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On the Overturning Instability of a Rectangular Rigid Block Under Ground Excitation

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Abstract: This paper deals with the rocking response of a free-standing rectangular rigid block subjected to a ground acceleration assuming that the friction between the block and the ground is large enough so that there is no sliding. Particular attention is focused on the minimum acceleration amplitude which may lead the block to overturning instability. The conditions of such a critical state are properly established. Subsequently, two distinct modes of overturning instability under a one-sine ground pulse are examined: (1) overturning without impact and (2) overturning after one impact occurring either before or after the ground excitation expires. The effect of initial conditions on the minimum amplitude acceleration is also discussed in connection with a one-cosine and a one-sine pulse. The proposed technique is applied to various examples covering all possible cases of overturning instability.

Keywords: Overturning instability, rigid block, impact, rocking motion, ground excitation, nonlinear and linearized analysis, and dynamic response.

1. INTRODUCTION

In recent years the attention of various researchers was focused on the rocking response of free-standing multi-drum columns carrying statues at their tip. A fundamental relative problem for the dynamic analysis of such a column-statue system is the rocking response of a free-standing rigid block for which various interesting studies have been presented lately [1-6]. This work is an extension of the last studies by presenting some new results *via* a simple and comprehensive analysis.

Consider a rectangular rigid block with dimensions 2b x 2h and total mass m which is in a vertical equilibrium position under its own weight (Fig. **1**). The angle $\alpha = \tan^{-1}(b/h)$ is the *stockiness* parameter (inverse of the slenderness ratio) of the block.

Depending on the *form* and *magnitude* of the ground excitation, the block may *translate* with the ground, *slide*, *rock* or *slide-rock.* Later work [1] on this subject showed that in addition to pure *sliding* and pure *rocking* a combined *slide-rock* mode of rigid body motion may also occur. This depends not only on the ratio $b/h=\tan\alpha$ and the static *friction* coefficient μ – as was believed in the past – but also on the *magnitude* of the ground excitation. Subsequently, it is assumed that the coefficient of friction μ is large enough so that there is no sliding.

Under a *positive* horizontal ground excitation (displacement or acceleration) who's magnitude is *sufficiently* large [5] the rigid block will initially rotate with a negative rotation $\theta < 0$ (Fig. **2a**), and if the block does not overturn it will eventually assume a *positive* rotation, and so on. However, as will be shown, if the positive excitation is *moderate* the rigid block will initially rotate with a positive rotation $\theta > 0$ (Fig. **2b**), and if it does not overturn it will eventually assume a negative rotation, and so on.

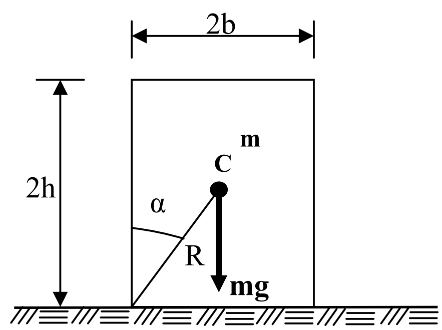

Fig. (1). Free-standing rectangular rigid block under its own weight.

Langrange's equations for rigid body motion of the above rectangular block for the cases of Fig. (**2a**) and Fig. (**2b**) are given by [7, 8].

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial K}{\partial \theta} + \frac{\partial \Omega}{\partial \theta} = 0 \qquad (1)$$

where K is the total kinetic energy and Ω the potential of the external force mg (the weight of the block) and θ the angle of the bock rotation measured from the vertical.

Case 1 (Fig. 2a)

According to the sign convention of Fig. (**2a**) the total horizontal displacement of the center of gravity C of the block, u_c, due to ground displacement u_g is [9].

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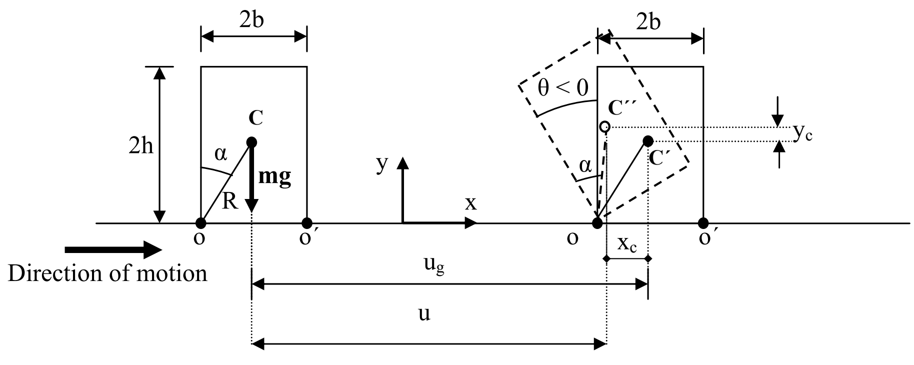


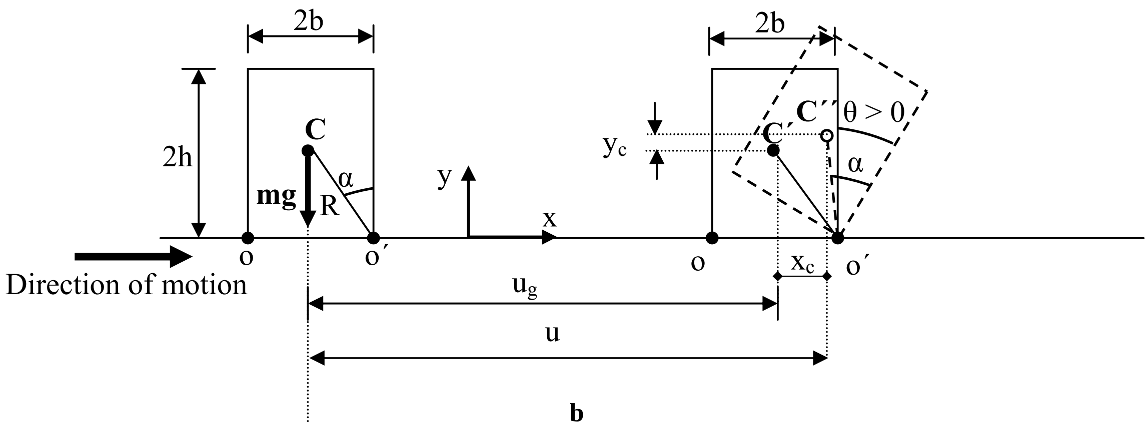


Fig. (2). a. Rigid body block displacement under ground excitation (rather suddenly applied), **b.** Rigid body block displacement under ground excitation (rather suddenly reduced).

$$u_c = u_g - x_c\,,\ x_c = b - R\sin(\alpha+\theta) \qquad (\theta(t)<0) \tag{2}$$

while the vertical displacement of C, is

$$y_c = R\left[\cos(\alpha+\theta) - \cos\alpha\right] \tag{3}$$

By virtue of relations (2) and (3) we get

$$\dot{u}_c = \dot{u}_g + R\dot{\theta}\cos(\alpha+\theta)\,,\ \dot{y}_c = -R\dot{\theta}\sin(\alpha+\theta) \tag{4}$$

The total kinetic energy due to the combined motion of the block (rotation and translation) is

$$K = \frac{1}{2}J_c\dot{\theta}^2 + \frac{1}{2}m\left(\dot{u}_c^{\,2} + \dot{y}_c^2\right) \tag{5}$$

where $J_c = mR^2/3$ is the polar moment of inertia of the block with respect to its center of gravity C.

Using relations (4), eq. (5) is written as follows

$$K = \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}m\dot{u}_g^{\,2} + m\dot{u}_g\dot{\theta}R\cos(\alpha+\theta) \tag{6}$$

where $J_o = J_c + mR^2 = 4mR^2/3$ is the polar moment of inertia of the block with respect to the pivot point O.

From eq. (6) it follows that

$$\frac{\partial K}{\partial\dot{\theta}} = J_o\dot{\theta} + m\dot{u}_g R\cos(\alpha+\theta)$$

and $\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{\theta}}\right) = J_o\ddot{\theta} + m\ddot{u}_g R\cos(\alpha+\theta) - m\dot{u}_g\dot{\theta}R\sin(\alpha+\theta)$. (7)

From eq. (6) we also obtain

$$\frac{\partial K}{\partial\theta} = -m\dot{u}_g\dot{\theta}R\sin(\alpha+\theta)\,. \tag{8}$$

Given that

$$\Omega = mgR[\cos(\alpha+\theta) - \cos\alpha] \tag{9}$$

then

$$\frac{\partial\Omega}{\partial\theta} = -mgR\sin(\alpha+\theta)\,. \tag{10}$$

Using relations (7), (8) and (10), eq. (1) becomes

$$J_o\ddot{\theta} + m\ddot{u}_g R\cos(\alpha+\theta) - mgR\sin(\alpha+\theta) = 0 \qquad (\theta(t)<0). \tag{11}$$

This is the equation of rigid body motion that can be also derived by taking equilibrium of moments of all forces with respect to the pivot point O (Fig. **3a**).

Case 2 (Fig. 2b)

The total horizontal displacement of the center of gravity C of the block u_c, due to ground displacement u_g, according to Fig. (**2b**), is

$$u_c = u_g + x_c\,,\ x_c = b - R\sin(\alpha-\theta) \tag{12}$$

while the vertical displacement of C is

$$y_c = R\left[\cos(\alpha-\theta) - \cos\alpha\right] \tag{13}$$

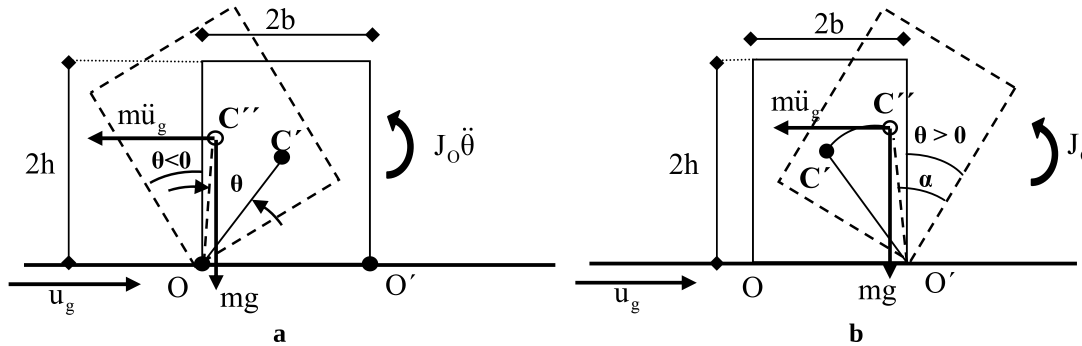


Fig. (3). a. Free body diagram of the block related to Fig. (**2a**), **b.** Free body diagram of the block related to Fig. (**2b**).

and the potential due to the external force mg is equal to

$$\Omega = mgR\left[\cos(\alpha-\theta)-\cos\alpha\right], \tag{14}$$

By virtue of relations (12) and (13) we get

$$\dot{u}_c = \dot{u}_g + R\dot{\theta}\cos(\alpha-\theta),\ \dot{y}_c = R\dot{\theta}\sin(\alpha-\theta). \tag{15}$$

The *kinetic energy* due to the combined rigid body motion of the block (rotation and translation about O′) using relations (15) is

$$K = \frac{1}{2}J_c\dot{\theta}^2 + \frac{m}{2}\left(\dot{u}_c^2+\dot{y}_c^2\right) = \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}m\dot{u}_g{}^2 + mR\dot{u}_g\dot{\theta}\cos(\alpha-\theta) \tag{16}$$

where $J_o = J_c + mR^2 = 4mR^2/3$ is the polar moment of inertia of the block about O′.

From eq. (16) it follows that

$$\frac{\partial K}{\partial\dot{\theta}} = J_o\dot{\theta} + mR\dot{u}_g\cos(\alpha-\theta)$$

and

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{\theta}}\right) = J_o\ddot{\theta} + mR\ddot{u}_g\cos(\alpha+\theta) + mR\dot{u}_g\dot{\theta}\sin(\alpha-\theta) \tag{17}$$

Moreover

$$\frac{\partial K}{\partial\theta} = mR\dot{u}_g\dot{\theta}\sin(\alpha-\theta). \tag{18}$$

Due to relation (14) we get

$$\frac{\partial\Omega}{\partial\theta} = mgR\sin(\alpha-\theta). \tag{19}$$

Using relations (17), (18) and (19), eq. (1) becomes

$$J_o\ddot{\theta} + mR\ddot{u}_g\cos(\alpha-\theta) + mgR\sin(\alpha-\theta) = 0\quad(\theta(t)>0). \tag{20}$$

This equation can also be derived by taking equilibrium of moments of all the above forces about O′ (Fig. **3b**).

Eqs. (11) and (20) can also be written as follows

$$\ddot{\theta} + p^2\left[\frac{\ddot{u}_g}{g}\cos(\alpha+\theta) - \sin(\alpha+\theta)\right] = 0,\ \theta(t)<0 \tag{21a}$$

$$\ddot{\theta} + p^2\left[\frac{\ddot{u}_g}{g}\cos(\alpha-\theta) + \sin(\alpha-\theta)\right] = 0,\ \theta(t)>0 \tag{21b}$$

where $p=\sqrt{3g/4R}$ is a measure of the dynamic characteristic of the block and not the oscillatory frequency under free vibration, because the oscillation frequency in not constant depending strongly on the vibration amplitude [10]. Note that owing to the difference in the last terms of eq. (21a, b) one can conclude that the *magnitude* of ground excitation in Fig. (**2a**) is significantly large (much larger to that corresponding to Fig. **2b**).

Regardless of the form of ground excitation there are two possible modes for overturning: (a) overturning without impact and (b) overturning with one impact. Referring to the case of Fig. (**2a**) the block may overturn under *very large* ground excitation with θ<0 *without impact* (mode 1). However, for a ground acceleration slightly smaller than the previous one, the block rotates in the reverse direction and impacts on point O′ before overturning with θ>0. The *minimum* acceleration amplitude corresponds to the *unstable static equilibrium* for which θ = α, $\dot{\theta}=0$ and $\ddot{\theta}=0$, implying $\ddot{u}_g=0$ due to eq. (21a); namely, we have overturning instability after one impact. Now referring to the case of Fig. (**2b**) the block may overturn (on the basis of minimum acceleration amplitude) through the unstable equilibrium position for which θ=α, $\dot{\theta}=0$ and $\ddot{\theta}=0$, implying $\ddot{u}_g=0$ due to eq. (21b). Namely, in this case overturning instability occurs without impact; otherwise for a smaller magnitude of excitation the block returns to its initial vertical equilibrium position.

The total energy E = K +U = K +V +Ω corresponding to eq. (21a) and (21b) is

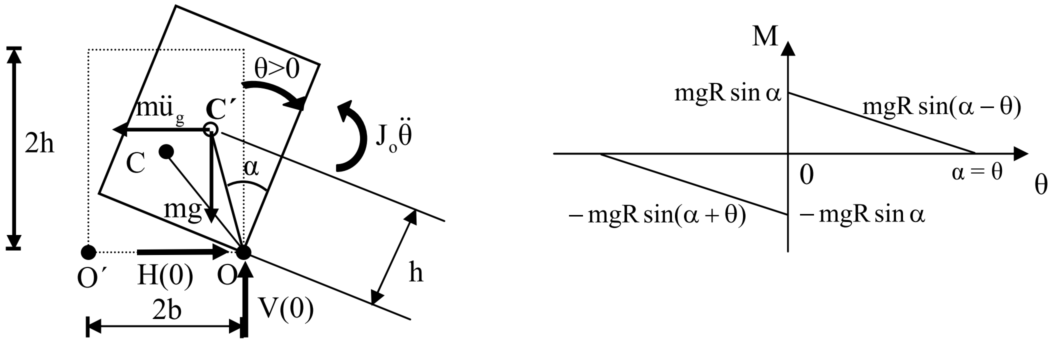
$$E = \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}m\dot{u}_g^2 + mR\dot{u}_g\dot{\theta}\cos(\alpha+\theta) + mgR\left[\cos(\alpha+\theta)-\cos\alpha\right],$$
$$\theta(t)<0 \tag{22a}$$

$$E = \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}m\dot{u}_g^2 + mR\dot{u}_g\dot{\theta}\cos(\alpha-\theta) + mgR\left[\cos(\alpha-\theta)-\cos\alpha\right],$$
$$\theta(t)>0. \tag{22b}$$

Clearly, replacing θ by -θ in eq. (22a) we obtain eq. (22b).

Condition for Initiation of Rocking Motion

Consider the rigid block shown in Fig. (**4a**) with ″stockiness″ α which can oscillate about the centers of rotation O and O′ when it is set to rocking motion. As assumed above the coefficient of friction is large enough so that there is no sliding. Fig. (**4b**) shows the *moment-rotation* relation during the rocking motion of a freely-standing rectangular block. The system has infinite stiffness until the magnitude



a

Fig. (4). a. Free body diagram of the block.

b

b. Moment M *versus* rotation θ.

of the applied moment reaches mgRsinα, and once the block is rocking its stiffness decreases gradually becoming zero when θ=α. During the oscillatory rocking motion, the moment-rotation relationship follows the above diagram without losing energy (i.e. enclosing any area). Energy is lost only due to friction (this case has been excluded because it has been assumed that there is no sliding) or during impact, when the angle of rotation *reverses*. The last case of loss of energy due to impact will be discussed in the next section.

When the block is in vertical equilibrium position the horizontal force H which is needed to initiate rocking motion is obtained from the condition

$$HR\cos\alpha \geq mgR\sin\alpha\,. \tag{23}$$

Since $H = m\ddot{u}_g = m\alpha_p$ (α_p is the maximum amplitude), then initiation of rocking yields

$$\alpha_p \geq g\tan\alpha$$

and $\alpha_{p,\min} = g\tan\alpha\,.$ (24)

This is the *minimum* amplitude of ground acceleration for the initiation of rocking motion.

Dynamic equilibrium (in horizontal and vertical direction) at the instant t=0, where θ(0)=0, gives (Fig. **5**)

$$H(0) = m\left(\ddot{u}_g + R\ddot{\theta}(0)\cos\alpha\right) = m\left(\lambda\alpha_\rho + h\ddot{\theta}(0)\right) \tag{25}$$

$$V(0) = m(g - R\ddot{\theta}(0)\sin\alpha) = m\left(g - b\ddot{\theta}(0)\right) \tag{26}$$

and moments about C

$$H(0)h - V(0)b + J_c\ddot{\theta}(0) = 0 \tag{27}$$

where $J_c = mR^2/3$ and $1 \geq \lambda \geq g\tan\alpha/\alpha_p$.

Substituting eq. (25) and eq. (26) into eq. (27) we get the angular acceleration at the instant t=0, i.e.

$$\ddot{\theta}(0) = p^2\sin\alpha\left(1 - \frac{\lambda\alpha_p}{g\tan\alpha}\right),\ p^2 = \frac{3g}{4R} \tag{28}$$

when rocking initiates.

To avoid sliding at t=0 we must have

$$\mu V(0) \geq H(0) \tag{29}$$

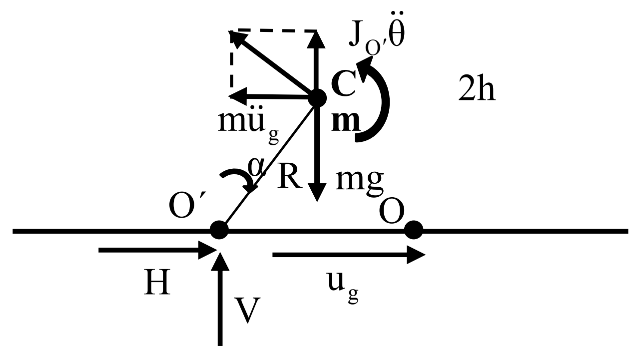


Fig. (5). Dynamic reactions acting on the point O′ at the initiation of rocking motion (t=0).

$$\text{or } \mu \geq \frac{H(0)}{V(0)}\,. \tag{30}$$

Inserting $\ddot{\theta}(0)$ from eq. (28) into eqs. (25) and (26), inequality (30) becomes [1, 2]

$$\mu \geq \frac{\lambda\alpha_\rho - \dfrac{3}{4}g\cos\alpha\sin\alpha\left(\dfrac{\lambda\alpha_\rho}{g\tan\alpha} - 1\right)}{g + \dfrac{3}{4}g\sin^2\alpha\left(\dfrac{\lambda\alpha_\rho}{g\tan\alpha} - 1\right)}\,. \tag{31}$$

This is the condition required for a block to enter the *rocking motion without sliding*.

According to previous work [1, 2], inequality (31) shows that, under some ground excitations with amplitude α_p, the condition for a block to enter rocking motion without sliding depends on the value α_p. However, for pulses in which acceleration increases gradually from zero (like one-sine pulse) the value of $\lambda\alpha_p$ at the initiation of rocking motion is equal to $g\tan\alpha$ and eq. (31) reduces to the expression defined from *static equilibrium*, i.e.

$$\mu \geq \frac{b}{h} = \tan\alpha\,. \tag{32}$$

Once the block enters rocking motion, both dynamic reactions H(t) and V(t) fluctuate with time. Hence, to avoid sliding during the entire rocking motion we must have at all times

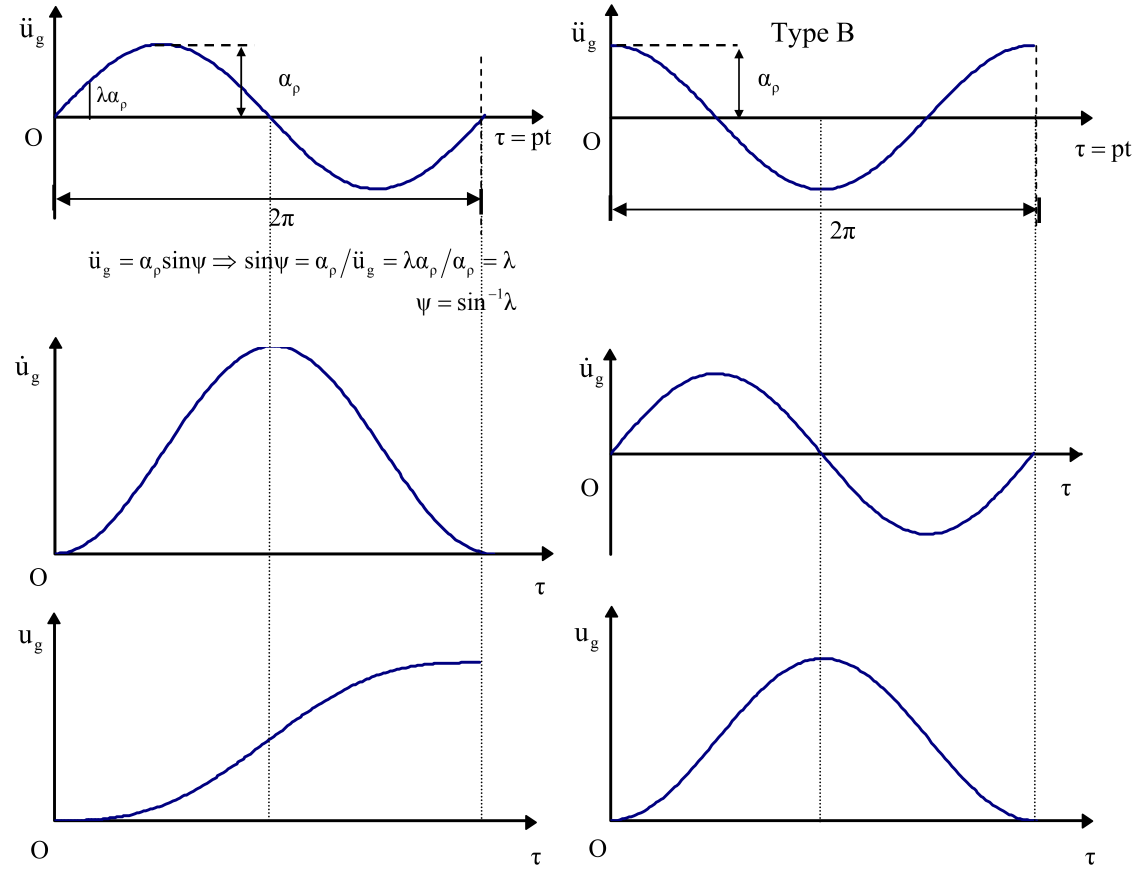


Fig. (6). Acceleration, velocity and displacement histories of one-sine (left) and one-cosine (right) pulse.

$$\mu > \frac{H(t)}{V(t)} \ . \tag{33}$$

In Fig. (**6**) one can see *acceleration*, *velocity* and *displacement* histories of one-sine pulse (left) and one-cosine pulse (right) [4]. In the first case (Type A) the ground acceleration is zero at the starting of motion and increases gradually. In contrast, in the second case (Type B) the ground acceleration assumes its maximum at the initiation of motion. Under other pulses, e.g. type-C_n pulses [3], the ground acceleration is finite at the initiation of rocking motion but assumes a value that is smaller than its maximum amplitude α_p.

2. Loss of Energy During Impact

When the angle of rotation *reverses* we assume that such a rotation continues smoothly from point O to point O′(Fig. **7**). We consider an *impact* without bouncing so that the block switches pivot points (from O to O′), while the angular momentum is *conserved*. If the coefficient of restitution is e the ratio of the angular speed of the block immediately after impact $\dot{\theta}_+$, to the angular speed immediately before impact $\dot{\theta}_-$ according to [11] is

$$e = \frac{\dot{\theta}_+}{\dot{\theta}_-} \ . \tag{34}$$

Clearly $|\dot{\theta}_-| > |\dot{\theta}_+|$. The energy lost is E_L, i.e.

$$E_L = \frac{1}{2} J_o \dot{\theta}_-^2 - \frac{1}{2} J_o \dot{\theta}_+^2 = \frac{2}{3} m R^2 \left(\dot{\theta}_-^2 - \dot{\theta}_+^2\right)$$

$$\text{or } E_L = \frac{2}{3} m R^2 \dot{\theta}_-^2 \left(1 - e^2\right). \tag{35}$$

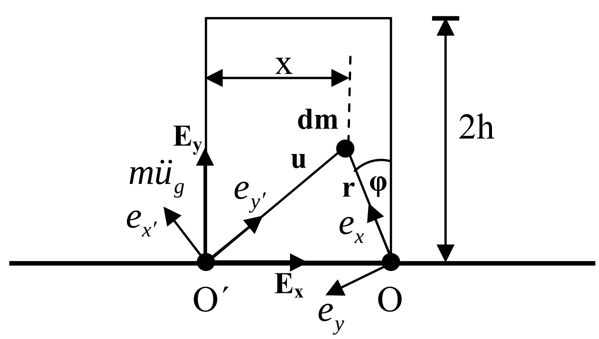


Fig. (7). Rectangular block under rocking motion just before it impacts on point O′.

Fig. (**7**) shows the rectangular block of uniform density ρ, [ρ(2hx2b)]=m, that is rotating about O and is about to impact at point O′. Consider firstly the *angular momentum* of the block about O′ before impact. A *mass element* dm located at a distance r from point O has a velocity $v_- = r\dot{\theta}_- e_y$. The position of the mass element is $u = 2bE_x + re_x$, and hence the angular momentum of the block B about O′ before the impact is

$$H_{-}^{O'} = \int_B u x v_{-}\, dm\,. \tag{36}$$

Taking into account that

$$e_x x e_y = |e_x||e_y|\sin\frac{\pi}{2} e_z = e_z\,,$$

$$E_x x e_y = -|e_x||e_y|\sin\phi \cdot e_z = -\sin\phi \cdot e_z$$

eq. (36) becomes

$$H_{-}^{O'} = \left(\int_B \left(r^2 - 2br\sin\varphi\right)dm\right)\dot{\theta}_{-}e_z\,. \tag{37}$$

Since $dm = \rho dxdy$, $r\sin\phi = 2b - x$ (38)

$$\text{then } H_{-}^{O'} = \left(\rho\int_0^{2b}\int_0^{2h}\left[r^2 - 2b\left(2b - x\right)\right]dxdy\right)\dot{\theta}_{-}e_z\,. \tag{39}$$

Given that

$$r^2 = y^2 + \left(2b - x\right)^2,\; r^2 - 2b\left(2b - x\right) = x^2 + y^2 - 2bx \tag{40}$$

then

$$\int_0^{2b}\int_0^{2h}\left[x^2 + y^2 - 2bx\right]dx\,dy = 4hb\left(\frac{4}{3}h^2 - \frac{2}{3}b^2\right). \tag{41}$$

Since $b = R\sin\alpha$

$$\text{and } 4hb\left(\frac{4}{3}h^2 - \frac{2}{3}b^2\right) = 4hb\left(\frac{4}{3}R^2 - 2R^2\sin^2\alpha\right) \tag{42}$$

by virtue of relations (41) and (42), eq(39) becomes

$$H^{O'} = \left(\rho\int_0^{2b}\int_0^{2b}\left[r^2 - 2b\left(2b - x\right)\right]dxdy\right)\dot{\theta}_{-}e_z$$

$$\text{or } H_{-}^{O'} = mR^2\left(\frac{4}{3} - 2\sin^2\alpha\right)\dot{\theta}_{-}e_z\,. \tag{43}$$

The position of the *infinitesimal mass element* dm immediately after impact is $u = re_x$, and its velocity $v_+ = r\dot{\theta}_+ e_y$. Thus, the *angular* momentum of the block B about O after impact is

$$H_{+}^{O'} = \int_B u x v_{+}\, dm \tag{44}$$

$$\text{or } H_{+}^{O'} = \left(\int_B r^2 dm\right)\dot{\theta}_{+}e_z = \frac{4}{3}mR^2\dot{\theta}_{+}e_z\,. \tag{45}$$

Conservation of angular momentum, $H_{-}^{O'} = H_{+}^{O'}$, due to relations (42) and (45), yields

$$\left(1 - \frac{3}{2}\sin^2\alpha\right)\dot{\theta}_{-} = \dot{\theta}_{+}$$

$$\text{or } e = \frac{\dot{\theta}_{+}}{\dot{\theta}_{-}} = 1 - \frac{3}{2}\sin^2\alpha\,. \tag{46}$$

This value of e (depending only on the geometry of the block) is the *maximum* value of the coefficient of restitution for which a block of stockiness α will undergo rocking motion [4]. Since e_{max} <1 the impact is *inelastic*. Since the angular momentum is not actually conserved during impact the value of e can only be thought of as a rough approximation because it's precise value depends on the contact region and the corresponding material properties. The value of e can be obtained experimentally. For the stockiness $\alpha = \tan^{-1}(b/h) = \tan^{-1}(0.11)$ of the statues in the Academy of Athens, eq. (46) gives e=0.982 [9].

Condition for Rocking Motion

When the block is rocking the horizontal and vertical reactions at point O and O′ are varying with time. Dynamic equilibrium in both directions yields (Figs. **3a** and **5**)

$$H(t) = m\left[\ddot{u}_g\left(t\right) - \ddot{x}_c\left(t\right)\right],\; V(t) = m\left[g - \ddot{y}_c(t)\right] \tag{47}$$

Since for θ(t)<0, $x_c = b - R\sin(\alpha + \theta)$, $\dot{x}_c = -R\dot{\theta}\cos(\alpha + \theta)$

$$\text{and } \ddot{x}_c = -\ddot{\theta}R\cos(\alpha + \theta) + \dot{\theta}^2 R\sin(\alpha + \theta) \tag{48}$$

Similarly

$$y_c = R\left[\cos\left(\alpha + \theta\right) - \cos\alpha\right],\; \dot{y}_c = -R\dot{\theta}\left[\sin\left(\alpha + \theta\right)\right]$$

$$\text{and } \ddot{y}_c = -\ddot{\theta}R\sin(\alpha + \theta) - \dot{\theta}^2 R\cos(\alpha + \theta) \tag{49}$$

where $\dot{\theta}(t)$ is the angular velocity of the block and $\ddot{\theta}(t)$ is the angular acceleration of the block given by eq. (21a) [2].

Substituting eqs. (47) into inequality (33) and using eqs. (11), (48) and (49) allows us to establish the condition for *excluding sliding* during the entire rocking motion, i.e.

$$\frac{H(t)}{V(t)} = \frac{\ddot{u}_g - p^2 R\left[\frac{\ddot{u}_g}{g}\cos^2\left(\alpha + \theta\right) - \frac{1}{2}\sin 2\left(\alpha + \theta\right) + \frac{\dot{\theta}^2}{p^2}\sin(\alpha + \theta)\right]}{g - p^2 R\left[\frac{\ddot{u}_g}{g}\sin 2\left(\alpha + \theta\right) - \sin^2\left(\alpha + \theta\right) - \frac{\dot{\theta}^2}{p^2}\cos(\alpha + \theta)\right]} \le \mu \tag{50}$$

At the initiation (t=0) of rocking, $\theta(0) = \dot{\theta}(0) = 0$, while – as shown above – the minimum acceleration must be $\ddot{u}_g = g\tan\alpha$. Clearly at t=0, eq. (50) gives

$$\frac{H(0)}{V(0)} = \frac{\ddot{u}_g}{g} = \tan\alpha \le \mu\,. \tag{51}$$

3. Linear Approximation Under One-Sine Pulse

On the basis of the type A of one- sine pulse (Fig. **6**, left) one can write the following ground acceleration

$$\ddot{u}_g(t) = \alpha_p\sin\left(\omega_p t + \psi\right),\; -\psi/\omega_p \le t \le \left(2\pi - \psi\right)/\omega_p\,, \tag{52}$$

otherwise, $\ddot{u}_g(t) = 0$.

Clearly

$$\ddot{u}_g(0) = \alpha_p\sin\psi = \lambda\alpha_p \text{ or } \psi = \sin^{-1}\lambda\,. \tag{53}$$

Since $\lambda\alpha_p = g\tan\alpha$, then

$$\psi = \sin^{-1}\left(\frac{g\tan\alpha}{\alpha_p}\right). \tag{54}$$

According to eq. (32) the block enters into pure rocking when

$$\tan\alpha = \frac{b}{h} < \mu. \tag{55}$$

For slender blocks with b/h≤0.25, eqs. (21a,b) can be linearized as follows

$$\ddot{\theta} - p^2\theta = p^2\alpha - p^2\frac{\alpha_p}{g}\sin\left(\omega_p t + \psi\right) \qquad (\theta(t)<0) \tag{56a}$$

$$\ddot{\theta} - p^2\theta = -p^2\alpha - p^2\frac{\alpha_p}{g}\sin\left(\omega_p t + \psi\right) \qquad (\theta(t)>0). \tag{56b}$$

Integration of eqs. (56a,b) gives

for $\theta(t) < 0$

$$\theta = A_1\sinh pt + A_2\cosh pt + \frac{1}{1+\omega_p^{\ 2}/p^2}\frac{\alpha_p}{g}\sin\left(\omega_p t+\psi\right) - \alpha \tag{57a}$$

for $\theta(t) > 0$

$$\theta = A_3\sinh pt + A_4\cosh pt + \frac{1}{1+\omega_p^2/p^2}\frac{\alpha_p}{g}\sin\left(\omega_p t+\psi\right) + \alpha\,. \tag{57b}$$

For the initial condition $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_o$ we get

$$A_1 = \frac{\dot{\theta}_0}{p} - \alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\,,\ A_2 = \theta_0 + \alpha - \frac{\alpha}{1+\omega_p^2/p^2}\,,$$

$$A_3 = \frac{\dot{\theta}_0}{p} - \alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\,,\ A_4 = \theta_0 - \alpha - \frac{\alpha}{1+\omega_p^2/p^2} \tag{58}$$

The derivatives of eqs. (57a,b) are

$$\frac{\dot{\theta}}{p} = A_1\cosh pt + A_2\sinh pt + \alpha\frac{\omega_p/p}{1+\omega_p^{\ 2}/p^2}\frac{\cos\left(\omega_p t+\psi\right)}{\sin\psi} \tag{59a}$$

$$\frac{\dot{\theta}}{p} = A_3\cosh pt + A_4\sinh pt + \alpha\frac{\omega_p/p}{1+\omega_p^{\ 2}/p^2}\frac{\cos\left(\omega_p t+\psi\right)}{\sin\psi}\,. \tag{59b}$$

Using eqs. (57a,b) and (59a,b) one can determine the minimum amplitude acceleration provided that one can establish a condition for overturning instability.

Condition for Overturning Instability

Attention is focused on the *minimum* amplitude acceleration leading to overturning instability. Regardless of the type of ground motion such a *critical state* may occur at some time t=t* only through the *unstable equilibrium position* $\theta(t^*) = \alpha$. Given that we are searching for the minimum amplitude acceleration which leads to overturning, the conditions for overturning instability are defined by

$$\theta(t^*) = \alpha\,,\ \dot{\theta}(t^*) = 0\,. \tag{60}$$

This is so because in such a critical case we may assume that the block can oscillate for a short period of time with an average amplitude θ(t)=α (e.g. about the unstable equilibrium position) and thus when $\dot{\theta}(t^*) = \alpha$ also $\theta(t^*) = 0$. Apparently for $t \le t^*$ the block is subjected to free vibrations with initial conditions at $t=t_o$ either the end conditions of the *forced* motion or the *impact* conditions.

Free Vibration

In this case eqs. (56a, b) become

$$\ddot{\theta} - p^2\theta = p^2\alpha \tag{61a}$$

$$\ddot{\theta} - p^2\theta = -p^2\alpha \tag{61b}$$

which upon integration yield

$$\theta = \bar{A}_1\sinh p(t-t_o) + \bar{A}_2\cosh p(t-t_o) - \alpha \tag{62a}$$

$$\theta = \bar{A}_3\sinh p(t-t_o) + \bar{A}_4\cosh p(t-t_o) + \alpha \tag{62b}$$

with corresponding angular velocities

$$\dot{\theta}(t_o)/p = \bar{A}_1\cosh p(t-t_o) + \bar{A}_2\sinh p(t-t_o) \tag{63a}$$

$$\dot{\theta}\left(t_o\right)/p = \bar{A}_3\cosh p(t-t_o) + \bar{A}_4\sinh p(t-t_o) \tag{63b}$$

where

$$\bar{A}_1 = \dot{\theta}(t_o)/p\,,\ \bar{A}_2 = \theta\left(t_0\right) + \alpha\,,\ \bar{A}_3 = \dot{\theta}(t_0)/p\,,\ \bar{A}_4 = \theta(t_0) - \alpha\,. \tag{64}$$

Subsequently, the particular case of overturning under the type A of a one-sine pulse ground acceleration is considered. Two modes of overturning instability are examined: Mode 1 with no impact and mode 2 with one impact.

Mode 1 (no impact)

Both cases shown in Fig. (**2a,b**) with their corresponding equations of free vibration motion occurring at $t \ge t_o = T_{ex} = \left(2\pi - \psi\right)/\omega_p$ are considered.

Application of the overturning instability criterion, eqs. (60), for the case of Fig. (**2a**) related to eqs. (62a & 63a), yields

$$\theta(t) = -\alpha = \bar{A}_1\sinh p(t-T_{ex}) + \bar{A}_2\cosh p(t-T_{ex}) - \alpha \tag{65a}$$

$$\dot{\theta}(t)/p = 0 = \bar{A}_1\cosh p(t-T_{ex}) + \bar{A}_2\sinh p(t-T_{ex}) \tag{65b}$$

where according to relations (64)

$$\bar{A}_1 = \dot{\theta}\left(T_{ex}\right)/p\,,\ \bar{A}_2 = \theta\left(T_{ex}\right) + \alpha\,. \tag{66}$$

Eqs. (65a,b) have the nontrivial solution

$$\bar{A}_1 = -\bar{A}_2 \tag{67}$$

if $\tanh p\left(t - T_{ex}\right) = 1$ which may occur at large time. Eq. (67) due to relation (66) yields

$$\dot{\theta}\left(T_{ex}\right) + p\left(\theta\left(T_{ex}\right) + \alpha\right) = 0 \tag{68}$$

where $\dot{\theta}\left(T_{ex}\right)$ and $\theta\left(T_{ex}\right)$ are determined from eqs. (57a) and (59a) which give

$$\theta(T_{ex}) = A_1\sinh pT_{ex} + A_2\cosh pT_{ex} - \alpha \tag{69a}$$

$$\dot{\theta}(T_{ex})/p = A_1\cosh pT_{ex} + A_2\sinh pT_{ex} + \frac{\alpha\,\omega_p/p}{1+\omega_p^2/p^2}\frac{1}{\sin\psi} \tag{69b}$$

where A_1 and A_2 are taken from relations (58) after setting $\theta(0)=\theta_o=0$ and $\dot\theta(0)=\dot\theta_o=0$, being equal to

$$A_1=-\alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\,,\ A_2=\frac{\alpha\,\omega_p^2/p^2}{1+\omega_p^2/p^2}\,. \tag{70}$$

Introducing expressions (69a, b) into eq(68) and using relations (70) we obtain [Zhang and Makris (2001)]

$$\cos\psi-\frac{\omega_p}{p}\sin\psi=e^{-\frac{(2\pi-\psi)}{\omega_p/p}}\,. \tag{71}$$

Eq. (71) yields the *minimum* amplitude acceleration $\alpha_p/g\alpha=1/\sin\psi$ for overturning instability for the case of Fig. (**2a**), a solution compatible with eq. (65a) implying θ(t)<0 from the beginning of the motion (as anticipated).

Similarly one can proceed for the case of Fig. (**2b**) related to eqs. (62b) and (63b). Application of the overturning instability criterion (60) leads to

$$\theta(t)=\alpha=\bar{A}_3\sinh p(t-T_{ex})+\bar{A}_4\cosh p(t-T_{ex})+\alpha \tag{72a}$$

$$\dot\theta(t)/p=0=\bar{A}_3\cosh p(t-T_{ex})+\bar{A}_4\sinh p(t-T_{ex}) \tag{72b}$$

where according to relation (64)

$$\bar{A}_3=\dot\theta\left(T_{ex}\right)/p\,,\ \bar{A}_4=\theta\left(T_{ex}\right)-\alpha\,. \tag{73}$$

Eqs. (72a,b) have the non-trivial solution

$$\bar{A}_3=-\bar{A}_4 \tag{74}$$

provided that $\tanh p\left(t-T_{ex}\right)=1$ occurring at large time. Eq. (74) due to relations (73) yields

$$\dot\theta\left(T_{ex}\right)+p\left(\theta\left(T_{ex}\right)-\alpha\right)=0 \tag{75}$$

where $\dot\theta\left(T_{ex}\right)$ and $\theta\left(T_{ex}\right)$ are determined from eqs. (57b) and (59b) which give

$$\theta(T_{ex})=A_3\sinh pT_{ex}+A_4\cosh pT_{ex}+\alpha \tag{76a}$$

$$\dot\theta(T_{ex})/p=A_3\cosh pT_{ex}+A_4\sinh pT_{ex}+\frac{\alpha\,\omega_p/p}{1+\omega_p^2/p^2}\cdot\frac{1}{\sin\psi} \tag{76b}$$

where A_3 and A_4 are taken from relations (58) after setting $\theta(0)=\theta_o=0$ and $\dot\theta(0)=\dot\theta_o=0$, i.e.

$$A_3=-\alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\,,\ A_4=-\frac{\alpha\left(2+\omega_p^2/p^2\right)}{1+\omega_p^2/p^2}\,. \tag{77}$$

Introducing expressions (76a,b) into eq(75) and using relations (77) we obtain the following result presented for the first time in the relevant literature,

$$\left(\frac{\omega_p}{p}+\frac{2p}{\omega_p}\right)\sin\psi+\cos\psi=e^{-\frac{(2\pi-\psi)}{\omega_p/p}}\,. \tag{78}$$

Eq. (78) yields the *minimum* amplitude ground acceleration, $\alpha_p/g\alpha=1/\sin\psi$, for overturning instability for the case of Fig. (**2b**). However, such a solution is not physically acceptable as it implies that Eq.72a is not valid yielding θ(t)<0 instead of θ(t)>0 according to case corresponding to Fig. (**2b**).

From both the above two cases it is concluded that for the one-sine ground excitation form only the case corresponding to Fig. (**2a**) is possible (i.e. eq. (71)) while that corresponding to Fig. (**2b**) (i.e. eq. (78)) is *physically unacceptable*. Note also that if the ground acceleration in eq. (52) is negative (i.e. $\ddot u_g=-\alpha_p\sin(\omega_p t+\psi)$) then eq. (71) is again valid (due to the symmetry of the block's in connection with the direction of the ground excitation). However, the important question which now arises is whether the case shown in Fig. (**2b**) may occur with a suddenly applied positive acceleration $\ddot u_g$ which decreases for t > 0. This case will be discussed at the end of this section in connection with the effect of initial conditions on the minimum amplitude acceleration.

Mode 2 (one impact)

Two cases are examined: in the 1st case impact occurs **before** the ground excitation expires (i.e. at $t_i<T_{ex}$) and in the 2nd case impact occurs **after** the excitation expires (i.e. at $t_i>T_{ex}$). However, in both cases the conditions of overturning instability (60) (occurring under the free vibrations regime) are still valid. Since we are looking for the minimum excitation amplitude, eq. (75) is also valid.

Case 1 ($t_i<T_{ex}$)

After one impact occurring before the excitation expires the block is rotating with $\theta(t)>0$. Hence, the equation of motion for $t_i\le t\le T_{ex}$ can be derived from equation (57b) and (59b), i.e.

$$\theta\left(t\right)=A_3\sinh p(t-t_i)+A_4\cosh p(t-t_i)+\frac{\alpha}{1+\omega_p{}^2/p^2}\frac{\sin\left(\omega_p t+\psi\right)}{\sin\psi}+\alpha \tag{79a}$$

$$\frac{\dot\theta(t)}{p}=A_3\cosh p\left(t-t_i\right)+A_4\sinh p\left(t-t_i\right)+\alpha\frac{\omega_p/p}{1+\omega_p{}^2/p^2}\frac{\cos\left(\omega_p t+\psi\right)}{\sin\psi}\,. \tag{79b}$$

For $t=T_{ex}$ eqs. (79a,b) become

$$\theta\left(T_{ex}\right)=A_3\sinh p(T_{ex}-t_i)+A_4\cosh p(T_{ex}-t_i)+\alpha \tag{80a}$$

$$\frac{\dot\theta(T_{ex})}{p}=A_3\cosh p(T_{ex}-t_i)+A_4\sinh p(T_{ex}-t_i)+\alpha\frac{\omega_p/p}{1+\omega_p{}^2/p^2}\frac{1}{\sin\psi} \tag{80b}$$

where A_3 and A_4 are determined from the conditions occurring at the instant, $t=t_i$, which implies $\theta(t_i)=0$ and $\dot\theta^{after}(t_i)=e\dot\theta^{before}(t_i)$, where $\dot\theta^{after}(t_i)=\dot\theta_+$, $\dot\theta^{before}(t_i)=\dot\theta_-$ and e is the *coefficient of restitution*. The last two initial conditions yield

$$A_4 = -\alpha - \frac{\alpha}{1+\omega_p^2/p^2}\frac{\sin(\omega_p t_i + \psi)}{\sin\psi} \quad (81)$$

and $$A_3 = \frac{e\dot{\theta}^{before}(t_i)}{p} - \alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\frac{\cos(\omega_p t_i+\psi)}{\sin\psi} \quad (82)$$

Subsequently, the expressions $\theta(T_{ex})-\alpha$ and $\dot{\theta}(T_{ex})/p$ taken from eqs. (80a,b) are introduced into eq. (75) which lead to

$$(A_3 + A_4)e^{p(T_{ex}-t_i)} = -\alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\frac{1}{\sin\psi} \quad (83)$$

where A_3 and A_4 are given in relations (81 & 82) as functions of $\dot{\theta}^{before}(t_i)$ and t_i (impact time) both of which will be determined from the previous *forced* motion regime for $t \le t_i$ associated with eqs. (57a) and (59a) from which for $\theta(t)<0$ we obtain

$$\theta(t_i) = A_1\sinh pt_i + A_2\cosh pt_i - \alpha + \frac{\alpha}{1+\omega_p^2/p^2}\frac{\sin(\omega_p t_i+\psi)}{\sin\psi} \quad (84)$$

$$\frac{\dot{\theta}^{before}(t_i)}{p} = A_1\cosh pt_i + A_2\sinh pt_i + \frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\frac{\cos(\omega_p t_i+\psi)}{\sin\psi} \quad (85)$$

where A_1 and A_2 are given in relation (70).

At the instant of impact, $\theta(t_i)=0$, which due to eq. (84) implies [4]

$$\tan\psi = \frac{\sin\omega_p t_i - \frac{\omega_p}{p}\sinh pt_i}{1+\frac{\omega_p^2}{p^2} - \frac{\omega_p^2}{p^2}\cosh pt_i - \cos\omega_p t_i}. \quad (86)$$

Eq. (83) by virtue of relations (81) and (82) is written as follows

$$\left(1+\frac{\omega_p^2}{p^2}\right)\left[\frac{e\dot{\theta}^{before}(t_i)}{\alpha p} - 1\right]\sin\psi - \frac{\omega_p}{p}\cos(\omega_p t_i+\psi)$$
$$-\sin(\omega_p t_i+\psi) = -\frac{\omega_p}{p}e^{-p(T_{ex}-t_i)} \quad (87a)$$

From eq. (85) after substituting A_1 and A_2, taken from relation (70), we obtain

$$\frac{\dot{\theta}^{before}(t_i)}{p} = \frac{\alpha\frac{\omega_p}{p}}{1+\frac{\omega_p^2}{p^2}}\left(\frac{\cos\omega_p t_i - \cosh pt_i}{\tan\psi} + \frac{\omega_p}{p}\sinh pt_i - \sin\omega_p t_i\right)$$
$$(87b)$$

Substituting the expression of $\dot{\theta}^{before}(t_i)$ from eq. (87b) into eq. (87a) we get

$$e(\cosh pt_i - \cos\omega_p t_i)\cos\psi +$$
$$\left[e\left(\sin\omega_p t_i - \frac{\omega_p}{p}\sinh pt_i\right) + \frac{p}{\omega_p} + \frac{\omega_p}{p}\right]\sin\psi +$$
$$+\cos(\omega_p t+\psi) + \frac{p}{\omega_p}\sin(\omega_p t+\psi) = e^{-p(Tex-t_i)}. \quad (88)$$

Solving eqs. (86&88) with respect to t_i and ψ, we determine the *minimum* amplitude acceleration $\alpha_p/g\alpha = 1/\sin\psi$. As it will be shown numerically eqs. (86&88) yield acceptable solutions for $t_i<T_{ex}$ and $0<\omega_p/p<4.8$. For $\omega_p/p>4.8$ eqs. (86&88) lead to physically *unacceptable* solutions corresponding to values of $t_i>T_{ex}$ which contradict the initial assumption ($t_i<T_{ex}$) used for their derivation.

Case 2 ($t_i > T_{ex}$)

Impact will occur during the free-vibration regime at some time $t=t_i$. Hence for $t_i \ge t \ge (2\pi-\psi)/\omega_p$ due to eqs. (65a,b) valid for $\theta(t)<0$ one can write the following equations

$$\theta(t) = \frac{\dot{\theta}(T_{ex})}{p}\sinh p(t-T_{ex}) + (\theta(T_{ex})+\alpha)\cosh p(t-T_{ex}) - \alpha \quad (89a)$$

$$\dot{\theta}^{before}(t)/p = \frac{\dot{\theta}(T_{ex})}{p}\cosh p(t-T_{ex}) + (\theta(T_{ex})+\alpha)\sinh p(t-T_{ex}) \quad (89b)$$

where $\dot{\theta}(T_{ex})$ and $\theta(T_{ex})$ are evaluated form eqs. (57a) and (59a) at $t=T_{ex}$. The resulting expressions after setting due to eq. (70)

$$A_1 = -\frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\cot\psi, \quad A_2 = \frac{\alpha\omega_p^2/p^2}{1+\omega_p^2/p^2} \quad (90)$$

are given by

$$\theta(T_{ex}) = -\frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\sinh pT_{ex} + \frac{\alpha\omega_p^2/p^2}{1+\omega_p^2/p^2}\cosh pT_{ex} - \alpha \quad (91a)$$

$$\dot{\theta}(T_{ex})/p = -\frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\cot\psi\cosh pT_{ex}$$
$$+\frac{\alpha\omega_p^2/p^2}{1+\omega_p^2/p^2}\sinh pT_{ex} + \frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\frac{1}{\sin\psi}. \quad (91b)$$

At the time of impact t_i, after taking into account that $\theta(t_i)=0$, eqs. (89a,b) by virtue of eqs. (91a,b) become

$$\frac{\omega_p/p}{1+\omega_p^2/p^2}\left[\left(\frac{\cosh pT_{ex}}{\tan\psi} - \frac{\omega_p}{p}\sinh pT_{ex} - \frac{1}{\sin\psi}\right)\sinh p(t_i-T_{ex}) + \right.$$
$$\left.\left(\frac{\sinh pT_{ex}}{\tan\psi} - \frac{\omega_p}{p}\cosh pT_{ex}\right)\cosh p(t_i-T_{ex})\right] = -1 \quad (92a)$$

and

$$\dot{\theta}^{before}(t_i) = \frac{\alpha\omega_p}{1+\omega_p^2/p^2}\left[\left(\frac{\omega_p}{p}\sinh pT_{ex} - \right.\right.$$

$$\left. - \frac{\cosh pT_{ex}}{\tan\psi} + \frac{1}{\sin\psi}\right)\cosh p\left(t_i - T_{ex}\right) +$$

$$\left. + \left(\frac{\omega_p}{p}\cosh pT_{ex} - \frac{\sinh pT_{ex}}{\tan\psi}\right)\sinh p\left(t_i - T_{ex}\right)\right] \qquad (92b)$$

The governing equation after the impact (i.e. for $t \geq t_i$ where $\theta(t) > 0$) is

$$\theta(t) = c_1 \sinh p\left(t - t_i\right) + c_2 \cosh p\left(t - t_i\right) + \alpha \qquad (93a)$$

$$\text{and } \frac{\dot{\theta}(t)}{p} = c_1 \cosh p\left(t - t_i\right) + c_2 \sinh p\left(t - t_i\right). \qquad (93b)$$

From the initial condition $\theta(t_i) = 0$ we get $c_2 = -\alpha$, while $c_1 = \dot{\theta}^{after}(t_i)/p = e\dot{\theta}^{before}(t_i)/p$. Introduction of the expressions of c_1 and c_2 into eqs. (93a,b) and application of the overturning conditions (60) at time t^* implying $\theta(t^*) = \alpha$, $\dot{\theta}(t^*) = 0$, we obtain

$$\frac{\dot{\theta}(t_i)}{p}\sinh p\left(t^* - t_i\right) - \alpha\cosh p\left(t^* - t_i\right) = 0 \qquad (94a)$$

$$\frac{\dot{\theta}(t_i)}{p}\cosh p\left(t^* - t_i\right) - \alpha\sinh p\left(t^* - t_i\right) = 0\,. \qquad (94b)$$

Elimination of $\sinh p\left(t^* - t_i\right)$ and $\cosh p\left(t^* - t_i\right)$ yields

$$\dot{\theta}^{after}(t_i) - p\alpha = 0$$

$$\text{and } e\dot{\theta}^{before}(t_i) - p\alpha = 0 \qquad (95)$$

where $\dot{\theta}^{before}(t_i)$ is taken from eq. (92b).

Combining eqs. (92b) and (95) we obtain

$$\frac{e\omega_p/p}{1+\omega_p^2/p^2}\left[\left(\frac{\omega_p}{p}\sinh pT_{ex} - \frac{\cosh pT_{ex}}{\tan\psi} + \frac{1}{\sin\psi}\right)\cosh p\left(t_i - T_{ex}\right) + \right.$$

$$\left. + \left(\frac{\omega_p}{p}\cosh pT_{ex} - \frac{\sinh pT_{ex}}{\tan\psi}\right)\sinh p\left(t_i - T_{ex}\right)\right] = 1 \qquad (96)$$

Eqs. (92a) and (96) can be solved with respect to ψ and t_i leading to the *minimum* amplitude acceleration for overturning instability, i.e. $\alpha_p/\alpha g = 1/\sin\psi$. Note that although in the above four cases the *dimensional* amplitude α_p depends on the angle α, the corresponding curves $\alpha_p/\alpha g$ *versus* ω_p/p are independent of α.

The Effect of Initial Conditions (Fig. 2b)

The occurrence of the case shown in Fig. (**2b**) will be discussed for a suddenly applied *positive* but *decreasing* ground acceleration, $\ddot{u}_g(t)$, in connection with the effect of initial (nontrivial) conditions $\theta(0) = \theta_0 \neq 0$, $\dot{\theta}(0) = \dot{\theta}_0 \neq 0$. For instance, one may assume the ground acceleration form $\ddot{u}_g(t) = \alpha_p \sin(\omega_p t + \pi/2 + \psi) = \alpha_p \cos(\omega_p t + \psi)$ which for $t=0$ yields: $\ddot{u}_g(0) = \alpha_p \cos\psi = \lambda\alpha_p$ where $\psi = \cos^{-1}\lambda$, $\alpha_p = g\tan\alpha/\cos\psi$ and $\alpha_p/g\tan\alpha = 1/\cos\psi$. Then eq(56b) becomes:

$$\ddot{\theta} - p^2\theta = -p^2\alpha - p^2\alpha\frac{\cos\left(\omega_p t + \psi\right)}{\cos\psi} \qquad (97)$$

whose integral is

$$\theta(t) = A_3 \sinh pt + A_4 \cosh pt + \frac{\alpha}{1+\omega_p^2/p^2}\frac{\cos(\omega_p t + \psi)}{\cos\psi} + \alpha \qquad (98)$$

$$\text{and } \frac{\dot{\theta}(t)}{p} = A_3 \cosh pt + A_4 \sinh pt - \frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\frac{\sin(\omega_p t + \psi)}{\cos\psi} \qquad (99)$$

where

$$A_3 = \frac{\dot{\theta}_0}{p} + \alpha\frac{\omega_p/p}{1+\omega_p^2/p^2}\tan\psi\,,\ A_4 = \theta_0 - \alpha\frac{2+\omega_p^2/p^2}{1+\omega_p^2/p^2} \qquad (100)$$

By virtue of eqs. (98&99) and taking into account that $T_{ex} = (3\pi/2 - \psi)/\omega_p$ we obtain

$$\theta(T_{ex}) = A_3 \sinh pT_{ex} + A_4 \cosh pT_{ex} + \alpha \qquad (101a)$$

$$\text{and } \frac{\dot{\theta}(T_{ex})}{p} = A_3 \cosh pT_{ex} + A_4 \sinh pT_{ex} + \frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\frac{1}{\cos\psi} \qquad (101b)$$

Introducing the above expressions of $\theta(T_{ex})$ and $\dot{\theta}(T_{ex})$ into the condition of overturning instability, eq. (75), we get

$$A_3 + A_4 = -\frac{\alpha\omega_p/p}{1+\omega_p^2/p^2}\frac{1}{\cos\psi}e^{-pT_{ex}} \qquad (102)$$

Eq. (102) using relations (100) yields

$$\left[2\frac{p}{\omega_p} + \frac{\omega_p}{p} - \frac{1}{\alpha}\left(\theta_o + \frac{\dot{\theta}_o}{p}\right)\left(\frac{p}{\omega_p} + \frac{\omega_p}{p}\right)\right]\cos\psi - \sin\psi = e^{-pT_{ex}} \qquad (103)$$

Eq. (103) can be solved with respect to ψ as function of ω_p/p for given values of θ_o and $\dot{\theta}_o$.

For the sake of comparison of eq. (103) with eq. (71) the last one, in order to take into account non-trivial initial condition $(\theta_o \neq 0$ and $\dot{\theta}_o \neq 0)$, is written as follows

$$\cos\psi - \left[\frac{\omega_p}{p} + \frac{1}{\alpha}\left(\theta_o + \frac{\dot{\theta}_o}{p}\right)\left(\frac{p}{\omega_p} + \frac{\omega_p}{p}\right)\right]\sin\psi = e^{-pT_{ex}} \qquad (104)$$

In case that θ_o and $\dot{\theta}_o$ are negative eq. (104) leads to a lower curve in the *versus* ω_p/p (than that of trivial initial conditions) which represents the *detrimental* effect of negative initial conditions as is shown below.

4. NUMERICAL RESULTS

Numerical results are presented in both *tabular* and *graphical* form in time series and phase plane portraits [12]. *Linearized* dynamic solutions are compared with their corresponding non-linear solutions. First we consider **mode 1** (no impact) based on eq.(71) for a one-sinus pulse (either positive or negative). In this respect Figs. (**8a** and **8b**) show time series (θ and $\dot{\theta}$ *versus* τ = pt, where τ is dimensionless time) for a block with p=2.14, α=0.25 and ω_p/p=2. Fig. (**8c**) presents more clearly the conditions for overturning instability ($\theta=\alpha$, $\dot{\theta}=0$) for the same block in terms of the phase-plane portrait. The *continuous* curve depicted in Figs. (**8a** to **8c**) corresponds to α_p = 2.35766gα (no overturning), whereas the *discontinuous* curve corresponds to overturning instability with α_p=2.35771gα. Similar results are shown in Figs. (**9a**) to 9c for this block when ω_p/p =4. Apparently, the block returns to its initial equilibrium position for α_p=5.3230gα, while overturning instability occurs for α_p=5.32303gα. Fig. (**10**) shows that the *minimum* amplitude acceleration for overturning instability for 0<ω_p/p<10 corresponds to **mode 2** (one impact) associated with eqs(86&88) when 0<ω_p/p<4.8 and eqs(92a&96) for 4.8<ω_p/p<6.7. Such a diagram coincides with that presented by Zhang and Makris (2001). Note that in Fig. (**10**) two physically unacceptable curves are also depicted based on eq. (78) (no impact) and eqs(86&88) (one impact) which although analytically derived in earlier work [4] have not been presented in graphical form. Eq. (78) is physically *unacceptable* since – as explained in Section 3 – yields θ(t)<0 instead of θ(t)>0 according to the case under discussion corresponding to Fig. (**2b**). Eqs(86&88) (one impact) lead to the above *unacceptable* solutions presented in graphical form for (t_i>T_{ex}) which contradict the initial assumption (i.e. t_i<T_{ex}).

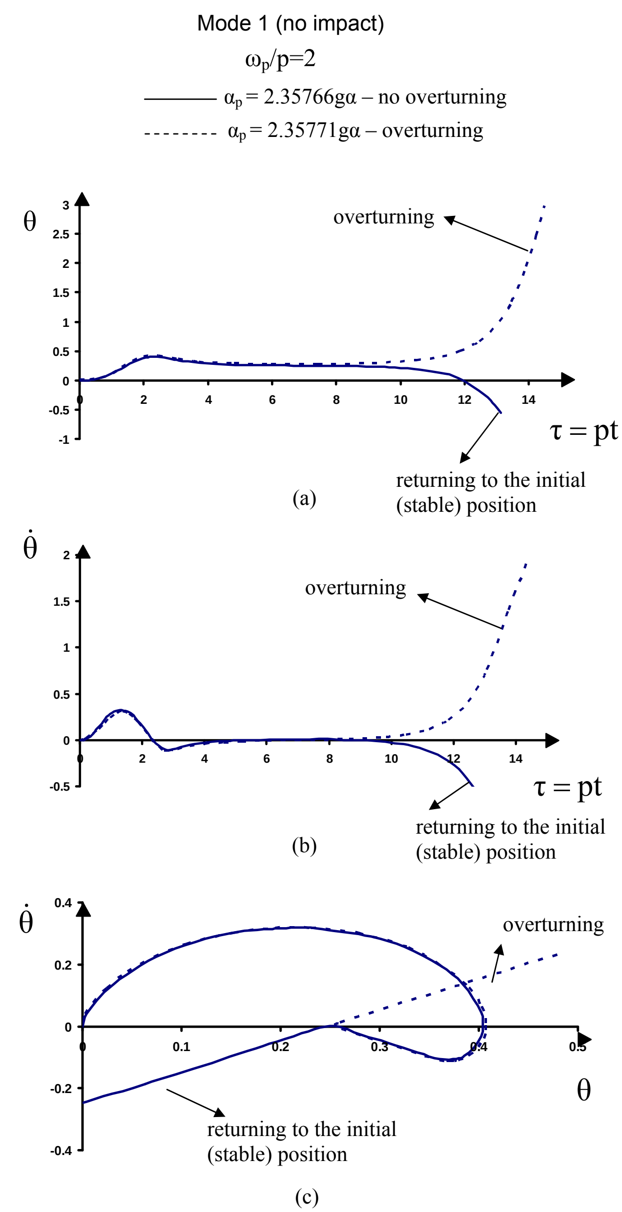


Fig. (8). Linearized solutions predicting the response of a rigid block (with p=2.14, α=0.25 and ω_p/p=2) under one-sine pulse ground excitation presented as: (**a**) θ *versus* τ=pt, (**b**) $\dot{\theta}$ *versus* τ=pt (where τ is dimensionless time) and (**c**) phase-plane portrait $\dot{\theta}$ *versus* θ.

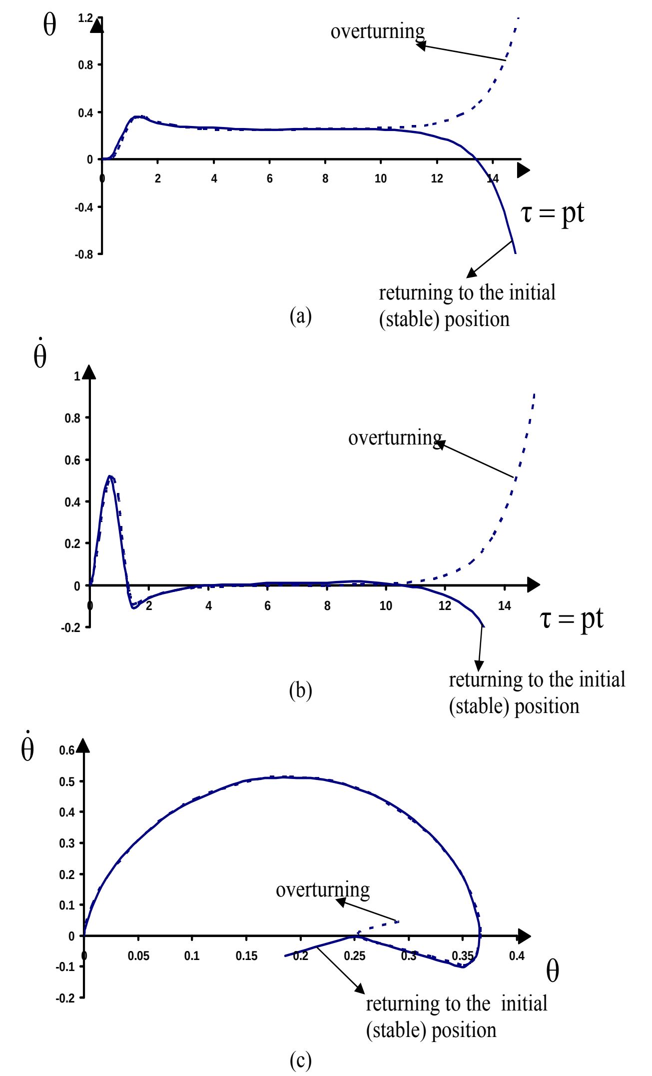


Fig. (9). Linearized solutions predicting the response of a rigid block (with p=2.14, α=0.25 and ω_p/p=4) under one-sine pulse ground excitation presented as: (**a**) θ *versus* τ=pt, (**b**) $\dot{\theta}$ *versus* τ=pt (where τ is dimensionless time) and (**c**) phase-plane portrait $\dot{\theta}$ *versus* θ.

In Fig. (**11**) the *linearized* solutions of Fig. (**10**) (minimum α_p / gα *versus* ω_p/p) are compared with the *nonlinear*

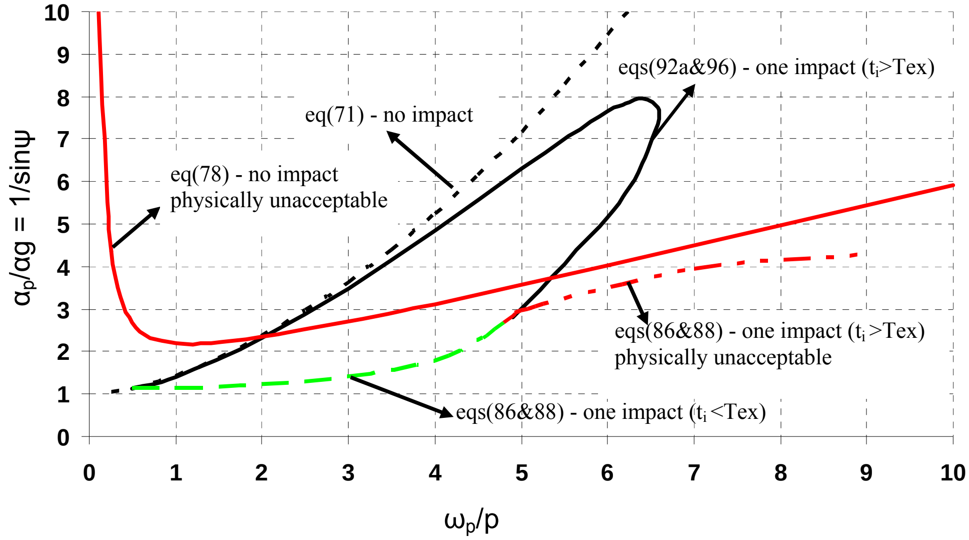


Fig. (10). The predicted values of minimum amplitude acceleration for overturning instability, $\alpha_p/\alpha g$ *versus* ω_p/p.

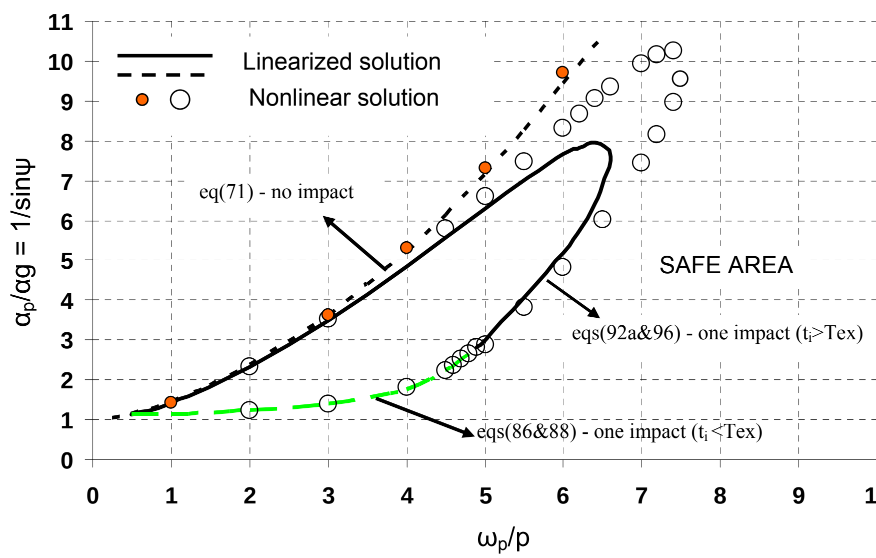


Fig. (11). Comparison of predicted values of minimum amplitude acceleration for overturning instability as obtained from the linearized solution and nonlinear analysis.

ones based on eqs. (21a,b) (solved numerically using Mathematica 6) being in good agreement.

Figs. (**12a,b,c**) show in terms of times series ($\theta(\tau)$ and $\dot{\theta}(\tau)$ *versus* τ) and phase-plane portrait ($\theta(\tau)$ *versus* $\dot{\theta}(\tau)$) the *overturning instability* corresponds to **Mode 2** (one impact) for the same rigid block with $\omega_p/p=3$ [eqs(86&88), $t_i<T_{ex}$]. Fig. (**12c**) illustrates the satisfaction of the overturning instability condition (i.e. $\theta=\alpha$, $\dot{\theta}=0$). Note that the *minimum* amplitude acceleration leading to overturning instability is $\alpha_p/\alpha g$ =0.71157, while for $\alpha_p/\alpha g$ = 0.71156 the block does not overturn but returns to its initial stable equilibrium position. Similar predictions to those presented in Fig. (**12**) are presented in Fig. (**13**) for the same rigid block but for ω_p/p=5.5 [eqs. (92a&96), $t_i>T_{ex}$].

Overturning instability associated with the response shown in Fig. (**2b**) is not possible to occur in the case of a suddenly applied positive (either increasing or decreasing) one-sinus or one-cosinus pulse in connection with trivial initial conditions. However, such a type of overturning instability may occur in cases of non-trivial initial conditions (θ_o=0.01, $\dot{\theta}_o \le 0.2$) in connection with a suddenly applied positive but decreasing one-cosine pulse. Thus, one may assume a small initial imperfection θ_o<0.01rad and a small initial angular velocity $\dot{\theta}_o \le 0.2$ rad/sec. Fig. (**14**) provides the minimum amplitude acceleration ($\alpha_p/\alpha g=1/\cos\psi$ *versus* ω_p/p) corresponding to Fig. (**2b**) for the case of a ground acceleration $\ddot{u}_g(t)=\alpha_p\cos(\omega_p t+\psi)$. In such a case (Mode

1, no impact) eq. (103) is more unfavorable compared to the predictions of eq. (71) and eq. (104) associated with $\ddot{u}_g(t) = \alpha_p \sin\left(\omega_p t + \psi\right)$.

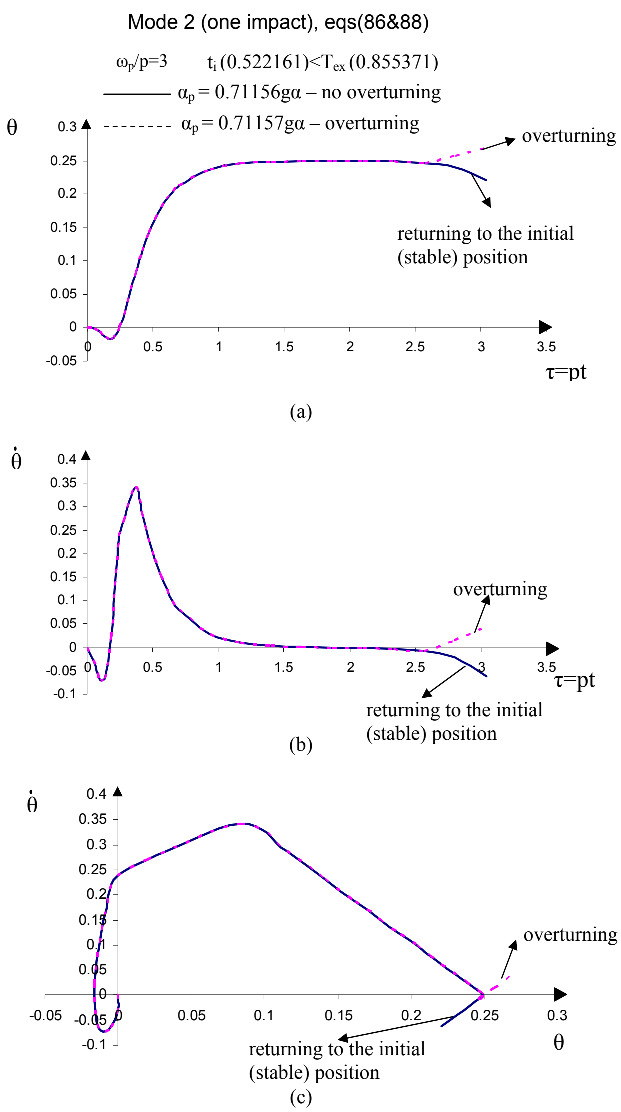


Fig. (12). Linearized solutions predicting the response of a rigid block (with p=2.14, α=0.25 and $\omega_p/p=3$) under one-sine pulse ground excitation presented as: (**a**) θ *versus* τ=pt, (**b**) $\dot{\theta}$ *versus* τ=pt (where τ ιs dimensionless time) and (**c**) phase-plane portrait $\dot{\theta}$ *versus* θ.

From all the above numerical results the most unfavorable (smallest) minimum amplitude ground acceleration (corresponding to $\omega_p/p=1$) is equal to $\alpha_p=\alpha g$ and (since $1/\sin\psi=1$) hence for α=0.25, α_p = 0.25g. This corresponds to a period of ground acceleration T_{ex}=T=(2π-ψ)/ω_p= 2.201sec. For $\omega_p/p=4$ we get $\alpha_p=2\alpha g=0.5g$ with corresponding period T=(2π-ψ)/ω_p=0.672sec. Both cases indicate in general safety against earthquake according to typical acceleration response spectra.The various curves shown in Fig. (**14**) can also be presented in terms of α_p/g *versus* the period of forcing excitation, T_{ex}=T. The corresponding curves α_p/g *versus* T are shown in Fig. (**15**). In order to investigate whether the maximum amplitude of ground acceleration are safe against regional earthquake hazard we will compare the lower curves of Fig. (**15**) with those of standard design codes for two types of soil foundation, type and A (rock or other rock-like geological formation) and B (deposits of very dense sand, gravel, or very stiff clay). From Figs. (**16a**, **b**) one can see that depicted values of *minimum* amplitude acceleration are *higher* than the corresponding values obtained from the response spectra of EC8 for soil foundation Type A and both cases of damping, ξ=3% and ξ=5%.

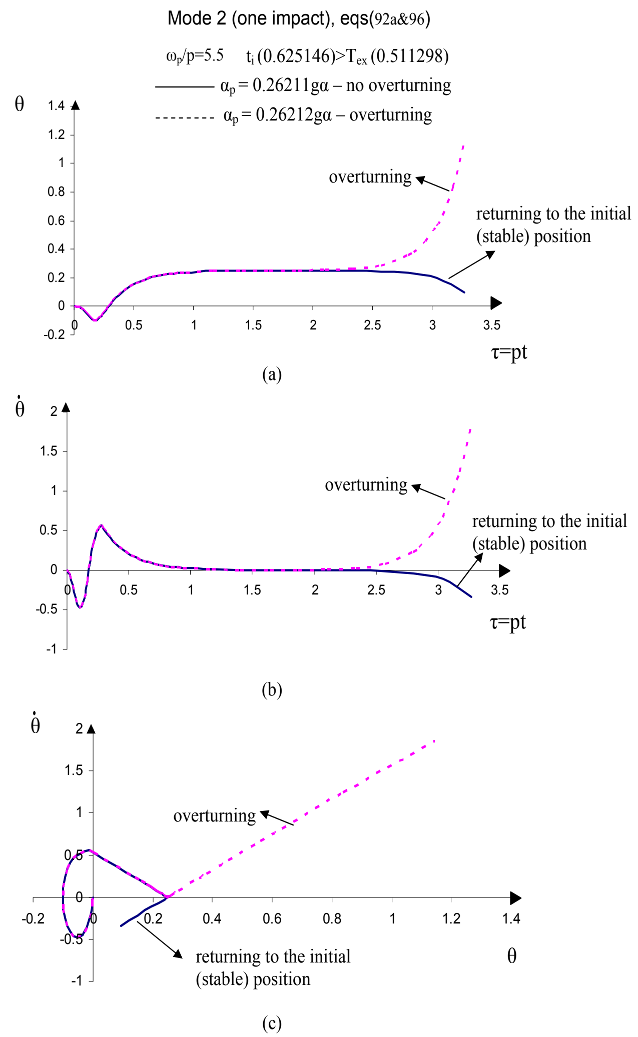


Fig. (13). Linearized solutions predicting the response of a rigid block with p=2.14, α=0.25 and $\omega_p/p=5.5$) under one-sine pulse ground excitation presented as: (**a**) θ *versus* τ=pt, (**b**) $\dot{\theta}$ *versus* τ=pt (where τ ιs dimensionless time) and (**c**) phase-plane portrait $\dot{\theta}$ *versus* θ.

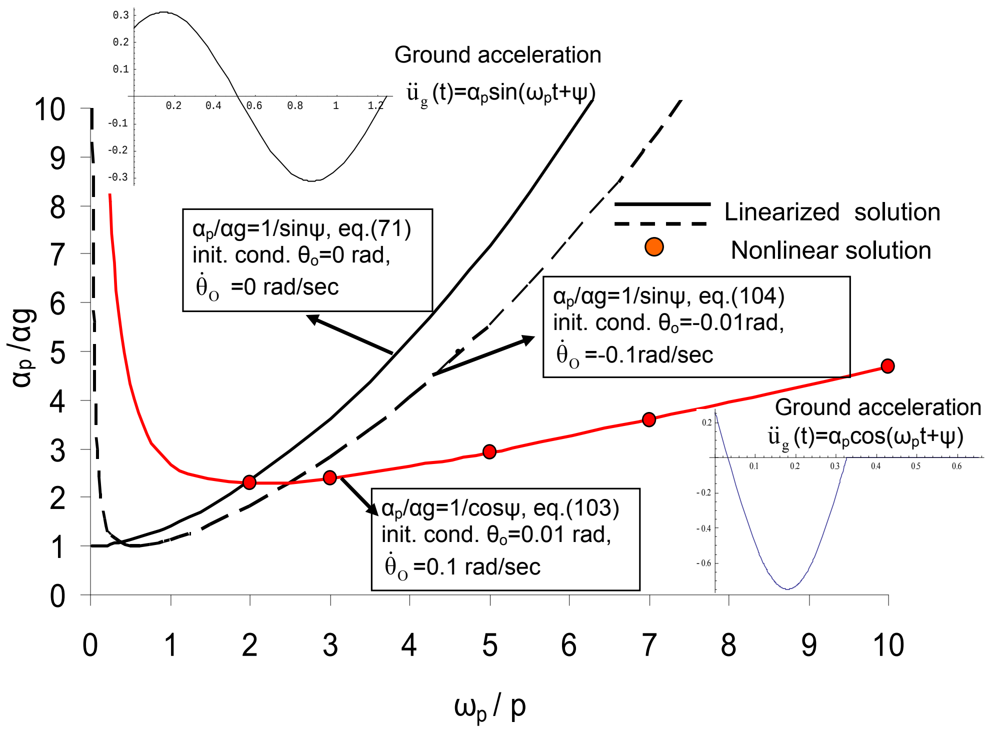


Fig. (14). The predicted values of minimum amplitude acceleration for overturning instability, $\alpha_p/\alpha g$ *versus* ω_p/p, for the trivial initial conditions $\theta_o = \dot{\theta}_o = 0$ and the nontrivial ones: $\theta_o = \pm 0.01$ rad, $\pm\dot{\theta}_o = 0.1$ rad/sec.

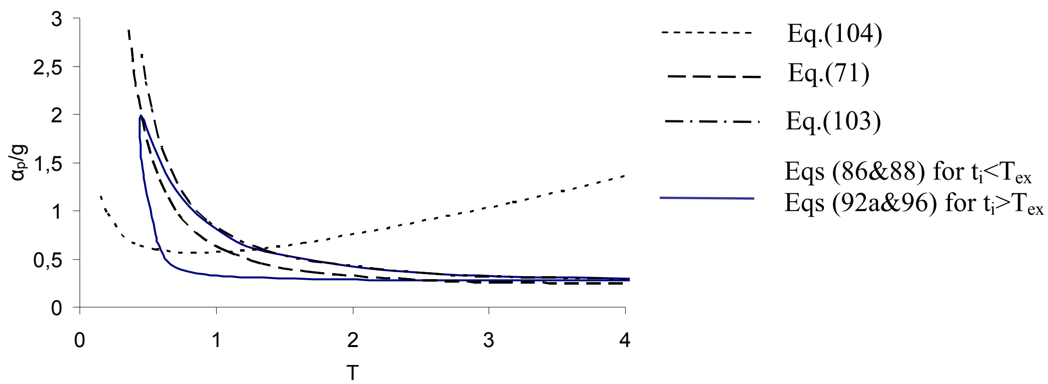


Fig. (15). Curves of minimum amplitude ground acceleration α_p/g *versus* T (period of forcing function).

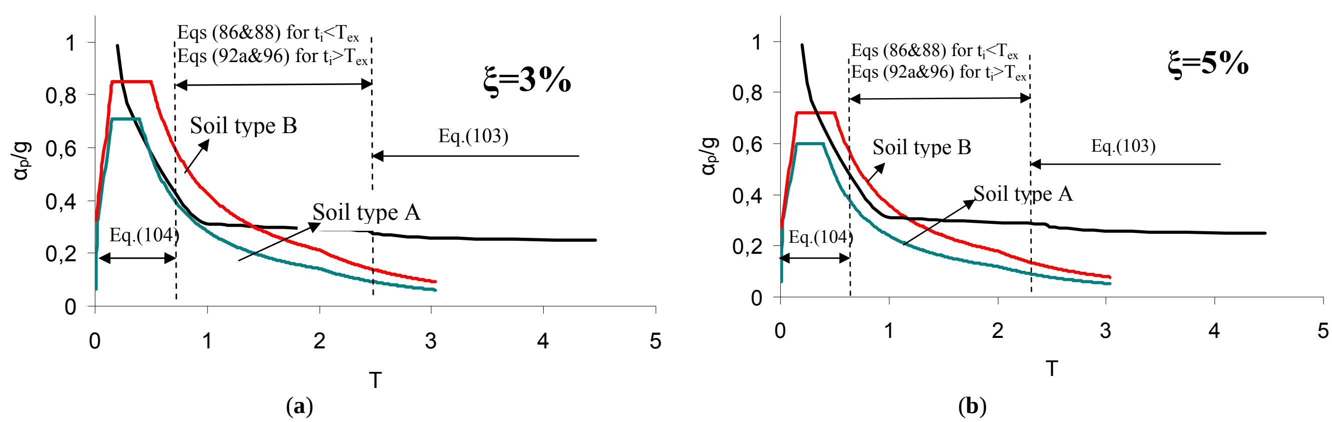


Fig. (16). Comparison of minimum amplitude acceleration α_p/g *versus* T (period of forcing function) with the corresponding predictions of EC8 for two types of soil foundation (A and B) and damping: (**a**) ξ=3% and (**b**) ξ=5%.

CONCLUSIONS

In this study dealing with the *overturning instability* of a free-standing rigid block under ground excitation and assuming that there is no sliding due to large friction, the following conclusions are worth mentioning:

1. The governing equations of rocking motion of the rigid block are comprehensively derived using energy considerations.
2. The condition of overturning instability occurring *via* the unstable equilibrium of the block is properly established by considering that such a critical state is related to the *minimum* amplitude ground acceleration. If this is so, one can assume that the block oscillates for a short time about the unstable equilibrium implying, thus, zero angular velocity.
3. A detailed linearized analysis under a one-sine pulse ground excitation either without impact or with one impact (occurring either before or after the ground acceleration expires) facilitates to understand the *mechanism* of such a type of instability. The comparison of the above results with those obtained by nonlinear dynamic analyses shows the regions of agreement and disagreement between linearized and nonlinear analyses.
4. New interesting, for structural design purposes, analytical and graphical results yielding the *minimum* amplitude ground acceleration are assessed covering various cases of overturning instability under a one-sine pulse. According to these results the *safe area* in the diagram $\alpha_p/g\alpha$ *versus* ω_p/p for the case shown in Fig. (**2a**) [either with mode 1 (no impact) or with mode 2 (one impact)] coincides with that presented in previous work [6]. The solution for overturning instability according to Fig. (**2b**) is physically *unacceptable* for a suddenly applied positive acceleration either of one-sine or one-cosine pulse. However, such a type of overturning instability may occur in case of non trivial initial conditions under a suddenly applied *positive* but *decreasing* one-cosine pulse.
5. Time series and mainly phase-plane portraits illustrate the rocking response and the subsequent overturning instability criterion of the block.
6. The effect of initial conditions on the minimum amplitude of ground acceleration in connection with a one-cosine pulse is discussed. Such a case may lead the block to overturning instability according to Fig. (**2b**).
7. The detrimental effect of initial conditions on the minimum amplitude acceleration in connection with a one-sine pulse is also assessed.
8. From a comparison of the last two cases one can conclude that the one cosine pulse leads to much more unfavorable results than those of the one-sinus pulse.
9. From all the above numerical results, the most unfavorable regarding the minimum amplitude ground acceleration indicate safety against earthquakes according to the predictions of the response spectra of Standard EC8 for both damping cases ξ = 3% and 5% for type A of soil foundation. However, this magnitude of α_p should be substantially reduced if the rigid block were supported on the top surface of a multi-drum column yielding loss of energy due to sliding and impact between drums. This implies seismic protection excluding overturning instability for the above types of ground excitation and both types of soil foundation and damping.

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