

A Numerical Algorithm for Solving Advection-Diffusion Equation with Constant and Variable Coefficients

S.G. Ahmed*

Department of Engineering Physics and Mathematics, Faculty of Engineering, Zagazig University, P.B. Box 44519, Zagazig, Egypt

Abstract: Advection-diffusion equation with constant and variable coefficients has a wide range of practical and industrial applications. Due to the importance of advection-diffusion equation the present paper, solves and analyzes these problems using a new finite difference equation as well as a numerical scheme. The developed scheme is based on a mathematical combination between Siemieniuch and Gradwell approximation for time and Dehghan's approximation for spatial variable. In the proposed scheme a special discretization for the spatial variable is made in such way that when applying the finite difference equation at any time level $(j + 1)$ two nodes from both ends of the domain are left. After that the unknowns at the two nodes adjacent to the boundaries are obtained from the interpolation technique. The results are compared with some available analytical solutions and show a good agreement.

Keywords: Advection-diffusion equation, Explicit finite difference techniques, Implicit finite difference techniques, Interpolation techniques.

1. INTRODUCTION

Advection-diffusion equation is one of the most important partial differential equations and observed in a wide range of engineering and industrial applications [1]. It has been used to describe heat transfer in a draining film [2], water transfer in soil [3], dispersion of tracers in porous media [4], contaminant dispersion in shallow lakes [5], the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere [6] and dispersion of dissolved salts in groundwater [7]. In the initial works while obtaining the analytical solutions of dispersion problems in the ideal conditions, the basic approach was to reduce the advection-diffusion equation into a diffusion equation by eliminating the advection term(s). It was done either by introducing moving coordinates see, Ogata Banks 1969 [8]; Harleman and Rumer 1963 [9]; Bear 1972 [10]; Guvanasen and Volker 1983 [11]; Aral and Liao 1972 [12] and Marshal *et al.*, 1996 [13]. Another direction is to transform advection-diffusion to diffusion equation only was by introducing another dependent variable see Banks and Ali 1964 [14]; Ogata 1970 [15]; Lai and Jurinak 1971 [16]; Marin 1974 [17] and Al-Niami and Rushton 1977 [18]. Some one-dimensional analytical solutions have been given, see Tracy 1995 [19] by transforming the nonlinear advection-diffusion into linear one for specific forms of the moisture contents vs pressure head and relative hydraulic conductivity vs pressure head curves which allow both two-dimensional and three-dimensional solutions to derived. Accurate numerical solution of the advection-diffusion

equation is usually characterized by a dimensionless parameter, called Peclet number. These results become increasingly difficult as the Peclet number increases due to onset of spurious oscillations or excessive numerical damping if finite difference [20] or finite element formulations are used [21]. Numerous innovative algorithms and methods can be found in the literatures [22-31]. In finding the analytical solutions many difficulties encountered such as the nonlinearities etc. also in the numerical solutions difficulties appear due to many reasons and herein we will give two for such reasons. Firstly, the nature of the governing equation, which includes first-order and second-order partial derivatives in space. Secondly, it is vital to construct an appropriate mesh to obtain a better approximation to the problem. However, the construction of an appropriate mesh is not easy task and sometimes the problem can not be solved because of the lack of mesh structure. In the present paper, the advection-diffusion equation with constant and variable coefficients is solved using a new scheme. The developed scheme is based on a mathematical combination between Siemieniuch and Gradwell approximation for time and Dehghan's approximation for spatial variable. The scheme also developed and based on a special discretization for spatial variable in such way that when applying the finite difference equation at any time level $(j + 1)$ two nodes from both ends of the domain are left and by assuming all unknowns became known at the old time level (j) , the new finite difference equation is applied at $m - 2$ internal nodes of the domain, where m are the total number of nodes inside the domain of interest. Then making use of interpolation technique to find the unknowns at two internal nodes left. After that the unknowns at the two nodes adjacent to the boundaries are obtained from the interpolation technique. Four examples with known analytical solutions are presented

*Address correspondence to this author at the Department of Engineering Physics and Mathematics, Faculty of Engineering, Zagazig University, P.B. Box 44519, Zagazig, Egypt; Tel: +20106392877; E-mail: sgahmed911@hotmail.com

to test the validity of the new finite difference equation and its new scheme. The first three examples are of constant coefficients and are chosen such that different cases between the advection and diffusion parameters are used to show their affect on the solution. The fourth example is for advection-diffusion equation with variable coefficients. It was for one-dimensional advection-diffusion equation in a longitudinal finite initially solute free domain and for dispersion problem. In this example the velocity of the flow considered spatially dependent due to inhomogeneity of the domain. Results due to the present method agree well with the analytical solutions and can be extend to cover higher dimensional problems.

2. MATHEMATICAL FORMULATION

In the present paper, the mathematical formulation for two different types of advection-diffusion equation will be presented. The first type is for advection-diffusion equation with constant coefficients while the second type is for the same equation but with variable coefficients.

Case (1): Advection-diffusion with constant coefficients:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

With initial condition

$$u(x,0) = f(x) \quad 0 \leq x \leq \ell \tag{2}$$

And boundary conditions

$$u(x = 0, t) = g_0(x) \quad 0 < t \leq T \tag{3}$$

$$u(x = \ell, t) = g_1(x) \quad 0 < t \leq T \tag{4}$$

Condition (3) or (4) may be replaced by:

$$\frac{\partial u(x,t)}{\partial x} = g_2(x) \quad 0 < t \leq T \tag{5}$$

In equations (2-5) the functions $f(x), g_0(x), g_1(x)$ and $g_2(x)$ are known.

Case (2): Advection-diffusion with variable coefficients:

As we mentioned before, advection-diffusion equation can be found in different types of practical applications, and so the variable coefficients will have different meaning from application to another. Therefore, the mathematical formulation for advection-diffusion herein, will be for solute dispersion problem. In this problem, the medium is considered inhomogeneous, therefore the velocity is considered depend on the position variable. The mathematical formulation for such application will be as follow:

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left(D(x,t) \frac{\partial C(x,t)}{\partial x} - u(x,t) C(x,t) \right) \tag{6}$$

With

$$u(x,t) = u_0(1 + ax) \tag{7}$$

$$D(x,t) = D_0(1 + ax)^2 \tag{8}$$

$$C(x,t) = 0, 0 \leq x \leq \ell, t = 0 \tag{9}$$

$$C(x,t) = C_0, x = 0, t > 0 \tag{10}$$

$$\frac{\partial C(x,t)}{\partial x} = 0, x = \ell, t \geq 0 \tag{11}$$

In fact the mathematical formulation in both cases is the same, only two equations are added in the second case and given by equations (7) and (8). This difference comes from the nature of the dispersion problem that is both flow velocity and dispersion should satisfy Darcy's law, therefore, Atul in [30] added these two relations to ensure satisfaction of Darcy's law.

3. THE PROPOSED FINITE DIFFERENCE EQUATION

The scheme starts by using the Siemieniuch-Gradwell time difference formula as follow:

$$\left(\frac{\partial u}{\partial t} \right)_{i,n} = A_{11}(u_{i-1,n+1} - u_{i-1,n}) + B_{11}(u_{i,n+1} - u_{i,n}) + C_{11}(u_{i+1,n+1} - u_{i+1,n}) \tag{12}$$

Next use the space difference formulas derived by Dehghan [29]:

$$\left(\frac{\partial u}{\partial x} \right)_{i,n} = A_{21}(u_{i+2,n} - u_{i,n}) + B_{21}(u_{i,n} - u_{i-2,n}) + C_{21}(u_{i+1,n} - u_{i-1,n}) \tag{13}$$

And

$$\left(\frac{\partial^2 u}{\partial x^2} \right)_{i,n} = A_{31}(u_{i+1,n} - 2u_{i,n} + u_{i-1,n}) + B_{31}(u_{i+2,n} - 2u_{i,n} + u_{i-2,n}) + C_{21}(u_{i+1,n} - u_{i-1,n}) \tag{14}$$

Substituting equations (12-14) into equation (1) and rearrange according the time level, lead to:

$$(A_{11}u_{i-1} + B_{11}u_i + C_{11}u_{i+1})^{n+1} = (\xi_1u_{i-2} + \xi_2u_{i-1} + \xi_3u_i + \xi_4u_{i+1} + \xi_5u_{i+2})^n \tag{15}$$

In which:

$$\xi_1 = \beta B_{21} + \alpha B_{31} \tag{16}$$

$$\xi_2 = A_{11} + \beta C_{21} + \alpha A_{31} \tag{17}$$

$$\xi_3 = B_{11} + \beta A_{21} - \beta B_{21} - 2\alpha A_{31} - 2\alpha B_{31} \tag{18}$$

$$\xi_4 = C_{11} - \beta C_{21} + \alpha A_{31} \tag{19}$$

$$\xi_5 = -\beta A_{21} + \alpha A_{31} \tag{20}$$

$$A_{11} = C_{11} = \frac{2 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] - \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]}{4(\Delta t)} \tag{21}$$

$$B_{11} = \frac{2 + \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right] - 2 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right]}{2(\Delta t)} \quad (22)$$

$$A_{21} = \frac{12 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] + 2 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^2 - 3 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right] - 2}{24(\Delta x)} \quad (23)$$

$$A_{31} = \frac{- \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^4 - 8 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^2 - 12 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right]^2 + 8 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right]}{6 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] (\Delta x)^2} \quad (24)$$

$$B_{21} = \frac{2 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^2 + 12 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] + 3 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right] - 2}{24(\Delta x)} \quad (25)$$

$$B_{31} = \frac{\left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^4 - 8 \left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^2 + 12 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right]^2 - 2 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right]}{6 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] (\Delta x)^2} \quad (26)$$

$$C_{21} = (-) \frac{\left[\beta \left(\frac{\Delta t}{\Delta x} \right) \right]^2 + 6 \left[\beta \left(\frac{\Delta t}{(\Delta x)^2} \right) \right] - 4}{6(\Delta x)} \quad (27)$$

The $(m - 2)$ linear system of equations obtained from applying equation (15) at the time levels $(j + 1)$ and (j) , respectively.

4. SOLUTION ALGORITHM

Suppose that the domain of the problem is divided into m internal points, see Fig. (1), then at each two successive times level apply equation (15), leads to a linear system of equations.

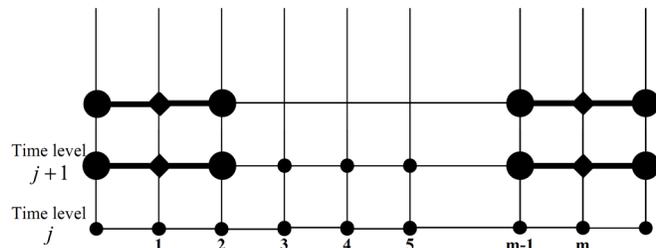


Fig. (1). Grid distribution.

The new scheme can be stated as follows:

1. Suppose that the domain $0 < x < \ell$ is divided into m internal nodes and numbered according to Fig. (1).
2. At time level (j) all temperature nodes should be known from previous time level and this represents the R.H.S. of equation (15).

3. At time level $(j + 1)$ apply the L.H.S. of equation (15) starting from node (2) and stop at node $(m - 1)$.
4. Solve the linear system of equation resulted from steps 3 and 4.
5. Use interpolation technique to find the temperature at the internal nodes 1 and (m) .
6. Repeat steps (2-5) up to the end of the time.

5. RESULTS AND DISCUSSION

The proposed finite difference equation and the numerical scheme are tested for their validity to solve advection-diffusion with constant and variable coefficients. Three different examples for advection-diffusion with constant coefficients are solved to study the effect of the parameters α and β on the results obtained, then another two different examples the first one is a one-dimensional advection-diffusion equation with variable coefficients and the second one is a two-dimensional case problem. Let us start with the first three examples of constant coefficients to study the effect of the parameters α and β

Problem (1): 1-D Advection-Diffusion with $\alpha = 0.01$ & $\beta = 1.0$

In this case problem, the constant coefficients appear in the advection-diffusion equation are $\alpha = 0.01$ & $\beta = 1.0$ associated with initial and boundary conditions respectively, given by:

$$u(x, 0) = f(x) = \exp\left(-\frac{(x + 0.5)^2}{0.00125}\right) \quad (28)$$

$$u(x = 0, t) = g_0(x) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(0.5 - t)^2}{0.00125 + 0.04t}\right) \quad (29)$$

$$u(x = 1, t) = g_1(x) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(1.5 - t)^2}{0.00125 + 0.04t}\right) \quad (30)$$

The exact solution is given by:

$$u(x, t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(x + 0.5 - t)^2}{0.00125 + 0.04t}\right) \quad (31)$$

The problem is solved at different cases for space size step and time size step. In Fig. (2), the results for two different space size step and two different time step size step are shown. It is very clear that a good agreement between the analytical solution and the present numerical results with minimum error obtained, and the error becomes clear when using large size step for time and space.

Problem (2): 1-D Advection-Diffusion with $\alpha = 1.0$ & $\beta = 0.1$

In this case problem, the constant coefficients appear in the advection-diffusion equation are $\alpha = 1.0$ & $\beta = 0.1$ associated with initial and boundary conditions respectively, given by:

$$u(x,t) = 0.0 \tag{32}$$

$$u(x=0,t) = g_0(x) = 300 \quad 0 < t \leq T \tag{33}$$

$$\frac{\partial u(x=6,t)}{\partial x} = g_1(x) \quad 0 < t \leq T \tag{34}$$

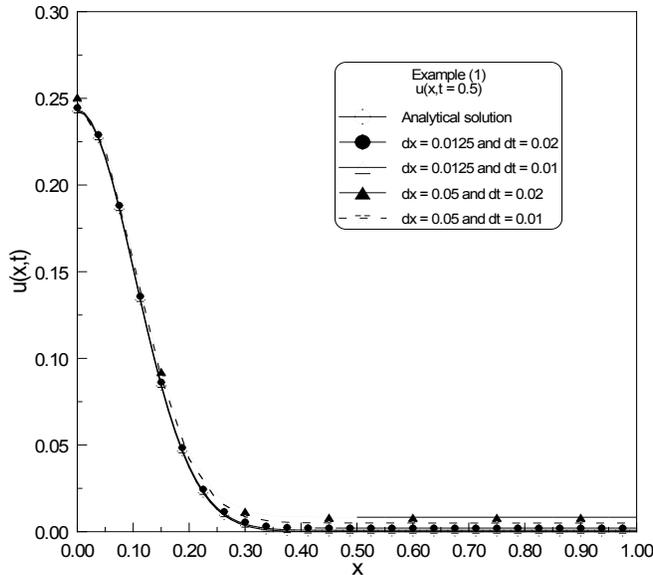


Fig. (2). Results of example (1).

In this case, the example is solved at three different time steps; 0.1, 1 and 3. The results are compared with the corresponding analytical solution as shown in Fig. (3). As it appears from the figure, a good agreement between the analytical solutions and the present results can be observed.

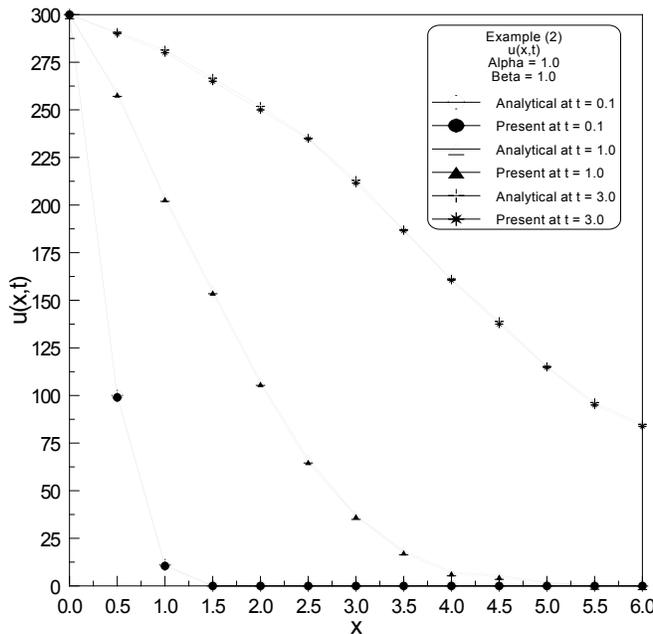


Fig. (3). Results of example (2).

Problem (3): 1-D Advection-Diffusion with $\alpha = 1.0$ & $\beta = 2.0$

In this case problem, the constant coefficients appear in the advection-diffusion equation are $\alpha = 1.0$ & $\beta = 2.0$

associated with initial and boundary conditions respectively, given by:

$$u(x,t) = 0.0 \tag{35}$$

$$u(x=0,t) = a \exp(bt) \quad 0 < t \leq T \tag{36}$$

$$u(x=6,t) = a \exp(bt + cx) \quad 0 < t \leq T \tag{37}$$

$$u(x,t=0) = a \exp(cx) \tag{38}$$

The analytical solution of this problem is given by:

$$u(x,t) = a \exp(bt + cx), \quad c = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta b}}{2\beta} \tag{39}$$

The temperature variation against spatial variable and at different times are shown in Fig. (4). It is clear from the figure, that a linear damping is occurred and the error becomes large by increasing the space and becomes clear at large times, but in all cases the error is acceptable and can be decreased by decreasing both space and time step size.

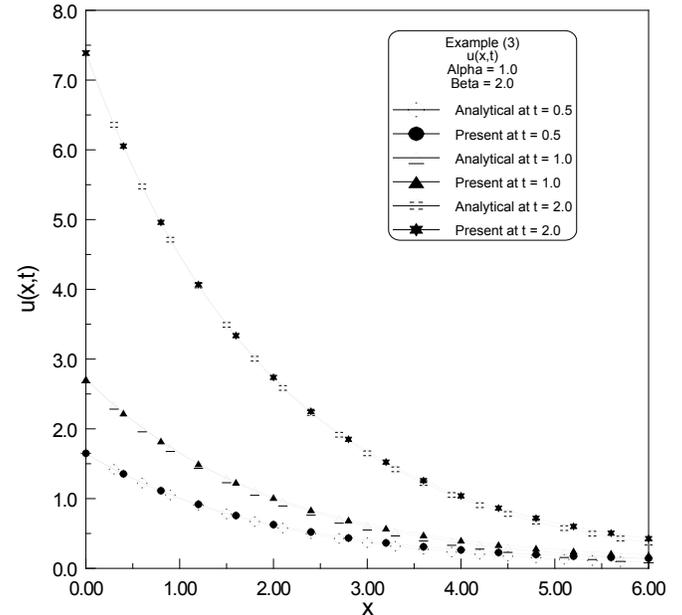


Fig. (4). Results of example (3).

Problem (4): 1-D Advection-Diffusion with variable coefficients

The fourth case problem is for the same equation with variable coefficients. The solute dispersion problem is taken as a practical example. The domain of the problem $0 \leq x \leq 1.0$ (Km) is inhomogeneous, therefore the velocity is considered dependent on the position variable. An important parameter called inhomogeneous parameter is taken into consideration and its effect on the results studied for two different values away from each others as shown in Figs. (5-7). Starting the results by Fig. (5) which shows the variation of the ratio $C/C_0, C_0 = 1$ against the spatial variable x at different times $0.1 \leq t \leq 1.0$ and at inhomogeneous parameter equals $a = 1.0$. As it is clear that the behavior in all curves is exponentially decaying and by going the time up the ratio increases at any spatial point. Following up the

results and turn up to Fig. (6) which shows the effect of the inhomogeneous parameter on similar results obtained in Fig. (5) but in this case the inhomogeneous parameter equals $a = 0.1$. Obviously the same behavior is obtained but the exponentially decaying behavior becomes more concave downward, i.e., at the same point x the ratio C/C_0 in case of $a = 1.0$ is greater when $a = 0.1$ and agrees very well with the practice. It is clear from both Figs. (5, 6) that by increasing the inhomogeneous parameter, the ratio C/C_0 increases and vice versa. Collecting these two figures in one figure, to clarify this observation which becomes very clear as shown in Fig. (7). Finally, from the solution of the above four test problems one can see that the proposed equation and scheme are simple in computation compared with other methods.

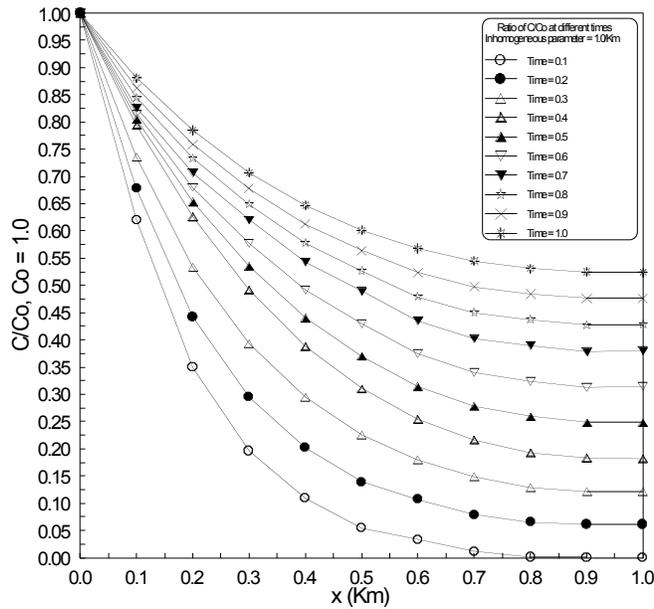


Fig. (5). Ratio of C/C_0 at different times and inhomogeneous parameter equals 1.0.

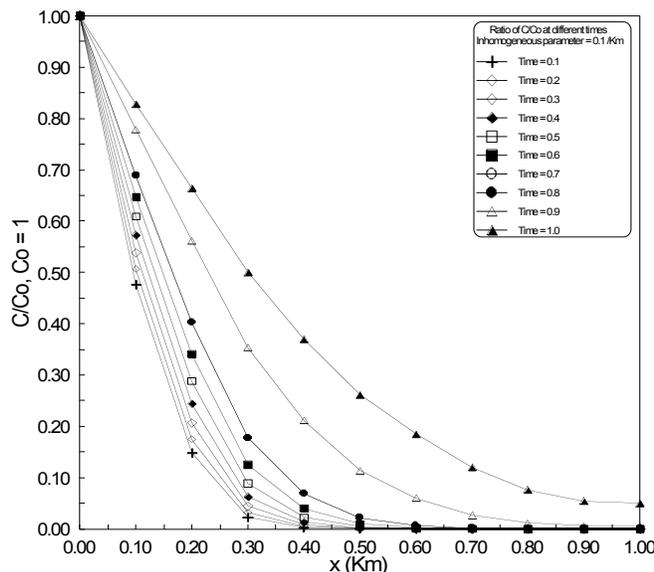


Fig. (6). Ratio of C/C_0 at different times and inhomogeneous parameter equals 0.1.

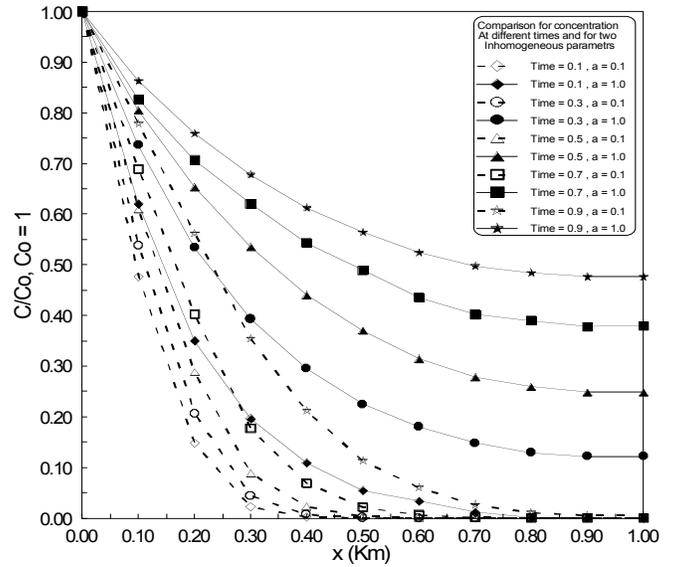


Fig. (7). Comparison between C/C_0 at different times and two different values for inhomogeneous parameter.

Problem (5): 2-D Advection-Diffusion with variable coefficients

In this test problem, a two-dimensional advection-diffusion equation with variable coefficients are solved and the obtained results are compared with the available analytical solution. The government differential equation is:

$$\frac{\partial u(x,y,t)}{\partial t} = k_x \frac{\partial^2 u(x,y,t)}{\partial x^2} + k_y \frac{\partial^2 u(x,y,t)}{\partial y^2} + v_x \frac{\partial u(x,y,t)}{\partial x} + v_y \frac{\partial u(x,y,t)}{\partial y} \quad (40)$$

$$0 \leq x,y \leq 1$$

The exact solution is given by;

$$u^{ex}(x,y,t) = a \left(\exp(bt + c_x x) + \exp(bt + c_y y) \right) \quad (41)$$

$$c_x = \frac{v_x \pm \sqrt{v_x^2 + 4k_x b}}{2k_x} \quad (42)$$

$$c_y = \frac{v_y \pm \sqrt{v_y^2 + 4k_y b}}{2k_y} \quad (43)$$

The initial and boundary conditions are obtained from the analytic solution as follows:

$$u(x,y,0) = u^{ex}(x,y,0) \quad (44)$$

$$u(0,y,t) = u^{ex}(0,y,t) \quad , \quad u(1,y,t) = u^{ex}(1,y,t) \quad (45)$$

$$u(x,0,t) = u^{ex}(x,0,t) \quad , \quad u(x,1,t) = u^{ex}(x,1,t) \quad (46)$$

The results for this test problem are shown in Figs. (8-13). In these figures a comparison between the results due to the present results and the analytical solution at three different times, $t = 0.1, 0.5, 0.9$ seconds. As it is clear from these figures, a good agreement between the two results are obtained.

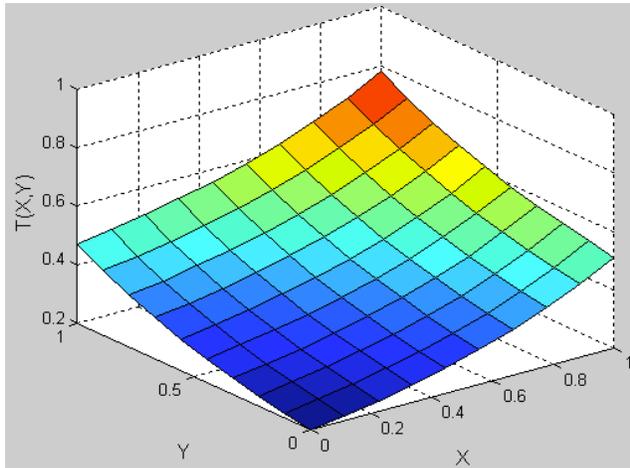


Fig. (8). Analytical solution at $t=0.1$.

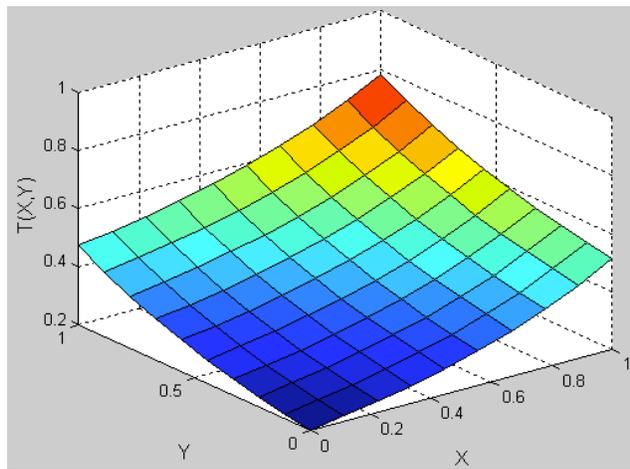


Fig. (9). Present at $t=0.1$.

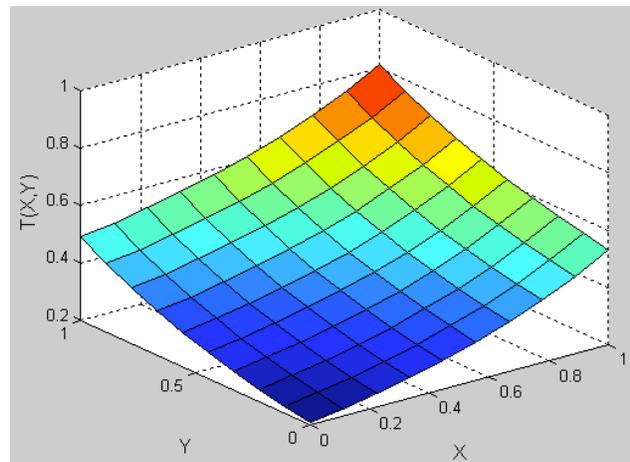


Fig. (10). Analytical at $t=0.5$.

6. CONCLUSION

In the present paper, a new finite difference equation is derived to solve linear advection-diffusion equation with both constant and variable coefficients. The derivation based on a mathematical combination between spatial and time discretisation from previous numerical methods. Also a new numerical scheme is developed to track the unknowns in the problem under consideration. Due to the practical

importance of the advection-diffusion equation, the present method can open the way to the researchers to this direction of method of computation due to easy manipulation and the high accuracy obtained as seen from the five test examples in one and two dimensions that solved in the present paper. Finally one can modify the present method by using different finite difference methods for approximating both spatial and time derivatives.

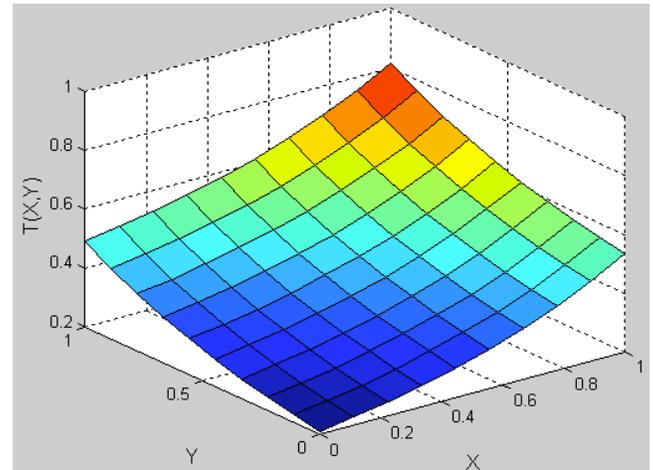


Fig. (11). Present at $t=0.5$.

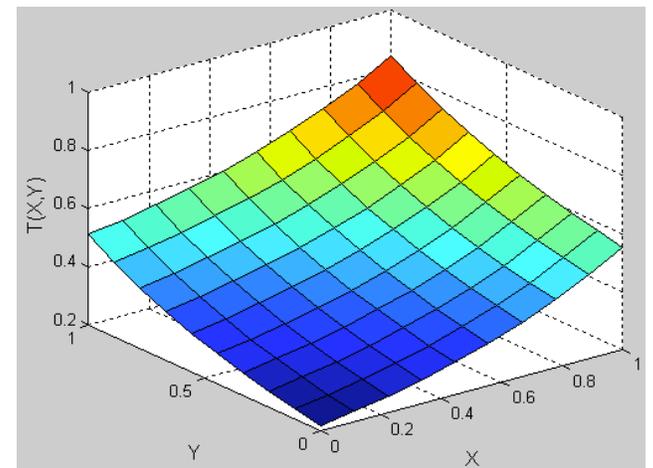


Fig. (12). Analytical at $t=0.9$.

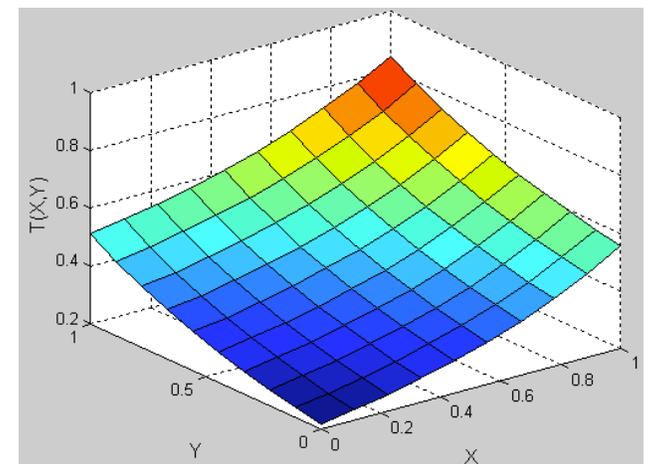


Fig. (13). Present at $t=0.9$.

NOMENCLATURE

$C(x, t)$ = Solute concentration at position x along the longitudinal direction at a time t

$D(x, t)$ = Solute dispersion

$u(x, t)$ = Medium's flow velocity

a = Inhomogeneous parameter, $a = \frac{b}{\ell}$

b = May be real constant

ACKNOWLEDGEMENTS

None declared.

CONFLICT OF INTEREST

None declared.

REFERENCES

- [1] Singh KM, Tanaka M. On exponential variable transformation based boundary element formulation for advection-diffusion problems. *Eng Anal Bound Elem* 2000; 24: 225-35.
- [2] Isenberg J, Gutfinger C. Heat transfer to a draining film. *Int J Heat Transfer* 1972; 16: 505-12.
- [3] Parlange JY. Water transport in soils. *Ann Rev Fluids Mech* 1980; 2: 77-102.
- [4] Fattah QN, Hoopes JA. Dispersion in anisotropic homogeneous porous media. *J Hydraul Eng* 1985; 111: 810-27.
- [5] Salmon JR, Ligett JA, Gallager RH. Dispersion analysis in homogeneous lakes. *Int J Numer Meth Eng* 1980; 15: 1627-42.
- [6] Zlatev Z, Berkowicz R, Prahm LP. Implementation of a variable stepsize variable formula in the time-integration part of a code for treatment of long-range transport of air pollutants. *J Comp Phys* 1984; 55: 278-301.
- [7] Guvanasen V, Volker RE. Numerical solution for solute transport in unconfined aquifers. *Int J Numer Meth Fluids* 1983; 3: 103-23.
- [8] Ogata A, Banks RB. A solution of the differential equation of longitudinal dispersion in porous media. *U.S. Geol Surv Prof Paper* 1961; 411-A: 1-9.
- [9] Harleman DRF, Rumer RE. Longitudinal and lateral dispersion in an isotropic porous medium. *J Fluid Mech* 1963; 385-94.
- [10] Bear J. *Dynamics of fluids in porous media*. New York: Am Elsevier Co. 1972; p. 764.
- [11] Guvanasen V, Volker RE. Experimental investigations of unconfined aquifer pollution from recharge basins. *Water Resour Res* 1983; 19(3): 707-17.
- [12] Aral MM, Liao B. Analytical solution for two-dimensional transport equation with time-dependent dispersion coefficients. *J Hydraul Eng* 1996; 1(1): 20-32.
- [13] Marshal TJ, Holmes JW, Rose CW. *Soil Physics*. 3rd ed. Cambridge: Cambridge University Press 1996.
- [14] Banks RB, Ali J. Dispersion and adsorption in porous media flow. *J Hydraul Div* 1964; 90: 13-31.
- [15] Ogata A. Theory of dispersion in granular media. *US Geol Sur Prof Paper* 1970; 411-1: 34.
- [16] Lai SH, Jurinak JJ. Numerical approximation of cation exchange in miscible displacement through soil columns. *Soil Sci Soc Am* 1971; 35: 894-9.
- [17] Al-Nami ANS, Rushton KR. Analysis of flow against dispersion in porous media. *J Hydraul Eng* 1977; 33: 87-97.
- [18] Spalding DB. A novel finite difference formulation for differential expressions involving both first and second derivatives. *Int J Numer Meth Eng* 1972; 4: 551-9.
- [19] Hughes TJR, Brooks AA. A theoretical framework for Petrov-Galerkin methods with discontinuous weighting functions: application to the streamline-upwind procedure. In: Gallagher RH, Norrie DH, Oden JT, Zienkiewicz OC, Eds. *Finite Elements in Fluids 4*, London: Wiley 1982; pp. 47-65.
- [20] Donea J, Giuliani S, Laval H. Quattapelle, time accurate solution of advection-diffusion problems by finite elements. *Comput Methods Appl Mech Eng* 1984; 45: 123-45.
- [21] Yu CC, Heinrich JC. Petrov-Galerkin methods for the time-dependent convective transport equation. *Int J Numer Meth Eng* 1986; 23: 883-901.
- [22] Westerink JJ, Shea D. Consistent higher degree Petrov-Galerkin methods for solution of the transient convection-diffusion equation. *Int J Numer Meth Eng* 1989; 29: 1077-101.
- [23] Park NS, Liggett JA. Taylor-least-squares finite element for two-dimensional advection-dominated unsteady advection-diffusion problems. *Int J Numer Meth In Fluids* 1990; 11: 21-38.
- [24] Morton KW. *Numerical solution of convection-diffusion equation*. London: Chapman & Hall 1996.
- [25] Kansa EJ. Multiquadrics: a scattered data approximation scheme with application to computational fluid-dynamics. *Comput Math Appl* 1990; 19: 147.
- [26] Thomas JW. *Numerical partial differential equations: finite difference methods*. Berlin, New York: Springer-Verlag 1995.
- [27] Dehghan M. Weighted finite difference techniques for the one-dimensional advection-diffusion equation. *Appl Math Comput* 2004; 147: 307-19.
- [28] Dehghan M. Quasi-implicit and two-level explicit finite-difference procedures for solving the one-dimensional advection equation. *Appl Math Comput* 2005; 167: 46-67.
- [29] Dehghan M. Numerical solution of three-dimensional advection-diffusion equation. *Appl Math Comput* 2004; 150: 5-19.
- [30] Dehghan M. Time-splitting procedures for the solution of the two-dimensional transport equation. *Kybernetes* 2007; 36: 791-805.
- [31] Dehghan M. On the numerical solution of the one-dimensional convection-diffusion equation. *Mathl Probl Eng* 2005; 61-74.

Received: September 19, 2011

Revised: November 14, 2011

Accepted: December 3, 2011

© S.G. Ahmed; Licensee Bentham Open.

This is an open access article licensed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted, non-commercial use, distribution and reproduction in any medium, provided the work is properly cited.