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An Adaptive Least-Squares Mixed Finite Element Method for Fourth-Order Elliptic Equations

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Abstract: A least-squares mixed finite element (LSMFE) method for the numerical solution of fourth-order elliptic equations is analyzed and developed in this paper. The *a posteriori* error estimator which is needed in the adaptive refinement algorithm is proposed. The local evaluation of the least-squares functional serves as *a posteriori* error estimator. The posteriori errors are effectively estimated.

Keywords: least-squares mixed finite element method, fourth-order elliptic equations, least-squares functional, *a posteriori* error.

I. INTRODUCTION

A general theory of the least-squares method has been developed by A K Aziz, R B Kellogg and A B Stephens in [1]. The most important advantage of the least-squares method leads to a symmetric positive definite problem. The least-squares mixed finite element method approaches a least-squares residual minimization is introduced. This method has an advantage which is not subject to the LBB condition [2]. Finite element methods of least-squares type have been studying in many fields recently (see, e.g., Stokes equation [2], Elliptic problem [3], Newtonian fluid flow problem [4], Transmission problems [5].

An adaptive least-squares mixed finite element method has been studied (see, e.g., the linear elasticity [6]). But the research about fourth-order elliptic equations which are widely used in hydrodynamics is not common. This paper mainly puts emphasis on an adaptive least-squares mixed finite element method for fourth-order elliptic equations. Our emphasis in this paper is on the performance of an adaptive refinement strategy based on the *a posteriori* error estimator inherent in the least-squares formulation by the local evaluation of the functional.

This paper is organized as follows. The least-squares formulation of the fourth-order elliptic equations is described in Section 2. It includes the coercivity properties of the least-squares variational formulation. Appropriate spaces for the finite element approximation and a generalization of the coercivity are shown in Section 2 to the discrete form is discussed in Section 3. The error estimates of the fourth-order elliptic equations are derived in Section 4. In Section 5, *a posteriori* error estimators which are needed in an adaptive refinement algorithm are composed with the least-squares functional, and the posteriori errors are effectively estimated. Finally, we summarize our findings and present conclusions in Section 6. In this paper, we define c to be a generic positive constant, ε be a generic small positive constant.

II. A LEAST-SQUARES FORMULATION OF FOURTH-ORDER ELLIPTIC EQUATIONS

We start from the equations of fourth-order elliptic in the form [7]:

$$\Delta^2 u = f \quad in\,\Omega, \tag{1}$$

$$u = 0 \quad \text{on}\,\partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on}\,\partial\Omega, \tag{3}$$

where $\Omega \subset R^n$ is a bounded domain, with boundary $\partial\Omega$. We shall consider an adaptive least-squares mixed finite element method for (1)-(3).

Now we set $\Delta u = -\sigma$, then, we have:

$$-\Delta\sigma = f \quad in\,\Omega, \tag{4}$$

$$\Delta u + \sigma = 0 \quad in\,\Omega, \tag{5}$$

$$u = 0 \quad \text{on}\,\partial\Omega, \tag{6}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on}\,\partial\Omega, \tag{7}$$

We introduce the Sobolev spaces:

$$H^1(\Omega) = \{p \in L^2(\Omega) : \nabla p \in L^2(\Omega)\},$$

$$H_0^m(\Omega) = \{v \in H^m(\Omega) : D^\alpha v\,|_{\partial\Omega} = 0, |\,\alpha\,| < m\}.$$

Now, let us define the least-squares problem: find $(\sigma, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$J(\sigma, u) = \inf_{q \in H^1(\Omega), v \in H_0^1(\Omega)} J(q, v), \tag{8}$$

where

$$J(q, v) = (\Delta q + f, \Delta q + f)_{0,\Omega} + (\Delta v + q, \Delta v + q)_{0,\Omega}. \tag{9}$$

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We introduce the least-squares functional:

$$F(\sigma,u)=\|\Delta\sigma+f\|_{0,\Omega}^2+\|\Delta u+\sigma\|_{0,\Omega}^2. \tag{10}$$

Taking variations in (9) with respect to q and v, the weak statement becomes : find

$(\sigma,u)\in H^1(\Omega)\times H_0^1(\Omega)$ such that:

$$B(\sigma,u;q,v)=-(f,\Delta v),\ \ (\forall v\in H_0^1(\Omega),\forall q\in H^1(\Omega)), \tag{11}$$

where

$$B(\sigma,u;q,v)=(\Delta\sigma,\Delta q)_{0,\Omega}+(\Delta u+\sigma,\Delta v+q)_{0,\Omega}. \tag{12}$$

Theorem 2.1. The bilinear form $B(\cdot,\cdot;\cdot,\cdot)$ is continuous and coercive. In other words, there exist positive constants α and β, such that

$$\begin{aligned}B(\sigma,u;q,v)\le\beta(\|\Delta\sigma\|_{0,\Omega}^2+\|\sigma\|_{0,\Omega}^2+\|\Delta u\|_{0,\Omega}^2)^{\frac{1}{2}}\\ (\|\Delta q\|_{0,\Omega}^2+\|q\|_{0,\Omega}^2+\|\Delta v\|_{0,\Omega}^2)^{\frac{1}{2}},\end{aligned} \tag{13}$$

$$B(q,v;q,v)\ge\alpha(\|\Delta q\|_{0,\Omega}^2+\|q\|_{0,\Omega}^2+\|\Delta v\|_{0,\Omega}^2), \tag{14}$$

holds for all $(\sigma,u),(q,v)\in H^1(\Omega)\times H_0^1(\Omega)$.

Proof: i) For the upper bound we have:

$$\begin{aligned}B(q,v;q,v)&=(\Delta q,\Delta q)_{0,\Omega}+(\Delta v+q,\Delta v+q)_{0,\Omega}\\&=\|\Delta q\|_{0,\Omega}^2+\|q+\Delta v\|_{0,\Omega}^2\\&\le C(\|\Delta q\|_{0,\Omega}^2+\|q\|_{0,\Omega}^2+\|\Delta v\|_{0,\Omega}^2).\end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

ii) For the lower bound.

$$\begin{aligned}B(q,v;q,v)&=(\Delta q,\Delta q)_{0,\Omega}+(\Delta v+q,\Delta v+q)_{0,\Omega}\\&=(\Delta q,\Delta q)_{0,\Omega}+(\Delta v,\Delta v)_{0,\Omega}+(q,q)_{0,\Omega}+2(\Delta v,q)_{0,\Omega}\\&\ge(\Delta q,\Delta q)_{0,\Omega}+(\Delta v,\Delta v)_{0,\Omega}+(q,q)_{0,\Omega}-2\varepsilon(\Delta v,q)_{0,\Omega}\\&\ge\|\Delta q\|_{0,\Omega}^2+\|q\|_{0,\Omega}^2+\|\Delta v\|_{0,\Omega}^2-\varepsilon(\delta\|\Delta v\|_{0,\Omega}^2+\frac{\|q\|_{0,\Omega}^2}{\delta})\\&=\|\Delta q\|_{0,\Omega}^2+(1-\frac{\varepsilon}{\delta})\|q\|_{0,\Omega}^2+(1-\varepsilon\delta)\|\Delta v\|_{0,\Omega}^2,\end{aligned}$$

So, we can select the positive constants ε and δ, satisfying

$$1-\varepsilon\delta>0,1-\frac{\varepsilon}{\delta}>0.$$

So we obtain

$$B(q,v;q,v)\ge\alpha(\|\Delta q\|_{0,\Omega}^2+\|q\|_{0,\Omega}^2+\|\Delta v\|_{0,\Omega}^2).$$

Then, we complete the proof.

Theorem 2.2. Let $f\in H^{-1}(\Omega)$. Then, (8) has a unique solution, and the solution is $(\sigma,u)\in H^1(\Omega)\times H_0^1(\Omega)$.

Proof: From Theorem2.1, we know that the bilinear form $B(\cdot,\cdot;\cdot,\cdot)$ is coercive and bounded on $H^1(\Omega)\times H_0^1(\Omega)$. Then the result follows from Lax-Milgram theorem.

III. FINITE ELEMENT APPROXIMATION

In principle, the least-squares mixed finite element approach simply consists of minimizing (10) in finite-dimensional subspaces $H_h(\Omega)\subset H^1(\Omega)$ and $M_h(\Omega)\subset H_0^1(\Omega)$. Suitable spaces are based on a triangulation T_h of Ω and consist of piecewise polynomials with sufficient continuity conditions.

Let T_h be a class qusi-uniform regular partition of Ω.

$$H_h(\Omega)=span\{\Phi(\cdot-X_1),\cdots,\Phi(\cdot-X_N)\}+P_m^d \tag{15}$$

where $\Phi:R^d\to R$ is a radial basis function, P_m^d denotes the space of polynomials of degree less than m and $X=(X_1,\cdots,X_N)\subseteq\Omega$ is a set of distinct nodes.

Consider Φ whose Fourier transform $\hat{\Phi}$ has the property in [8]:

$$C_1(1+\|\omega\|)^{-2\varsigma}\le\hat{\Phi}\le C_2(1+\|\omega\|)^{-2\varsigma}, \tag{16}$$

with positive constants C_1 and C_2.

The least-squares functional:

$$F_h(\sigma,u)=\sum_{T\in T_h}(\|\Delta\sigma+f\|_{0,T}^2+\|\Delta u+\sigma\|_{0,T}^2). \tag{17}$$

Minimizing the functional (17) is equivalent to the following variational problem: find $\sigma_h\in H_h$ and $u_h\in M_h$ such that

$$B_h(\sigma_h,u_h;q,v)=-(f,\Delta v), \tag{18}$$

holds for all $(q,v)\in H_h(\Omega)\times M_h(\Omega)$.

The discrete bilinear form $B_h(\cdot,\cdot;\cdot,\cdot)$ is defined as follows:

$$B_h(\sigma_h,u_h;q,v)=\sum_{T\in T_h}((\Delta\sigma_h,\Delta q)_{0,T}+(\Delta u_h+\sigma_h,\Delta v+q)_{0,T}), \tag{19}$$

where

$(\sigma_h,u_h)\in H_h(\Omega)\times M_h(\Omega)$ **,** $(q,v)\in H_h(\Omega)\times M_h(\Omega)$.

Theorem 3.1. The bilinear $B_h(\cdot,\cdot;\cdot,\cdot)$ is continuous and coercive, i.e. there exist positive constants α_h and β_h such that

$$\begin{aligned}B(\sigma_h,u_h;q,v)\le\beta_h(\sum_{T\in T_h}(\|\Delta\sigma_h\|_{0,T}^2+\|\sigma_h\|_{0,T}^2+\|\Delta u_h\|_{0,T}^2))^{\frac{1}{2}}\\ (\sum_{T\in T_h}(\|\Delta q\|_{0,T}^2+\|q\|_{0,T}^2+\|\Delta v\|_{0,T}^2))^{\frac{1}{2}},\end{aligned} \tag{20}$$

$$B(q,v;q,v)\ge\alpha_h\sum_{T\in T_h}(\|\Delta q\|_{0,T}^2+\|q\|_{0,T}^2+\|\Delta v\|_{0,T}^2), \tag{21}$$

which holds for all

$(\sigma_h, u_h) \in H_h(\Omega) \times M_h(\Omega)$, $(q, v) \in H_h(\Omega) \times M_h(\Omega)$.

Proof: i) For the upper bound we have

$$\begin{aligned} B_h(q,v;q,v) &= \sum_{T\in T_h}((\Delta q, \Delta q)_{0,T} + (\Delta v + q, \Delta v + q)_{0,T}) \\ &= \sum_{T\in T_h}(\|\Delta q\|^2_{0,T} + \|q+\Delta v\|^2_{0,T}) \\ &\le C\sum_{T\in T_h}(\|\Delta q\|^2_{0,T} + \|q\|^2_{0,T} + \|\Delta v\|^2_{0,T}). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 3.1.

ii) For the lower bound,

$$\begin{aligned} B_h(q,v;q,v) &= \sum_{T\in T_h}((\Delta q,\Delta q)_{0,T} + (\Delta v+q,\Delta v+q)_{0,T}) \\ &= \sum_{T\in T_h}((\Delta q,\Delta q)_{0,T} + (\Delta v,\Delta v)_{0,T} + (q,q)_{0,T} + 2(\Delta v,q)_{0,T}) \\ &\ge \sum_{T\in T_h}((\Delta q,\Delta q)_{0,T} + (\Delta v,\Delta v)_{0,T} + (q,q)_{0,T} - 2\varepsilon_1(\Delta v,q)_{0,T}) \\ &\ge \sum_{T\in T_h}(\|\Delta q\|^2_{0,T} + \|q\|^2_{0,T} + \|\Delta v\|^2_{0,T} - \varepsilon_1(\delta_1\|\Delta v\|^2_{0,T} + \frac{\|q\|^2_{0,T}}{\delta_1})) \\ &= \sum_{T\in T_h}(\|\Delta q\|^2_{0,T} + (1-\varepsilon_1\delta_1)\|\Delta v\|^2_{0,T} + (1-\frac{\varepsilon_1}{\delta_1})\|q\|^2_{0,T}). \end{aligned}$$

So, we can select the positive constants ε_1 and δ_1, satisfying

$$1-\varepsilon_1\delta_1 > 0, 1-\frac{\varepsilon_1}{\delta_1} > 0.$$

We obtain

$$B(q,v;q,v) \ge \alpha_h \sum_{T\in T_h}(\|\Delta q\|^2_{0,T} + \|q\|^2_{0,T} + \|\Delta v\|^2_{0,T}).$$

Then we complete the proof.

Theorem 3.2. Let $f \in H^{-1}(\Omega)$. Then, (18) has a unique solution, and the solution is $(\sigma_h, u_h) \in H_h(\Omega) \times M_h(\Omega)$.

Proof: From Theorem3.1, we know that the bilinear form $B_h(\cdot,\cdot;\cdot,\cdot)$ is coercive and bounded on $H_h(\Omega)\times M_h(\Omega)$. Then the result follows from Lax-Milgram theorem.

IV. ERROR ESTIMATES

The error estimates of the second-order elliptic problem have studied by Kim *et al.* [9]. In this section, we discuss the error estimates of the fourth-order elliptic equations.

Assume the domain Ω is convex, from the general finite element approximation theory we have the estimate [8]:

Lemma 4.1. Assume $\omega \in H^k(\Omega)$, Φ satisfies (16) with $\varsigma \ge k > d/2 + m.$ Let $H_h(\Omega)$ be given by (15). Then there exists a function $s \in H_h(\Omega)$ such that for $x \in \Omega$, the estimate

$$\|\omega - s\|_{m,\Omega} \le ch^{k-m}\|\omega\|_{k,\Omega} \tag{22}$$

is valid if h is sufficiently small.

We defined the:

$$B(\sigma_h, u_h; q, v) = (\Delta\sigma_h, \Delta q)_{0,\Omega} + (\Delta u_h + \sigma_h, \Delta v + q)_{0,\Omega}. \tag{23}$$

Since the exact solution (u,σ) satisfy (12), using the condition (18), we get the following property:

$$\begin{aligned} B(\sigma-\sigma_h, u-u_h; q, v) &= (\Delta(\sigma-\sigma_h), \Delta q)_{0,\Omega} + (\Delta(u-u_h) \\ &\quad +(\sigma-\sigma_h), \Delta v + q)_{0,\Omega} \\ &= 0, (\forall q \in H_h(\Omega), \forall v \in M_h(\Omega)) \end{aligned}$$

Now we are ready to derive the following error estimation.

Theorem 4.2. Suppose that $u \in H^k(\Omega)$ and $\sigma \in H^k(\Omega)$ are the solutions of (12), and $u_h \in H_h(\Omega)$ and $\sigma_h \in H_h(\Omega)$ are the solutions of (23). Then for sufficiently small h, we have the error estimation

$$\begin{aligned} &\|\sigma-\sigma_h\|^2_{0,\Omega} + \|\Delta(\sigma-\sigma_h)\|^2_{0,\Omega} + \|\Delta(u-u_h) \\ &\|^2_{0,\Omega} \le ch^{2(k-2)}(\|u\|^2_{k,\Omega} + \|\sigma\|^2_{k,\Omega}) \end{aligned} \tag{24}$$

Proof: From (12), we have:

$$\begin{aligned} B(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h) &= (\Delta(u-u_h)+(\sigma-\sigma_h),\Delta(u-u_h) \\ &\quad +(\sigma-\sigma_h))_{0,\Omega} + (\Delta(\sigma-\sigma_h),\Delta(\sigma-\sigma_h))_{0,\Omega} \\ &= \|\Delta(u-u_h)+(\sigma-\sigma_h)\|^2_{0,\Omega} \\ &\quad + \|\Delta(\sigma-\sigma_h)\|^2_{0,\Omega} \\ &\le c(\|\Delta(\sigma-\sigma_h)\|^2_{0,\Omega} + \|\Delta(u-u_h)\|^2_{0,\Omega} \\ &\quad + \|\sigma-\sigma_h\|^2_{0,\Omega}). \end{aligned}$$

From (14), we obtain the following inequality:

$$\begin{aligned} &\|\Delta(\sigma_I-\sigma_h)\|^2_{0,\Omega} + \|\sigma_I-\sigma_h\|^2_{0,\Omega} + \|\Delta(u_I-u_h)\|^2_{0,\Omega} \\ &\le B(\sigma_I-\sigma_h,u_I-u_h;\sigma_I-\sigma_h,u_I-u_h) \\ &= B(\sigma-\sigma_I,u-u_I;\sigma_I-\sigma_h,u_I-u_h) \\ &\le (\Delta(u-u_I)+(\sigma-\sigma_I),\Delta(u_I-u_h)+(\sigma_I-\sigma_h))_{0,\Omega} \\ &\quad +(\Delta(\sigma-\sigma_I),\Delta(\sigma_I-\sigma_h))_{0,\Omega} \\ &\le (\|\Delta(\sigma-\sigma_I)\|^2_{0,\Omega} + \|\Delta(u-u_I)\|^2_{0,\Omega} + \|\sigma-\sigma_I\|^2_{0,\Omega})^{\frac{1}{2}} \\ &\quad (\|\Delta(\sigma_I-\sigma_h)\|^2_{0,\Omega} + \|\Delta(u_I-u_h)\|^2_{0,\Omega} + \|\sigma_I-\sigma_h\|^2_{0,\Omega})^{\frac{1}{2}}, \end{aligned}$$

So we have

$$\begin{aligned} &\|\Delta(\sigma_I-\sigma_h)\|^2_{0,\Omega} + \|\Delta(u_I-u_h)\|^2_{0,\Omega} + \|\sigma_I-\sigma_h\|^2_{0,\Omega} \\ &\le \|\Delta(\sigma-\sigma_I)\|^2_{0,\Omega} + \|\Delta(u-u_I)\|^2_{0,\Omega} + \|\sigma-\sigma_I\|^2_{0,\Omega} \end{aligned}$$

From above the inequalities, we have:

$$\|\Delta(\sigma-\sigma_h)\|_{0,\Omega}^2+\|\sigma-\sigma_h\|_{0,\Omega}^2+\|\Delta(u-u_h)\|_{0,\Omega}^2$$
$$\le\|\Delta(\sigma-\sigma_I)\|_{0,\Omega}^2+\|\Delta(u-u_I)\|_{0,\Omega}^2+\|\sigma-\sigma_I\|_{0,\Omega}^2$$
$$+\|\Delta(\sigma_I-\sigma_h)\|_{0,\Omega}^2+\|\Delta(u_I-u_h)\|_{0,\Omega}^2+\|\sigma_I-\sigma_h\|_{0,\Omega}^2$$
$$\le 2(\|\Delta(\sigma-\sigma_I)\|_{0,\Omega}^2+\|\Delta(u-u_I)\|_{0,\Omega}^2+\|\sigma-\sigma_I\|_{0,\Omega}^2)$$

where we used Lemma4.1, we have the following inequality:

$$\|\sigma-\sigma_h\|_{0,\Omega}^2+\|\Delta(\sigma-\sigma_h)\|_{0,\Omega}^2+\|\Delta(u-u_h)\|_{0,\Omega}^2$$
$$\le ch^{2(k-2)}(\|u\|_{k,\Omega}^2+\|\sigma\|_{k,\Omega}^2)$$

Then we complete the proof.

V. POSTIERIORI ERROR ESTIMATION

One of the main motivations for using least-squares finite element approaches is the fact that the element-wise evaluation of the functional serves as an *a posteriori* error estimator.

A posteriori estimate attempt to provide quantitatively accurate measures of the discretization error through the so-called *a posteriori* error estimators which are derived by using the information obtained during the solution process. In recent years, the use of *a posteriori* error estimators has become an efficient tool for assessing and controlling computational errors in adaptive computations [10].

Now we defined the least-squares functional:

$$F_h(\sigma_h,u_h)=\sum_{T\in T_h}(\|\Delta\sigma_h+f\|_{0,T}^2+\|\Delta u_h+\sigma_h\|_{0,T}^2). \tag{25}$$

where $(\sigma_h,u_h)\in H_h(\Omega)\times M_h(\Omega)$.

We have

$$F_h(\sigma-\sigma_h,u-u_h)=\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)+f\|_{0,T}^2$$
$$+\|\Delta(u-u_h)+\sigma-\sigma_h\|_{0,T}^2).$$

So we define the posteriori estimator as following:

$$F_h(\sigma-\sigma_h,u-u_h)=\sum_{T\in T_h}\eta^2. \tag{26}$$

Theorem 5.1. Let $f\in H^{-1}(\Omega)$, The least-squares functional constitutes an *a posteriori* error estimator. In other words, for

$$\eta^2=\|\Delta(\sigma-\sigma_h)+f\|_{0,T}^2+\|\Delta(u-u_h)+\sigma-\sigma_h\|_{0,T}^2$$

there exist positive constants α_T and β_T such that

$$\sum_{T\in T_h}\eta^2\le\beta_T\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)\|_{0,T}^2+\|\sigma-\sigma_h\|_{0,T}^2+\|\Delta(u-u_h)\|_{0,T}^2), \tag{27}$$

$$\sum_{T\in T_h}\eta^2\ge\alpha_T\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)\|_{0,T}^2+\|\sigma-\sigma_h\|_{0,T}^2+\|\Delta(u-u_h)\|_{0,T}^2). \tag{28}$$

which holds for all $(\sigma_h,u_h)\in H_h(\Omega)\times M_h(\Omega)$.

Proof: From (26) and $f\in H^{-1}(\Omega)$, we know

$$\sum_{T\in T_h}\eta^2=F_h(\sigma-\sigma_h,u-u_h)$$
$$=\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)+f\|_{0,T}^2+\|\Delta(u-u_h)+\sigma-\sigma_h\|_{0,T}^2)$$
$$=C\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)\|_{0,T}^2+\|\Delta(u-u_h)+\sigma-\sigma_h\|_{0,T}^2)$$
$$=CB_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h).$$

From Theorem 3.1, we have:

$$B_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h)\le\beta_T\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)\|_{0,T}^2$$
$$+\|\sigma-\sigma_h\|_{0,T}^2+\|\Delta(u-u_h)\|_{0,T}^2),$$
$$B_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h)\ge\alpha_T\sum_{T\in T_h}(\|\Delta(\sigma-\sigma_h)\|_{0,T}^2$$
$$+\|\sigma-\sigma_h\|_{0,T}^2+\|\Delta(u-u_h)\|_{0,T}^2).$$

The positive constants $\alpha_T=c\alpha_h$ and $\beta_T=c\beta_h$, this completes the proof.

Remark: The mesh is adapted based on *a posteriori* error estimate of the fourth-order elliptic equations. We use a mesh optimization procedure to compute the size of elements in the new mesh, based on the computed *a posteriori* error estimate η.

The mesh is adapted using the mesh modification procedures developed by Li *et al.* [11]. This requires the specification of a mesh metric field to define the desired element size and shape distribution from the computed η. The mesh is then adapted to satisfy the prescribed metric field by the processes of refinement, coarsening and re-alignment.

Adaptive refinement strategies consist in refining those triangles with the largest values of η.

VI. SUMMARY AND CONCLUSIONS

As the fourth-order elliptic equations belong to high-order partial differential equations which possess complex numerical structure, and the select of finite element spaces is difficult, so the research about the fourth-order elliptic equations is still quite few. This paper describes an adaptive least-squares mixed finite elements method for the fourth-order elliptic equations for the first time, constructes *a posteriori* error estimator by the least-squares functional, and estimates the posteriori errors effectively by composed bilinear form.

We describe an adaptive least-squares mixed finite element procedure for solving the fourth-order elliptic equations in this paper. The procedure uses a least-squares mixed finite element formulation and adaptive refinement based on *a posteriori* error estimate. The method is applied to study the continuous and coercivity of the fourth-order elliptic equations.

In this paper, we applied relatively standard *a posteriori* error estimation technique to solve the fourth-order elliptic equations adaptively.

This paper provides theory foundation for numerical computation in plate bending and fluid dynamics.

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