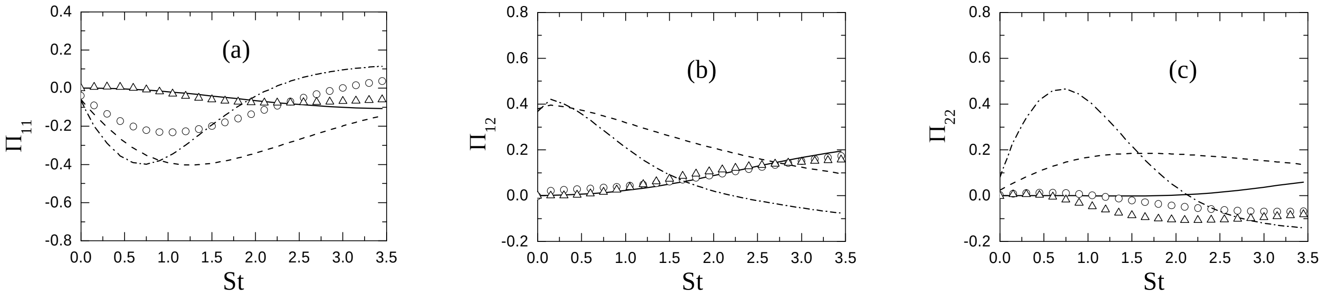

Fig. (1). Evolution of the pressure-strain correlations Π_{11} (**a**), Π_{12} (**b**) and Π_{22} (**c**) for compressible homogeneous shear flow (M_{g0} = 48).
— : RDT prediction; –.–: Hamba model before calibration; Δ Δ: Hamba model after calibration; ---: Marzougui *et al*. model before calibration; o o: Marzougui *et al*. model after calibration.

Differences remain significant for Π_{11} after calibration. It is important to clarify that these results are the best which we obtained with this systematic procedure of calibration and we think that this model is not able to give better results, due in part to the scalar character of coefficients C_3 and C_4. After Hamba's model calibration, we obtained good agreement with RDT for Π_{11} and Π_{12}. Little differences remain for Π_{22}. In addition, Fig. (**2**) shows that calibration of the linear part of pressure-dilatation Π_d in Hamba model improves the comparison with RDT in the compressible regime (M_{g0}=48).

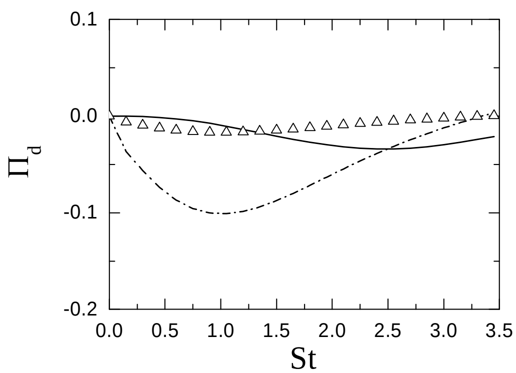

Fig. (2). Evolution of the pressure-dilatation correlation term Π_d for compressible homogeneous shear flow (M_{g0} = 48).
— : RDT prediction; –.–: Hamba model before calibration; Δ Δ : Hamba model after calibration.

Effectively, the calibration procedure applied to Marzougui *et al.* and Hamba models provides a net improvement.

6. CONCLUSION

Rapid-distortion-theory (RDT) has largely showed its relevance and its utility to identify compressibility effects which allow us to evaluate compressible models of turbulence in the field of RDT validity for small values of *St*. Thus, a systematic evaluation of recently proposed compressible turbulence models has been conducted with the RDT code which solves linearized equations for compressible homogeneous shear turbulence. Evaluation concerns linear parts of pressure-strain correlations Π_{11}, Π_{12} and Π_{22} models of Hamba [3] and Marzougui *et al.* [4]. These models do not correctly predict compressible turbulence. Thus, a numerical calibration of coefficients included in these models is carried out to improve their performances in the compressible regime (M_{g0}=48). After calibration, a good agreement with RDT is obtained for Π_{11} and Π_{12} with Hamba model and for Π_{12} with Marzougui *et al.* model.

In conclusion, the numerical procedure of calibration provides broadly a net improvement for pressure-strain correlations predicted by these models.

ABBREVIATIONS

RDT = rapid-distortion-theory

DNS = direct numerical simulation

LRR = Launder, Reece and Rodi model of turbulence

NOMENCLATURE

M_g = gradient Mach number

M_t = turbulent Mach number

S = shear rate

Π_{ij} = pressure-strain correlation

Π_d = pressure-dilatation correlation

u_i = velocity fluctuation

p = pressure fluctuation

$\overline{\rho}$ = mean density

ν = kinematic viscosity

γ = ratio of the specific heats c_p and c_v

λ_{ij} = mean velocity gradient

c = mean speed of sound

$q^2/2$ = turbulent kinetic energy

b_{ij} = anisotropy tensor

ε_0 = initial total (solenoidal and dilatational) dissipation rate of turbulent kinetic energy

R_{et0} = initial turbulent Reynolds number

r = rapidity of the shear

REFERENCES

[1] Riahi M, Chouchane L, Lili T. A study of compressibility effects on structure of homogeneous sheared turbulence. Eur Phys J Appl Phys 2007; 39: 67-75.

[2] Simone A, Coleman GN, Cambon C. The effect of compressibility on turbulent shear flow: a rapid-distortion-theory and direct-numerical-simulation study. J Fluid Mech 1997; 330: 307-38.

[3] Hamba F. Effect of pressure fluctuations on turbulence growth compressible homogenous shear flow. Phys Fluids 1999; 11: 1623-35.

[4] Marzougui H, Khlifi H, Lili T. Extension of the Launder, Reece and Rodi model on compressible homogeneous shear flow. Eur Phys J B 2005; 45: 147-54.

[5] Launder BE, Reece GJ, Rodi W. Progress in the development of a Reynolds-stress turbulence closure. J Fluid Mech 1975; 68: 537-66.

[6] Sarkar S, Erlebacher G, Hussaini M, Kreiss HO. The analysis and modelling of dilatational terms in compressible turbulence. J Fluid Mech 1991; 27: 473-93.

[7] Sarkar S. The pressure dilatation correlation in compressible flows. Phys Fluids 1992; A 4: 2674-82.

[8] Sarkar S. The stabilizing effect of compressibility in turbulent shear flow. J Fluid Mech 1995; 282: 163-86.

[9] Cambon C, Coleman GN, Mansour NN. Rapid distortion analysis and direct simulation of compressible homogeneous turbulence at finite Mach number. J Fluid Mech 1993; 257: 641-65.

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