Open Access

A Bird's Eye View of $f(R)$-Gravity

Salvatore Capozziello[1,*], Mariafelicia De Laurentis[1] and Valerio Faraoni[2]

[1]*Dip. di Scienze Fisiche, Università di Napoli "Federico II" and INFN Sez. di Napoli, Compl. Universitario Monte S. Angelo, Ed. N, Via Cinthia, I-80126 Napoli, Italy*

[2]*Physics Department, Bishop's University, Sherbrooke, Québec, Canada J1M 1Z7*

Abstract: The currently observed accelerated expansion of the Universe suggests that cosmic flow dynamics is dominated by some unknown form of dark energy characterized by a large negative pressure. This picture comes out when such a new ingredient, beside baryonic and dark matter, is considered as a source in the r.h.s. of the field equations. Essentially, it should be some form of un-clustered, non-zero vacuum energy which, together with (clustered) dark matter, should drive the global cosmic dynamics. Among the proposals to explain the experimental situation, the "*concordance model*", addressed as ΛCDM, gives a reliable snapshot of the today observed Universe according to the CMBR, LSS and SNeIa data, but presents dramatic shortcomings as the "*coincidence and cosmological constant problems*" which point out its inadequacy to fully trace back the cosmological dynamics. On the other hand, alternative theories of gravity, extending in some way General Relativity, allow to pursue a different approach giving rise to suitable cosmological models where a late-time accelerated expansion can be achieved in several ways. This viewpoint does not require to find out candidates for dark energy and dark matter at fundamental level (they have not been detected up to now), it takes into account only the "observed" ingredients (i.e. gravity, radiation and baryonic matter), but the l.h.s. of the Einstein equations has to be modified. Despite of this modification, it could be in agreement with the spirit of General Relativity since the only request is that the Hilbert-Einstein action should be generalized asking for a gravitational interaction acting, in principle, in different ways at different scales. We survey the landscape of $f(R)$ theories of gravity in their various formulations, which have been used to model the cosmic acceleration as alternatives to dark energy and dark matter. Besides, we take into account the problem of gravitational waves in such theories. We discuss some successes of $f(R)$-gravity (where $f(R)$ is a generic function of Ricci scalar R), theoretical and experimental challenges that they face in order to satisfy minimal criteria for viability.

Keywords: Alternative theories of gravity, dark energy, dark matter, gravitational radiation.

1. INTRODUCTION

Theories of gravity, alternative to Einstein's General Relativity (GR), have been proposed to cure the problems of the standard cosmological model and, above all, because they arise in quantizations of gravity. These alternative gravitational theories constitute at least an attempt to formulate a semi-classical scheme in which GR and its most successful features can be recovered. One of the most fruitful approaches thus far has been that of *Extended Theories of Gravity* (ETGs), which have become a paradigm in the study of the gravitational interaction. ETGs are based on corrections and extensions of Einstein's theory. The paradigm consists, essentially, of adding higher order curvature invariants and/or minimally or non-minimally coupled scalar fields to the dynamics; these corrections emerge from the effective action of quantum gravity [1].

Further motivation to modify GR arises from the problem of fully implementing Mach's principle in a theory of gravity, which leads one to contemplate a varying gravitational coupling. Mach's principle states that the local inertial frame is determined by the average motion of distant astronomical objects [2]. This fact would imply that the gravitational coupling here and now is determined by the distant distribution of matter, and it can be scale-dependent and related to some scalar field. As a consequence, the concept of ``inertia" and the Equivalence Principle have to be revised. Brans-Dicke theory [3] constituted the first consistent and complete theory alternative to Einstein's GR. Brans-Dicke theory incorporates a variable gravitational coupling strength whose dynamics are governed by a scalar field non-minimally coupled to the geometry, which implements Mach's principle in the gravitational theory [3-5].

Independent motivation for extending gravity comes from the fact that every unification scheme of the fundamental interactions, such as Superstring, Supergravity, or Grand Unified Theories exhibit effective actions containing non-minimal couplings to the geometry or higher order terms in the curvature invariants. These contributions are one-loop or higher loop corrections in the high-curvature regime approaching the full, and still unknown, quantum gravity regime [1]. Specifically, this scheme was adopted in the study of quantum field theory on curved spacetime and it was found that interactions between quantum scalar fields

*Address correspondence to this author at the Dip. di Scienze Fisiche, Università di Napoli "Federico II" and INFN Sez. di Napoli, Compl. Universitario Monte S. Angelo, Ed. N, Via Cinthia, I-80126 Napoli, Italy; Tel: ++39-081-676496; Fax: ++39-081-676346;
E-mail: capozziello@na.infn.it

and background geometry, or gravitational self-interactions, yield such corrections to the Einstein-Hilbert Lagrangian [6]. Moreover, it has been realized that these corrective terms are inescapable in the effective action of quantum gravity close to the Planck energy [7]. Of course, all these approaches do not constitute a full quantum gravity theory, but are needed as working schemes toward it.

In summary, higher order terms in the invariants of the Riemann tensor, such as R^2, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\mu R$, or $R\upsilon^k R$, and non-minimal coupling terms between scalar fields and geometry such as $\phi^2 R$, have to be added to the effective gravitational Lagrangian when quantum corrections are introduced. These terms occur also in the effective Lagrangian of string or Kaluza-Klein theories when a mechanism of compactification of extra spatial dimensions is used [8].

From a conceptual point of view, there is no *a priori* reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar R minimally coupled with matter [9]. Furthermore, the idea has been proposed that there are no exact laws of physics, in the sense that the effective Lagrangians describing physical interactions could be stochastic functions at the microscopic level. This property would imply that local gauge invariances and the associated conservation hold only in the low energy limit and the fundamental constants of physics can vary [10].

Besides fundamental physics motivations, all these theories have been the subject of enormous attention in cosmology due to the fact that they naturally exhibit an inflationary behaviour which can overcome the shortcomings of the GR-based standard cosmological model. The cosmological scenarios arising from ETGs seem realistic and capable of reproducing observations of the the cosmic microwave background (CMB) [11-13]. It has been shown that, by means of conformal transformations, the higher order and non-minimally coupled terms can be related to Einstein gravity with one or more scalar fields minimally coupled to9 gravity [14-18].

Higher order terms always appear as contributions of even order in the field equations. For example, the term R^2 produces fourth order equations [19], $R\Box R$ gives sixth order equations [18, 20], $R\mu^2 R$ eighth order equations [21], and so on. By means of a conformal transformation, any second order derivative term corresponds to a scalar field [1]. Fourth-order gravity corresponds to Einstein gravity with one scalar field, sixth-order gravity to Einstein gravity with two scalar fields, *etc.* [18, 22]. It is also possible to show that $f(R)$ gravity is equivalent not only to a scalar-tensor theory, but also to GR plus an ideal fluid [23]. This feature becomes interesting if multiple inflationary events are desired, because an early inflationary stage could select very large scale structures (observed as clusters of galaxies today), while a later inflationary epoch could select smaller scale structures (observed as galaxies today) [20], with each inflationary era corresponding to the dynamics of a scalar field. Finally, these extended schemes could naturally solve the graceful exit problem bypassing the shortcomings of known inflationary models [13, 24].

In addition to the revision of standard cosmology at early epochs with the concept of inflation, a new approach is necessary also at late epochs. ETGs could play a fundamental role also in this context. In fact, the increasing bulk of data accumulated in the past few years have nurtured a new cosmological model referred to as the *Concordance Model*. The Hubble diagram of type Ia Supernovae (hereafter SNeIa) measured by both the Supernova Cosmology Project [25] and the High-z Team [26] up to redshifts $z:1$, has been the first piece of evidence that the universe is currently undergoing a phase of accelerated expansion. Balloon-born experiments, such as BOOMERanG [27] and MAXIMA [28], have detected the first and second peak in the anisotropy spectrum of the CMB radiation indicating that the geometry of the universe is spatially flat. In conjunction with constraints on the matter density parameter Ω_M coming from galaxy clusters, these data indicate that the universe is dominated by an unclustered fluid with negative pressure, generically dubbed *dark energy*, which is able to drive the accelerated expansion. This picture has been further strengthened by the recent precise measurements of the CMB spectrum obtained by the WMAP experiment [29-31], and by the extension of the SNeIa Hubble diagram to redshifts higher than one [32]. An overwhelming flood of papers has appeared following this observational evidence, presenting a great variety of models trying to explain this phenomenon. The simplest explanation is the well known cosmological constant Λ [33]. Although it is the best fit to most of the available astrophysical data [29], the ΛCDM model fails in explaining why the inferred value of Λ is so tiny (120 orders of magnitude smaller) in comparison with the typical vacuum energy values predicted by particle physics and why its energy density is comparable to the matter density today (the *coincidence problem*).

As a tentative solution, many authors have replaced the cosmological constant with a scalar field rolling down its potential and giving rise to the model referred to as *quintessence* [34, 35]. Even when successful in fitting the data, the quintessence approach to dark energy is still plagued by the coincidence problem since the dark energy and matter densities evolve differently and reach comparable values for a very limited portion of the cosmic evolution coinciding at the present era. To be more precise, the quintessence dark energy is tracking matter and evolves in the same way for a long time. But then, at late times, somehow it has to change its behavior from tracking the dark matter to dominating as a cosmological constant. This is the coincidence problem of quintessence.

Moreover, the origin of this quintessence scalar field is unknown, leaving a great uncertainty on the choice of the scalar field potential. The subtle and elusive nature of dark energy has led many authors to look for completely different scenarios able to give a quintessential behavior without the need for exotic components. To this end, it is worth stressing

[1]The dynamics of these scalar fields are governed given by a second order Klein-Gordon-like equation.

that the acceleration of the universe only calls for a dominant component with negative pressure, but does not tell us anything about the nature and the number of cosmic fluids filling the universe. This consideration suggests that it could be possible to explain the accelerated expansion by introducing a single cosmic fluid with an equation of state causing it to act like dark matter at high densities and dark energy at low densities. An attractive feature of these models, usually referred to as *Unified Dark Energy* (UDE) or *Unified Dark Matter* (UDM) models, is that such an approach naturally solves, al least phenomenologically, the coincidence problem. Interesting examples are the generalized Chaplygin gas [36], the tachyon field [37] and the condensate cosmology [38]. A different class of UDE models has been proposed [39] in which a single fluid is considered: its energy density scales with the redshift in such a way that the radiation-dominated era, the matter era, and the accelerating phase can be naturally achieved. These models are very versatile since they can be interpreted both in the framework of UDE models and as a two-fluid scenario with dark matter and scalar field dark energy. The main advantage of this approach is that a suitable generalized equation of state can be always obtained and observational data can be fitted.

There is a yet different way to address the problem of the cosmic acceleration. As stressed in [40], it is possible that the observed acceleration is not the manifestation of another ingredient of the cosmic pie, but rather the first signal of a breakdown of our understanding of the laws of gravitation in the infrared limit. From this point of view, it is tempting to modify the Friedmann equations to see whether it is possible to fit the astrophysical data with models comprising only standard matter. Interesting examples of this kind are the Cardassian expansion [41] and DGP gravity [42]. In the same framework it is possible to find alternative schemes in which a quintessential behavior is obtained by taking into account effective models coming from fundamental physics and giving rise to generalized or higher order gravity actions [43] (see [44] for a comprehensive review). For instance, a cosmological constant term may be recovered as a consequence of a non-vanishing torsion field, leading to a model consistent with both the SNeIa Hubble diagram and Sunyaev-Zel'dovich data of galaxy clusters [45]. SNeIa data could also be efficiently fitted including higher order curvature invariants in the gravitational Lagrangian [46, 47]. These alternative models provide naturally a cosmological component with negative pressure whose origin is related to the cosmic geometry, thus overcoming the problems linked to the physical significance of the scalar field.

The large number of cosmological models which constitute viable candidates to explain the observed accelerated expansion is evident from this short overview. On the one hand, this overabundance of models signals the fact that only a limited number of cosmological tests are available to discriminate between competing theories and, on the other hand, it shows that we are facing an urgent degeneracy problem. It is useful to remark that both the SNeIa Hubble diagram and the angular size-redshift relation of compact radio sources [48] are distance-based probes of cosmological models, so systematic errors and biases could be iterated. From this point of view, it is interesting to search for tests based on time-dependent observables. For example, one can take into account the *lookback time* to distant objects since this quantity can discriminate between different cosmological models. The lookback time is observationally estimated as the difference between the present age of the universe and the age of a given object at redshift z. Such an estimate is possible if the object is a galaxy observed in more than one photometric band since its color is determined by its age as a consequence of stellar evolution. It is thus possible to get an estimate of the galaxy age by measuring its magnitude in different bands and then using stellar evolutionary codes to choose the model that best reproduces the observed colors.

Coming to the weak-field-limit approximation, which essentially means considering Solar System scales, ETGs are expected to reproduce GR which, in any case, is firmly tested only in this limit [49]. This fact is a matter of debate since several relativistic theories do not reproduce exactly the Einsteinian results in the Newtonian approximation but, in some sense, generalize them. As first noticed by Stelle [50], an R^2-theory gives rise to Yukawa-like corrections in the Newtonian potential. This feature could have interesting physical consequences; for example, certain authors claim to explain the flat rotation curves of galaxies by using such terms [51]. Others [52] have shown that a conformal theory of gravity is nothing but a fourth-order theory containing such terms in the Newtonian limit. Besides, an apparent, anomalous, long-range acceleration in the data analysis of the Pioneer 10/11, Galileo, and Ulysses spacecrafts could be framed in a general theoretical scheme by taking into account corrections to the Newtonian potential [53].

In general, any relativistic theory of gravitation yields corrections to the Newtonian and post-Newtonian (PPN) potentials (e.g., [54]) which test the theory [49]. Furthermore, the newborn *gravitational lensing astronomy* [55] is generating additional tests of gravity over small, large, and very large scales which soon will provide direct measurements for the variation of the Newtonian coupling [56], the potential of galaxies, clusters of galaxies and several other features of self-gravitating systems. Such data, very likely, will be capable of confirming or ruling out the physical consistency of GR or of any ETG. In summary, the general features of ETGs are that the Einstein field equations are modified in two ways: $i)$ the geometry can be non-minimally coupled to some scalar field, and/or $ii)$ higher than second order derivatives of the metric appear. In the first case we deal with scalar-tensor theories of gravity; in the second case we have higher order theories. Combinations of non-minimally coupled and higher order terms can emerge as contributions to effective Lagrangians; then we have higher order-scalar-tensor theories of gravity.

From the mathematical point of view, the problem of reducing generalized theories to an Einstein-like form has been extensively discussed. Under suitable regularity conditions on the Lagrangian and using a Legendre transformation on the metric, higher order theories take the form of GR in which one or more scalar field(s) source of the gravitational field (see, e .g., [9, 57-59]). On the other hand, as discussed above, the mathematical equivalence between models with variable gravitational coupling and Einstein gravity has been studied using suitable conformal transformations [60, 61]. A debate on the physical meaning of these conformal transformations seems to be ongoing ([62] and references therein). Several authors claim a physical difference between Jordan frame (higher order theories and/or variable gravitational couplings) since there is experimental and observational evidence suggesting that the Jordan frame is better suited for matching solutions and data. Others state that the true physical frame is the Einstein one according to the energy theorems [59]. However, the discussion is open and no definitive conclusion seems to have been reached. The problem becomes more involved at the semiclassical and quantum level, and should be faced from a more general point of view---the Palatini approach to gravity could be useful to this goal.

The Palatini approach to gravitational theories was first introduced and analyzed by Einstein himself [63], but was named as a consequence of an historical misunderstanding [64, 65].

The fundamental idea of the Palatini formalism is to consider the torsion-free connection $\Gamma^{\mu}_{\alpha\beta}$ entering the definition of the Ricci tensor, to be independent of the spacetime metric $g_{\mu\nu}$. The Palatini formulation of the standard Einstein-Hilbert theory turns out to be equivalent to the purely metric theory. This property follows from the fact that the field equations for the connection $\Gamma^{\mu}_{\alpha\beta}$, considered to be independent of the metric, produce the Levi-Civita connection of the metric $g_{\mu\nu}$. As a consequence, there is no reason to impose the Palatini variational principle instead of the metric variational principle in the Einstein Hilbert theory. However, the situation changes if we consider the ETGs, which depend on functions of the curvature invariants (such as $f(R)$ theories) or couple non-minimally to some scalar field. In these cases the Palatini and the metric variational principles provide different field equations and the theories thus derived differ [59, 66]. The relevance of the Palatini approach for cosmological applications in this framework has been recently demonstrated [43-44, 67].

From the physical point of view, considering the metric $g_{\mu\nu}$ and the connection $\Gamma^{\mu}_{\nu\alpha}$ as independent fields means to decouple the metric structure of spacetime and its geodesic structure (the connection $\Gamma^{\mu}_{\alpha\beta}$, in general, is not the Levi-Civita connection of $g_{\mu\nu}$). The causal structure of spacetime is governed by $g_{\mu\nu}$ while the spacetime trajectories of particles are governed by $\Gamma^{\mu}_{\alpha\beta}$. This decoupling enriches the geometric structure of spacetime and generalizes the purely metric formalism. This metric-affine structure of spacetime is naturally translated, by means of the Palatini field equations, into a bi-metric structure of spacetime. Besides the physical metric $g_{\mu\nu}$, another metric $\tilde{g}_{\mu\nu}$ appears. This new metric is related, in the case of $f(R)$ gravity, to the connection. The connection $\Gamma^{\mu}_{\alpha\beta}$ turns out to be the Levi-Civita connection of $\tilde{g}_{\mu\nu}$ and provides the geodesic structure of spacetime.

For non-minimally coupled interactions in the gravitational Lagrangian in scalar-tensor theories, the new metric $\tilde{g}_{\mu\nu}$ is related to the non-minimal coupling; $\tilde{g}_{\mu\nu}$ can be related to a different geometric and physical aspect of the gravitational theory. Thanks to the Palatini formalism, the non-minimal coupling and the scalar field, entering the evolution of the gravitational fields, are separated from the metric structure of spacetime. The situation mixes when we consider the case of higher order-scalar-tensor theories. Due to these features, the Palatini approach could contribute to clarify the physical meaning of conformal transformations [69].

In this review paper, without claiming for completeness, we want to give a survey on the formal and phenomenological aspects of ETGs in metric and Palatini approaches, considering the cosmological and astrophysical applications of some ETG models. The layout is the following. The field equations for generic ETGs are derived in Sec.2. Specifically, we discuss metric, Palatini and metric-affine approaches. In Sec. 3 the e quivalence of metric and Palatini $f(R)$ gravities with Brans-Dicke theories are discussed. In Sec. 4 we introduce theoretical and experimental viability of $f(R)$-gravity. Briefly, we discuss on the correct cosmological dynamics and on the instabilities for a particular case of $f(R)$. After we discuss the precence of ghost fields and the wealk field limit for metric approach. Finally we consider the growth of cosmological perturbations and the Chauchy problem. Cosmological applications are considered in Sec.5-6. We show that dark energy and the dark matter can be addressed as "curvature effects", if ETGs (in particular $f(R)$ theories) are considered. We work out some cosmological models comparing the solutions with data coming from observational surveys. As further result in Sec. 7. , we show that also the stochastic cosmological background of gravitational waves can be "tuned" by ETGs. This fact could open new perspective also in the problems of detection of gravitational waves which should be investigated not only in the standard GR-framework. Discussion and conclusions are drawn in Sec.8.

2. THE THREE VERSIONS OF *f* (*R*)- GRAVITY

In this survey we focus on $f(R)$- gravity (see [70] for a more comprehensive discussion and a list of references, and [71] for short introductions to the subject). In these theories the Einstein-Hilbert action[2]

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,R + S^{(m)} \tag{1}$$

is modified to

[2]Here R is the Ricci curvature of the metric tensor $g_{\mu\nu}$, which has determinant g, G is Newton's constant, and $\kappa \equiv 8\pi G$. We mostly follow the notations of Ref. [72].

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(R) + S^{(m)}\,, \tag{2}$$

where $f(R)$ is a non-linear function of its argument and $S^{(m)}$ is the matter part of the action. Actually, there are two variational principles that one can apply to the Einstein-Hilbert action in order to derive Einstein's equations: the standard metric variation and a less standard variation dubbed Palatini variation. In the latter the metric and the connection are assumed to be independent variables and one varies the action with respect to both of them, under the important assumption that the matter action does not depend on the connection. The choice of the variational principle is usually referred to as a formalism, so one can use the terms metric (or second order) formalism and Palatini (or first order) formalism. However, even though both variational principles lead to the same field equation for an action whose Lagrangian is linear in R, this is no longer true for a more general action. Therefore, it is intuitive that there will be two version of $f(R)$-gravity, according to which variational principle or formalism is used. Indeed this is the case: $f(R)$-gravity in the metric formalism is called *metric* $f(R)$-*gravity* and $f(R)$-gravity in the Palatini formalism is called *Palatini* $f(R)$-*gravity*.

Finally, there is actually even a third version of $f(R)$-gravity: *metric-affine* $f(R)$-*gravity*. This comes about if one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. Clearly, metric affine $f(R)$-gravity is the most general of these theories and reduces to metric or Palatini $f(R)$-gravity if further assumptions are made. In this section we will present the actions and field equations of all three versions of $f(R)$- gravity and point out their difference. We will also clarify the physical meaning behind the assumptions that discriminate them.

Then brefly has we show above three versions of $f(R)$-gravity have been studied:

- metric (or second order) formalism;
- Palatini (or first order) formalism;

and

- metric-affine gravity.

These families of theories are discussed in the following.

2.1. Metric $f(R)$- Gravity

In the metric formalism the action is

$$S_{metric} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(R) + S^{(m)}, \tag{3}$$

and its variation with respect to $g^{\mu\nu}$ yields, after some manipulations and modulo surface terms, the field equation

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + \kappa T_{\mu\nu}\,, \tag{4}$$

with a prime denoting differentiation with respect to R, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric, and $\Box \equiv \nabla^\mu\nabla_\mu$. Fourth order derivatives of the metric appear in the first two terms on the right hand side, justifying the alternative name "fourth order gravity" used for this class of theories.

By taking the trace of eq. (4) one obtains

$$3\Box f'(R) + Rf'(R) - 2f(R) = \kappa T\,, \tag{5}$$

where $T \equiv T^\alpha{}_\alpha$ is the trace of the energy-momentum tensor of matter. This second order differential equation for $f'(R)$ is qualitatively different from the trace of the Einstein equation $R = -\kappa T$ which, instead, constitutes an algebraic relation between T and the Ricci scalar, displaying the fact that $f'(R)$ is a dynamical (scalar) degree of freedom of the theory. This is already an indication that the field equations of $f(R)$ theories will admit a larger variety of solutions than Einstein's theory. As an example, we mention here that the Jebsen-Birkhoff's theorem, stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, no longer holds in metric $f(R)$ gravity. Without going into details, let us stress that $T = 0$ no longer implies that $R = 0$, or is even constant. Eq. (5) will turn out to be very useful in studying various aspects of $f(R)$ gravity, notably its stability and weak-field limit. For the moment, let us use it to make some remarks about maximally symmetric solutions. Recall that maximally symmetric solutions lead to a constant Ricci scalar. For $R = constant$ and $T_{\mu\nu} = 0$, eq. (5) reduces to

$$f'(R)R - 2f(R) = 0, \tag{6}$$

which, for a given f, is an algebraic equation in R. If $R = 0$ is a root of this equation and one takes this root, then eq. (4) reduces to $R_{\mu\nu} = 0$ and the maximally symmetric solution is Minkowski spacetime. On the other hand, if the root of eq. (6) is $R = C$, where C is a constant, then eq. (4) reduces to $R_{\mu\nu} = g_{\mu\nu}C/4$ and the maximally symmetric solution is de Sitter or anti-de Sitter space depending on the sign of C, just as in GR with a cosmological constant. Another issue that should be stressed is that of energy conservation. In metric $f(R)$ gravity the matter is minimally coupled to the metric. One can, therefore, use the usual arguments based on the invariance of the action under diffeomorphisms of the spacetime manifold [coordinate transformations $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ followed by a pullback, with the field ξ^μ vanishing on the boundary of the spacetime region considered, leave the physics unchanged, see [72] to show that $T_{\mu\nu}$ is divergence-free. The same can be done at the level of the field equations: a "brute force" calculation reveals that the left hand side of eq. (4) is divergence-free (generalized Bianchi identity) implying that $\nabla_\mu T^{\mu\nu} = 0$.

The field equation (4) can be rewritten as form of Einstein equations with an effective stress-energy tensor to the right hand side. Specifically, as

$$G_{\mu\nu} = \kappa\left(T_{\mu\nu} + T_{\mu\nu}^{(eff)}\right) \quad (7)$$

where

$$T_{\mu\nu}^{(eff)} = \frac{1}{\kappa}\left[\frac{f(R) - Rf'(R)}{2}g_{\mu\nu} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] \quad (8)$$

is an effective energy-momentum tensor constructed with geometric terms. Since $T_{\mu\nu}^{(eff)}$ is only a formal energy-momentum tensor, it is not expected to satisfy any of the energy conditions deemed reasonable for physical matter, in particular the effective energy density cannot be expected to be positive-definite. An effective gravitational coupling $G_{eff} \equiv G/f'(R)$ can be defined in a way analogous to scalar-tensor gravity. It is apparent that $f'(R)$ must be positive for the graviton to carry positive kinetic energy.

Motivated by the recent cosmological observations, we adopt the spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric to describe the universe,

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \quad (9)$$

where a is the scale factor. Then, the field equations of metric $f(R)$ cosmology become

$$H^2 = \frac{\kappa}{3f'(R)}\left[\rho^{(m)} + \frac{Rf'(R) - f(R)}{2} - 3H\dot{R}f''(R)\right], \quad (10)$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'(R)}\left[P^{(m)} + f'''(R)\left(\dot{R}\right)^2 + 2H\dot{R}f''(R) + \ddot{R}f''(R) + \frac{f(R) - Rf'(R)}{2}\right], \quad (11)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and an overdot denotes differentiation with respect to the comoving time t. The corresponding phase space is a 2-dimensional curved manifold embedded in a 3-dimensional space and with a rather complicated structure [73].

2.2. Palatini $f(R)$- Gravity

In the Palatini version of $f(R)$ gravity, both the metric $g_{\mu\nu}$ and the connection $\Gamma^\mu_{\nu\gamma}$ are regarded as independent variables. In other words, the connection is not the metric connection of $g_{\mu\nu}$. While in GR the metric and Palatini variations produce the same field equations (i.e., the Einstein equations), for non-linear Lagrangians one obtains two different sets of field equations.[3]

Palatini $f(R)$ gravity was proposed as an alternative to dark energy, on the same footing as metric $f(R)$ models. The original model advanced for this purpose was based on the specific form $f(R) = R - \mu^4/R$ [67].

The Palatini action is

$$S_{Palatini} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(\tilde{R}) + S^{(m)}\left[g_{\mu\nu}, \psi^{(m)}\right], \quad (12)$$

where a distinction needs to be made between two different Ricci tensors contained in the theory. $R_{\mu\nu}$ is constructed from the metric connection of the (unique) physical metric $g_{\mu\nu}$, while $\tilde{R}_{\mu\nu}$ is the Ricci tensor of the non-metric connection $\Gamma^\mu_{\nu\gamma}$ and defines the scalar $\tilde{R} \equiv g^{\mu\nu}\tilde{R}_{\mu\nu}$. The matter part of the action does not depend explicitly from the connection $\Gamma^\mu_{\alpha\beta}$, but only from the metric and the matter fields, which we collectively label as $\psi^{(m)}$.

By varying the Palatini action (12) one obtains the field equation

$$f'(\tilde{R})\tilde{R}_{\mu\nu} - \frac{f(\tilde{R})}{2}g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (13)$$

in which no second covariant derivative of f' appears, in contrast with eq. (4). An independent variation with respect to the connection yields

$$\tilde{\nabla}_\sigma\left(\sqrt{-g}\,f'(\tilde{R})g^{\mu\nu}\right) - \tilde{\nabla}_\sigma\left(\sqrt{-g}\,f'(\tilde{R})g^{\sigma(\mu}\right)\delta^{\nu)}_\gamma = 0, \quad (14)$$

where $\tilde{\nabla}_\gamma$ denotes the covariant derivative associated to the (non-metric) connection $\Gamma^\mu_{\alpha\beta}$.

By tracing eqs. (13) and (14) we obtain

$$f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = \kappa T \quad (15)$$

and

$$\tilde{\nabla}_\gamma\left(\sqrt{-g}\,f'(\tilde{R})g^{\mu\nu}\right) = 0, \quad (16)$$

respectively. Eq. (16) is interpreted as stating that $\tilde{\nabla}_\gamma$ is the covariant derivative of the "new" metric tensor

$$\tilde{g}_{\mu\nu} \equiv f'(\tilde{R})g_{\mu\nu} \quad (17)$$

conformally related to $g_{\mu\nu}$. Eq. (15) is an algebraic (or trascendental, according to the functional form of $f(R)$) equation for $f'(\tilde{R})$, not a differential equation describing its evolution. Therefore, $f'(R)$ is a non-dynamical quantity, in contrast to what happens in metric $f(R)$ gravity. The lack of dynamics has consequences which are discussed below. It is possible to eliminate the non-metric connection from the field equations by rewriting them as

$$G_{\mu\nu} = \frac{\kappa}{f'}T_{\mu\nu} - \frac{1}{2}\left(R - \frac{f}{f'}\right)g_{\mu\nu} + \frac{1}{f'}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' - \frac{3}{2(f')^2}\left[\nabla_\mu f\nabla_\nu f' - \frac{1}{2}g_{\mu\nu}\nabla_\gamma f\nabla^\gamma f'\right]. \quad (18)$$

[3]By imposing that the metric and Palatini variations generate the same field equations, Lovelock gravity is selected [74]. GR is a special case of Lovelock theory.

2.3. Metric-affine $f(R)$- Gravity

The third family of $f(R)$ theories, metric-affine $f(R)$ gravity [75], is characterized by the fact that also the matter part of the action depends explicitly on the connection Γ, as described by the action

$$S_{affine} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f\left(\tilde{R}\right) + S^{(m)}\left[g_{\mu\nu}, \Gamma^{\mu}_{\nu\gamma}, \psi^{(m)}\right]. \tag{19}$$

$\Gamma^{\mu}_{\alpha\beta}$ is possibly a non-symmetric connection, which would lead to torsion associated with matter and to a reincarnation of torsion theories. The latter were introduced in view of elementary particles, rather than cosmology, by coupling the spin of elementary particles to the torsion. The study of metric-affine $f(R)$ gravity has not been completed yet, in particular its cosmological consequences have not been fully elucidated. It is for this reason that our discussion will be limited to metric and Palatini $f(R)$ gravity in what follows.

3. EQUIVALENCE OF METRIC AND PALATINI $f(R)$- GRAVITIES WITH BRANS-DICKE THEORIES

In the same way that one can make variable redefinitions in classical mechanics in order to bring an equation describing a system to a more attractive, or easy to handle, form (and in a very similar way to changing coordinate systems), one can also perform field redefinitions in a field theory, in order to rewrite the action or the field equations.

There is no unique prescription for redefining the fields of a theory. One can introduce auxiliary fields, perform renormalizations or conformal transformations, or even simply redefine fields to one's convenience. It is important to mention that, at least within a classical perspective such as the one followed here, two theories are considered to be dynamically equivalent if, under a suitable redefinition of the gravitational and matter fields, one can make their field equations coincide. The same statement can be made at the level of the action. Dynamically equivalent theories give exactly the same results when describing a dynamical system which falls within the purview of these theories. There are clear advantages in exploring the dynamical equivalence between theories: we can use results already derived for one theory in the study of another, equivalent, theory.

The term "*dynamical equivalence*" can be considered misleading in classical gravity. Within a classical perspective, a theory is fully described by a set of field equations. When we are referring to gravitation theories, these equations describe the dynamics of gravitating systems. Therefore, two dynamically equivalent theories are actually just different representations of the same theory (which also makes it clear that all allowed representations can be used on an equal footing).

The issue of distinguishing between truly different theories and different representations of the same theory (or dynamically equivalent theories) is an intricate one. It has serious implications and has been the cause of many misconceptions in the past, especially when conformal transformations are used in order to redefine the fields (e.g., the Jordan and Einstein frames in scalar-tensor theory). In what follows, we review the equivalence between metric and Palatini $f(R)$ gravity with specific theories within the Brans-Dicke class with a potential.

Metric $f(R)$ gravity is equivalent to an $\omega = 0$ Brans-Dicke theory[4] when $f''(R) \neq 0$ [3], while Palatini modified gravity is equivalent to one with $\omega = -3/2$. The equivalence has been rediscovered several times over the years, often in the context of particular theories [76].

3.1. Metric Formalism

It has been noticed quite early that metric quadratic gravity can be cast into the form of a Brans-Dicke theory and it did not take long for these results to be extended to more general actions which are functions of the Ricci scalar of the metric . Let us present this equivalence in some detail.

We will work at the level of the action but the same approach can be used to work directly at the level of the field equations. We begin with metric $f(R)$ gravity. Let $f''(R)$ be non-vanishing and consider the action (2); by using the auxiliary scalar field $\phi = R$, it is easy to see that the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\psi(\phi)R - V(\phi)\right] + S^{(m)} \tag{20}$$

with

$$\psi(\phi) = f'(\phi), \qquad V(\phi) = \phi f'(\phi) - f(\phi) \tag{21}$$

is equivalent to the previous one. It is trivial that (20) reduces to (2) if $\phi = R$. Vice-versa, the variation of (20) with respect to $g^{\mu\nu}$ yields

$$G_{\mu\nu} = \frac{1}{\psi}\left(\nabla_\mu\nabla_\nu\psi - g_{\mu\nu}\Box\psi - \frac{V}{2}g_{\mu\nu}\right) + \frac{\kappa}{\psi}T_{\mu\nu}\,. \tag{22}$$

The variation with respect to ϕ, instead, gives us

$$R\frac{d\psi}{d\phi} - \frac{dV}{d\phi} = \left(R - \phi\right)f''(\phi) = 0\,, \tag{23}$$

from which it follows that $\phi = R$ because $f'' \neq 0$. The scalar field $\phi = R$ is clearly a dynamical quantity which obeys the trace equation

$$3f''(\phi)\Box\phi + 3f'''(\phi)\nabla^\alpha\phi\nabla_\alpha\phi + \phi f'(\phi) - 2f(\phi) = \kappa T \tag{24}$$

and is massive. Its mass squared

$$m_\phi^2 = \frac{1}{3}\left(\frac{f_0'}{f_0''} - R_0\right) \tag{25}$$

is computed in the analysis of small perturbations of de Sitter space (here a zero subscript denotes quantities evaluated at the constant curvature R_0 of the de Sitter background). It is

[4]The Brans-Dicke action for general values of the Brans-Dicke parameter ω is $S_{BD} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla^\gamma\phi\nabla_\gamma\phi - V(\phi)\right] + S^{(m)}$.

convenient to consider, instead of ϕ, the scalar $\psi \equiv f'(\phi)$ obeying the evolution equation

$$3\Box\psi + 2U(\psi) - \psi\frac{dU}{d\psi} = \kappa T \,, \tag{26}$$

where $U(\psi) = V(\phi(\psi)) - f(\phi(\psi))$.

To summarize, metric $f(R)$ gravity contains a scalar degree of freedom and the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\psi R - U(\psi)\right] + S^{(m)} \,, \tag{27}$$

is identified as an $\omega = 0$ Brans-Dicke theory. This theory ("massive dilaton gravity") was introduced in the 1970's in order to generate a Yukawa term in the Newtonian limit [77], and then abandoned. The assumption $f'' \neq 0$ is interpreted as the requirement of invertibility of the change of variable $R \to \psi(R)$.

3.2. Palatini Formalism

In Palatini modified gravity the equivalence with a Brans-Dicke theory is discovered in a way similar to that of the metric formalism. Beginning with the action (12) and defining $\phi \equiv \tilde{R}$ and $\psi \equiv f'(\phi)$, it is seen that, apart from an irrelevant boundary term, the action can be rewritten as

$$S_{Palatini} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\psi R + \frac{3}{2\psi}\nabla^\gamma\psi\nabla_\gamma\psi - V(\psi)\right] + S^{(m)} \tag{28}$$

in terms of the metric $g_{\mu\nu}$ and its Ricci tensor $R_{\mu\nu}$. Here we have used the property that, since $\tilde{g}_{\mu\nu} = \psi\, g_{\mu\nu}$, the Ricci curvatures of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ satisfy the relation

$$\tilde{R} = R + \frac{3}{2\psi}\nabla^\gamma\psi\nabla_\gamma\psi - \frac{3}{2}\Box\psi \,. \tag{29}$$

The action (28) is easily identified as a Brans-Dicke theory with Brans-Dicke parameter $\omega = -3/2$.

4. THEORETICAL AND EXPERIMENTAL VIABILITY OF *f* (*R*)- GRAVITY

In order to be acceptable, $f(R)$ theories should not only reproduce the current acceleration of the universe, but they must also satisfy the constraints imposed by Solar System and terrestrial experiments on relativistic gravity, and they must obey certain minimal requirements for theoretical viability. More precisely, these families of theories must:

- possess the correct cosmological dynamics;
- be free from instabilities and ghosts;
- attain the correct Newtonian and post-Newtonian limits;
- originate cosmological perturbations compatible with the observations of the CMB and with large scale structure surveys; and
- possess a well-formulated and well-posed initial value problem.

If a single one of these criteria is not met the theory should be regarded as unviable. In the following we examine how $f(R)$ gravity performs with regard to these criteria.

4.1. Correct Cosmological Dynamics

According to the tenets of standard cosmology, an acceptable cosmological model must contain an early inflationary era (or possibly another mechanism) solving the horizon, flatness, and monopole problems and generating density perturbations, followed by a radiation- and then a matter-dominated era. The present accelerated epoch then begins, possibly explained by $f(R)$ gravity. The future universe usually consists of an eternal de Sitter attractor, or ends in a Big Rip singularity [87]. Smooth transitions between different eras are necessary. The exit from the radiation era, in particular, was believed to be impossible in many models [78], but this proved to be not true. In fact, exit from the radiation or any era can be obtained as follows. In the approach dubbed "designer $f(R)$ gravity" in [74], the desired expansion history of the universe can be obtained by specifying the desired scale factor $a(t)$ and integrating an ordinary differential equation for the function $f(R)$ that produces the chosen $a(t)$ [79]. In general, the solution to this ODE is not unique and can assume a form that appears rather contrived in comparison with simple forms adopted in most popular models.

4.2. Instabilities

The choice $f(R) = R - \mu^4/R$ with $\mu : H_0 : 10^{-33}$ eV is again the prototypical example model to discuss instabilities. Shortly after it was advanced as an explanation of the cosmic acceleration, this model was found to suffer from the pernicious "Dolgov-Kawasaki" instability [80]. This type of instability was later shown to be common to any metric $f(R)$ theory with $f''(R) < 0$ ([81]) and the extension to even more general gravitational theories has been discussed [82]. Let us parametrize the deviations from GR as

$$f(R) = R + \varepsilon\varphi(R) \tag{30}$$

with $\varepsilon > 0$ a small constant with the dimensions of a mass squared and φ dimensi onless. The trace equation for the Ricci scalar R becomes

$$\Box R + \frac{\varphi'''}{\varphi''}\nabla^\gamma R\nabla_\gamma R + \left(\frac{\varepsilon\varphi' - 1}{3\varepsilon\varphi''}\right)R = \frac{\kappa T}{3\varepsilon\varphi''} + \frac{2\varphi}{3\varphi''} \,. \tag{31}$$

By expanding around a de Sitter background and writing the metric *locally* as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{32}$$

and the scalar R as

$$R = -\kappa T + R_1 \,, \tag{33}$$

with R_1 a perturbation, the first order trace equation translates into the dynamical equation for R_1

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''}\dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''}\vec{\nabla}T \cdot \vec{\nabla}R_1$$
$$+ \frac{1}{3\varphi''}\left(\frac{1}{\varepsilon} - \varphi'\right)R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{\left(\kappa T\varphi^2 + 2\varphi\right)}{3\varphi''}. \tag{34}$$

The expression containing ε^{-1} dominates the last term on the left hand side, giving the effective mass squared of R_1

$$m^2 \simeq \frac{1}{3\varepsilon\varphi''}. \tag{35}$$

Therefore, the theory is stable if $f''(R) > 0$ and unstable if $f''(R) < 0$. Strictly speaking, GR is excluded by the assumption $f'' \neq 0$, but the well-known stability of this case can easily be included by writing the stability criterion for metric $f(R)$ gravity as $f'' \geq 0$.

To go back to the example model of [80] $f(R) = R - \mu^4/R$, this is unstable because $f'' < 0$. The small scale μ determines the time scale for the onset of this instability as $:10^{-26}$ s [80], making this an explosive instability.

A physical interpretation of this stability criterion is the following [83]: the effective gravitational coupling is $G_{eff} = G/f'(R)$ and, if $dG_{eff}/dR = -f''G/(f')^2 > 0$ (corresponding to $f'' < 0$), then G_{eff} increases with R and a large curvature causes gravity to become stronger and stronger, which in turn causes a larger R, in a positive feedback loop. If instead $dG_{eff}/dR < 0$, then a negative feedback stops the growth of the gravitational coupling.

What about Palatini $f(R)$ gravity? Since this formalism contains only second order field equations and the trace equation $f'(\tilde{R})\tilde{R} - 2f(\tilde{R}) = \kappa T$ is not a differential equation but rather a non-dynamical equation, as noted above, there is no Dolgov-Kawasaki instability [84].

The discussion of metric $f(R)$ instabilities presented above is based on the local expansion (32) and, therefore, is limited to short wavelength modes (compared to the curvature radius). However, it can be extended to the longest wavelengths in the case of a de Sitter background [85]. This extension requires a more complicated formalism because long modes introduce inhomogeneities and are affected by the notorious gauge-dependence problems of cosmological perturbations. A covariant and gauge-invariant formalism is needed here. One proceeds by assuming that the background space is de Sitter and by considering the general action

$$S = \int d^4x\sqrt{-g}\left[\frac{f(\phi, R)}{2} - \frac{\omega(\phi)}{2}\nabla^\gamma\phi\nabla_\gamma\phi - V(\phi)\right] \tag{36}$$

containing $f(R)$ and scalar-tensor gravity as special cases, and mixtures of them. The field equations originating from this action become, in a FLRW background space,

$$H^2 = \frac{1}{3f'}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{Rf' - f}{2} + V - 3H\dot{f}\right), \tag{37}$$

$$\dot{H} = \frac{-1}{2f'}\left(\omega\dot{\phi}^2 + \ddot{f}' - H\dot{f}'\right), \tag{38}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\frac{d\omega}{d\phi}\dot{\phi}^2 - \frac{\partial f}{\partial\phi} + 2\frac{dV}{d\phi}\right) = 0. \tag{39}$$

de Sitter space is a solution of the field equations provided that the conditions

$$6H_0^2 f_0' - f_0 + 2V_0 = 0, \qquad f_0' = 2V_0', \tag{40}$$

are satisfied. An analysis of inhomogeneous perturbations of small amplitude and arbitrary wavelengths [85] using the covariant and gauge-invariant Bardeen-Ellis-Bruni-Hwang formalism [86] in Hwang's version [87] for alternative gravitational theories yields the stability condition in the zero momentum limit

$$\frac{(f_0')^2 - 2f_0 f_0''}{f_0' f_0''} \geq 0, \tag{41}$$

This is the stability condition of de Sitter space in metric $f(R)$ gravity with respect to *inhomogeneous* perturbations and coincides with the corresponding stability condition with respect to *homogeneous* perturbations [83].

The equivalence between metric $f(R)$ gravity and an $\omega = 0$ Brans-Dicke theory holds also at the level of perturbations; doubts advanced to this regard have now been resolved. The stability condition of de Sitter space with respect to inhomogeneous perturbations in $\omega = 0$ Brans-Dicke theory is given again by eq. (41), while that for stability with respect to homogeneous perturbations is

$$\frac{(f_0')^2 - 2f_0 f_0''}{f_0'} \geq 0. \tag{42}$$

This inequality is again equivalent to (41) if stability against *local* perturbations (*i.e.*, $f_0'' > 0$) is also required. Hence, metric $f(R)$ gravity and $\omega = 0$ Brans-Dicke theory are equivalent also with regard to perturbations.

Beyond the linear approximation, metric $f(R)$ theories have been shown to be susceptible to non-linear instability, potentially threatening the possibility of constructing models of relativistic stars in strong $f(R)$ gravity. Inside compact objects with spherical symmetry, a singularity could develop if R becomes large [89]. Avoiding this singularity requires some degree of fine-tuning. Various authors have contended that this problem can be cured by adding, for example, a quadratic term αR^2 to the action as first [88, 90]. This problem needs further study, since it could be the biggest challenge left for metric $f(R)$ theories.

4.3. Ghost Fields

Ghosts are massive states of negative norm which ruin unitarity and appear frequently in attempts to quantize

Einstein's theory. Fortunately, $f(R)$ gravity theories are free of ghosts. More general ETGs of the form $f\left(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma}, ...\right)$, in general, are plagued by the presence of ghosts. A possible exception under certain conditions studied in [91] is provided by theories in which the extra terms are restricted to appear in the Gauss-Bonnet combination $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma}$, as in $f = f(R, G)$. Then, the field equations reduce to second order equations without ghosts [92, 93].

4.4. The Weak-Field Limit for Metric $f(R)$ Gravity

After errors and omissions in the early treatments of the weak-field limit of metric and Palatini modified gravity, a satisfactory discussion of the particular model $f(R) = R - \mu^4/R$ in the metric formalism appeared [94], followed by the generalization to arbitrary forms of the function $f(R)$ [95, 96].

One studies the PPN parameter γ which is constrained by light deflection experiments in the Solar System. The goal consists of finding the weak-field solution of the field equations and, using this solution, computing the parameter γ. A static, spherically symmetric, non-compact body which constitutes a perturbation of a background de Sitter universe is considered, as described by the line element

$$ds^2 = -\left[1 + 2\Psi(r) - H_0^2 r^2\right]dt^2 + \left[1 + 2\Phi(r) + H_0^2 r^2\right]dr^2 + r^2 d\Omega^2 \quad (43)$$

in Schwarzschild coordinates, with $d\Omega^2$ being the line element on the unit 2-sphere. Ψ and Φ are post-Newtonian potentials with small amplitudes, i.e., $|\Psi(r)|, |\Phi(r)| << 1$, and small (non-cosmological) scales such that $H_0 r << 1$ are considered. The Ricci scalar is expanded around the constant curvature of the background de Sitter space as $R(r) = R_0 + R_1$. The PPN parameter γ is then given by $\gamma = -\Phi(r)/\Psi(r)$ [49]. The analysis relies upon three assumptions [96]:

1. $f(R)$ is analytical at R_0;

2. $mr << 1$, where m is the effective mass of the scalar degree of freedom of the theory. In other words, this scalar field is assumed to be light and with a range larger than the size of the Solar System (there are no experimental constraints on scalars with range $m^{-1} < 0.2$ mm).

3. The matter composing the spherical body has negligible pressure, $P \simeq 0$ and $T = T_0 + T_1 \simeq -\rho$.

While it is easy to satisfy the first and the last assumptions, the second one is more tricky, as discussed below. The trace equation (5) turns into

$$\nabla^2 R_1 - m^2 R_1 = \frac{-\kappa\rho}{3f_{0''}}, \quad (44)$$

regulating the Ricci scalar perturbation, where

$$m^2 = \frac{(f_{0'})^2 - 2f_0 f_{0''}}{3f_{0'} f_{0''}} \quad (45)$$

is the effective mass squared of the scalar, which reproduces the expression derived in the gauge-invariant stability analysis of de Sitter space and in propagator calculations.

If $mr << 1$, the solution of the linearized field equations is

$$\Psi(r) = \frac{-\kappa M}{6\pi f_0'}\frac{1}{r}, \quad (46)$$

$$\Phi(r) = \frac{\kappa M}{12\pi f_0'}\frac{1}{r}, \quad (47)$$

and the PPN parameter γ is given by

$$\gamma = \frac{-\Phi(r)}{\Psi(r)} = \frac{1}{2}. \quad (48)$$

This value manifestly violates the experimental bound [97]

$$|\gamma - 1| < 2.3 \cdot 10^{-5}. \quad (49)$$

This violation would mark the demise of metric $f(R)$ gravity were it not for the fact that the second assumption necessary to perform this calculation is usually not satisfied. In fact, mr fails to be smaller than unity due to the *chameleon effect*. This effect consists of a dependence of the effective mass m on the spacetime curvature or, alternatively, on the matter density of the surroundings. The scalar degree of freedom can have a short range (for example, $m > 10^{-3}$ eV, corresponding to a range $\lambda < 0.2$ mm) at Solar System densities, escaping the experimental constraints, and have a long range at cosmological densities, which allows it to have an effect on the cosmological dynamics [93, 98]. While the chameleon effect may seem a form of fine-tuning, one should bear in mind that $f(R)$ gravity is complicated and the effective range does indeed depend on the environment. The chameleon mechanism is not arranged, but is built into the theory and is well-known and accepted in quintessence models, in which it was originally discovered [99]. It has been studied for many forms of the function $f(R)$ which pass the observational tests. For example, the model

$$f(R) = R - (1-n)\mu^2\left(\frac{R}{\mu^2}\right)^n \quad (50)$$

is compatible with the PPN limits if $\mu : 10^{-50}$ eV$: 10^{-17} H_0$ [98]. To understand how this model can work it is sufficient to note that a correction $: R^n$ to the Einstein-Hilbert Lagrangian R with $n < 1$ will eventually dominate as $R \rightarrow 0^+$. The model (50) agrees with the experimental data but could be essentially indistinguishable from a dark energy model. Discriminating between dark energy and $f(R)$ models, or between modified gravity scenarios should be possible on the basis of the growth history of cosmological perturbations.

4.5. Growth of Cosmological Perturbations

Since the spatially homogeneous and isotropic FLRW metric solves the field equations of many gravitational theories, the expansion history of the universe by itself cannot discriminate between various ETGs. However, the growth of structures depends on the theory of gravity considered and has the potential to achieve this goal. A typical study is that of Ref. [100]; these authors postulate an expansion history $a(t)$ characteristic of the ΛCDM model and find that vector and tensor modes are not affected by $f(R)$ corrections to Einstein gravity, to lowest order, and can be neglected, whereas scalar modes do depend on the theory chosen. In [100] the stability condition $f''(R) > 0$ discussed above for scalar perturbations is also recovered. It is found there that $f(R)$ corrections lower the large angle anisotropies of the cosmic microwave background and produce correlations between cosmic microwave background and galaxy surveys which are different from those obtained in dark energy models. A rigorous and mathematically self-consistent approach to the problem of cosmological perturbations in $f(R)$-gravity as been developed using covariant and gauge-invariant quantities in [101-103].

The study of structure formation in modified gravity is still uncomplete and, most of the times, is carried out within specific $f(R)$ models. Insufficient attention has been paid to the fact that some of these models are already ruled out because they contradict the weak-field limit or the stability conditions. A similar situation is found in Palatini models which, for this reason, will not be discussed here with regard to their weak-field limit and cosmological perturbations.

4.6. The Initial Value Problem

A physical theory is required to make predictions and, therefore, it must have a well-posed Cauchy problem. GR satisfies this requirement for "reasonable" forms of matter [72]. The well-posedness of the initial value problem for vacuum $f(R)$ gravity was briefly discussed for special metric models a long time ago [104]. Owing to the equivalence between $f(R)$ gravity and scalar-tensor gravity when $f''(R) \neq 0$, the initial value problem of $f(R)$ gravity is reduced to the one for Brans-Dicke gravity with $\omega = 0$ or $-3/2$. The Cauchy problem was shown to be well-posed for particular scalar-tensor theories in [104, 105] but a general analysis has been completed only relatively recently [106, 107]. A separate treatment, however, was necessary for $\omega = 0, -3/2$ Brans-Dicke theory.

We begin by defining the basic concepts employed: a system of $3+1$ equations is said to be *well-formulated* if it can be written as a system of equations of only first order in both temporal and spatial derivatives. Assume that this system can be cast in the full first order form

$$\partial_t \vec{u} + M^i \nabla_i \vec{u} = \vec{S}(\vec{u}), \tag{51}$$

where $\vec{u}$ collectively denotes the fundamental variables h_{ij}, K_{ij}, *etc.* introduced below, M^i is called the *characteristic matrix* of the system, and $\vec{S}(\vec{u})$ describes source terms and contains only the fundamental variables but not their derivatives. Then, the initial value formulation is *well-posed* if the system of PDEs is *symmetric hyperbolic* (*i.e.*, the matrices M^i are symmetric) and *strongly hyperbolic* if $s_i M^i$ has a real set of eigenvalues and a complete set of eigenvectors for any 1-form s_i, and obeys some boundedness conditions [108].

To summarize the results of [109], the Cauchy problem for metric $f(R)$ gravity is well-formulated and is well-posed in vacuo and with "reasonable" forms of matter (*i.e.*, perfect fluids, scalar fields, or the Maxwell field). For Palatini $f(R)$ gravity, instead, the Cauchy problem is well-formulated [110] but not well-posed in general, due to the presence of higher derivatives of the matter fields in the field equations and to the fact that it is impossible to eliminate them [109]. However, as it was remarked in [111], the Cauchy problem for Palatini is still well-posed in vacuo and when the trace of the matter energy-momentum tensor vanishes or it is a constant. On the other hand, it is possible to show the well-formulation and the well-position as soon as the source of the field equations is perfect-fluid matter [111].

As an alternative, the Brans-Dicke theory equivalent to Palatini $f(R)$ gravity can be mapped into its Einstein frame representation. In this conformal frame the redefined Brans-Dicke field couples minimally to gravity and non-minimally to matter [112] and the non-dynamical role of this scalar is even more obvious [112].

The problems with Palatini $f(R)$ gravity manifest themselves from a completely different angle when one tries to match static interior and exterior solutions with spherical symmetry [113].[5] The field equations are second order PDEs for the metric components and, since f is a function of $\tilde{R}$, which in turn is an algebraic function of T due to eq. (15), the right hand side of eq. (18) contains second derivatives of T. Now, T contains derivatives of the matter fields up to first order, hence eq. (18) contains derivatives of the matter fields up to third order. This property is very different from the familiar situation of GR and most of its extensions, in which the field equations contain only first order derivatives of the matter fields. A consequence of this dependence on lower order derivatives of the matter fields is that, in these theories the metric is generated by an integral over the matter sources and discontinuities in the matter fields and their derivatives are not accompanied by unphysical discontinuities of the metric. In Palatini $f(R)$ gravity, instead, the algebraic dependence of the metric on the matter fields creates unacceptable discontinuities in the metric and singularities in the curvature, which were discovered in

[5]Other problems of Palatini $f(R)$ gravity were reported and discussed in [114, 115].

[113]. Both the failure of the initial value problem and the presence of curvature singularities with matter fields can be ascribed to the non-dynamical nature of the scalar degree of freedom and to the fact that the latter is related algebraically to T. A possible cure consists of modifying the gravitational sector of the Lagrangian in such a way that the order of the field equations is raised.

5. DARK ENERGY AS CURVATURE

Let us now show, by some straightforward arguments, how $f(R)$-gravity can be related to the problem of dark energy. The field equations (4) may be recast in the Einstein-like form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{(eff)} + T_{\mu\nu}/f'(R) \tag{52}$$

with $T^{(eff)}$ given by eq. (8) and in which matter couples non-minimally to the geometry through the term $1/f'(R)$. As noted above, the appearance of $f'(R)_{;\mu\nu}$ in $T_{\mu\nu}^{(eff)}$ makes eq. (52) a fourth order equation (unless $f(R) = R$, in which case the curvature stress - energy tensor $T_{\alpha\beta}^{(eff)}$ vanishes identically and (52) reduces to the second order Einstein equation). As is clear from eq. (52), the curvature stress-energy tensor $T_{\mu\nu}^{(eff)}$ formally plays the role of a source in the field equations and its effect is the same as that of an effective fluid of purely geometrical origin. However, one can also consider the Palatini approach [66, 75], in which the Einstein equations can still be rewritten as effective Einstein equations containing a fluid of geometric origin.

In principle, the scheme outlined above provides all the ingredients needed to tackle the dark side of the universe. Depending on the scale considered, the effective curvature fluid can play the role of both dark matter and dark energy. From the cosmological point of view, in the standard framework of a spatially flat homogenous and isotropic universe, the cosmological dynamics are determined by the energy budget through the Friedmann equations. In particular, the cosmic acceleration is achieved when the right hand side of the acceleration equation remains positive. In units in which $8\pi G = c = 1$ this means

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho_{tot} + 3P_{tot}\right), \tag{53}$$

where the subscript tot denotes the sum of the curvature fluid and the matter contributions to the energy density and pressure. The acceleration condition $\ddot{a} > 0$ for a dust-dominated model is

$$\rho_{eff} + \rho_M + 3P_{eff} < 0 \tag{54}$$

or

$$w_{eff} < -\frac{\rho_{tot}}{3\rho_{eff}}. \tag{55}$$

Then, the effective quantities

$$\rho_{eff} = \frac{8}{f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3H\dot{R}f''(R)\right\}, \tag{56}$$

and

$$w_{eff} = -1 + \frac{\ddot{R}f''(R) + \dot{R}\left[\dot{R}f'''(R) - Hf''(R)\right]}{\left[f(R) - Rf'(R)\right]/2 - 3H\dot{R}f''(R)} \tag{57}$$

play a key role in determining the dynamics of the universe. To gain insight into the dynamics, one can begin by neglecting ordinary matter and studying the power-law form $f(R) = f_0 R^n$ (with n a real number), which represents a straightforward generalization of Einstein's GR corresponding the $n = 1$ limit. This choice yields power-law solutions for the scale factor $a(t)$ which provide a good fit to the SNeIa data and are in good agreement with the estimated age of the universe in the range $1.366 < n < 1.376$ [46]. The same kind of analysis can be carried out in the presence of ordinary matter, but in this case, numerical solution of the field equations is required. Then, it is still possible to confront the Hubble flow described by such a model with the Hubble diagram of SNeIa.

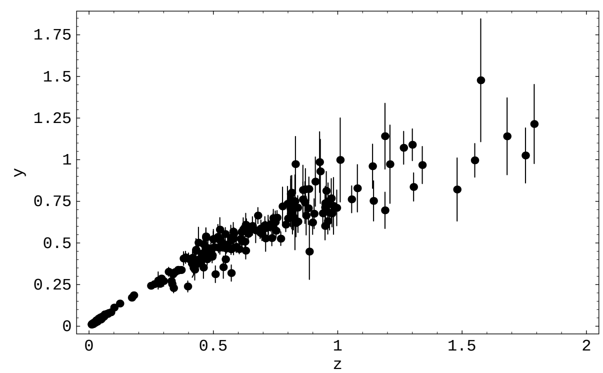

Fig. (1). The Hubble diagram of twenty radio galaxies together with the "gold" SNeIa sample is plotted versus the redshift z, as suggested in [116]. The best-fit curve corresponds to the $f(R)$ gravity model without dark matter.

The fit to the data is remarkably good (see Fig. **1**) improving the χ^2 value and it fixes the best-fit value at $n = 3.46$ if the baryons contribute to the energy density by $\Omega_b \approx 0.04$, in agreement with the prescriptions if Big Bang nucleosynthesis. The inclusion of dark matter does not modify the fit appreciably, supporting the assumption that dark matter is not essential in this model. From the evolution of the Hubble parameter in terms of redshift, one can even calculate the age of the universe t_{univ}. The best-fit value $n = 3.46$ provides $t_{univ} \approx 12.41$ Gyr. Of course, $f(R) = f_0 R^n$ gravity represents only a toy model generalization of Einstein's theory. Here we only suggest that several cosmological and astrophysical results can be well reproduced in the realm of a power-law extended gravity model. This approach allows flexibility in the value of the exponent n, although it would be preferable to determine a

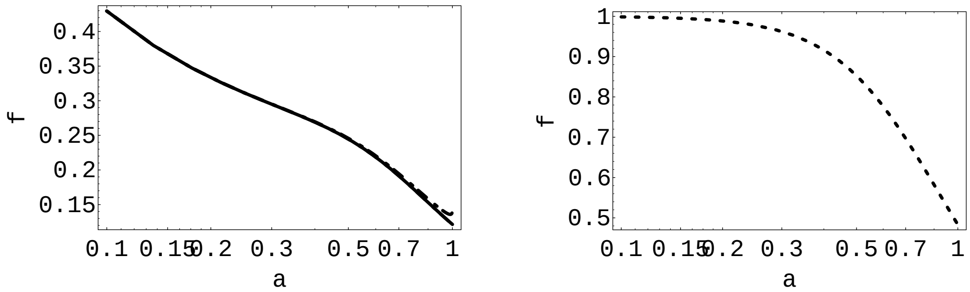

Fig. (2). The evolution of the growth index f in terms of the scale factor. The left panel corresponds to modified gravity, in the case $\Omega_m = \Omega_{bar} : 0.04$, for the SNeIa best fit model with $n=3.46$. The right panel shows the same evolution in the ΛCDM model. In the case of R^n gravity it is shown also the dependence on the scale k. The three cases $k=0.01, 0.001$, and 0.0002 have been eamined, and only the last of these three cases revelas a very small deviation from the leading behavior.

model capable of working at various scales. Furthermore, we do not expect to be able to reproduce the entire cosmological phenomenology by means of a simple power-law model, which is not sufficiently versatile [78]. For example, it can be easily demonstrated that this model fails when it is analyzed with respect to its ability of providing the correct evolutionary conditions for the perturbation spectra of matter overdensities [117]. This point is typically regarded as one of the most important arguments suggesting the need for darm matter. If one wants to discard this component, it is crucial to match the observational results related to the large-scale structure of the universe with the CMB. These carry the imprints of the initial matter spectrum at late times and at early times, respectively. It is important that the quantum spectrum of primordial perturbations, which provide the seeds of matter perturbations, can be recovered in the framework of R^n gravity. In fact, the model $f(R) \propto R + R^2$ can represent a viable model with respect to CMB data and is a good candidate for cosmological inflation. To obtain the matter power spectrum suggested by this model, we resort to the equation for the matter contrast obtained in Ref. [117] for fourth order gravity. This equation can be deduced in the Newtonian conformal gauge for the perturbed metric [117].

$$ds^2 = -(1+2\psi)dt^2 + a^2(1+2\phi)\Sigma_{i=1}^{3}(dx^i)^2. \tag{58}$$

In GR, it is $\phi = -\psi$ because there is no anisotropic stress; in general, this relation breaks down in ETGs and the non-diagonal components of the field equations yield new relations between the potentials ϕ and ψ. In $f(R)$ gravity, due to the non-vanishing $f_{R;i;j}$ with $i \neq j$, the $\phi - \psi$ relation becomes scale-dependent. Instead of the perturbation equation for the matter contrast δ, we provide here its evolution in terms of the growth index $s \equiv d\ln\delta / d\ln a$, a quantity measured at $z : 0.15$:

$$s'(a) - \frac{s(a)^2}{a} + \left[\frac{2}{a} + \frac{1}{a}E'(a)\right]s(a) - \frac{1-2Q}{2-3Q}\cdot\frac{3\Omega_m a^{-4}}{nE(a)^2\widetilde{R}^{n-1}} = 0, \tag{59}$$

where $E(a) = H(a)/H_0$, $\widetilde{R}$ is the dimensionless Ricci scalar, and

$$Q = -\frac{2f_{RR}c^2k^2}{f_R a^2}. \tag{60}$$

For $n=1$, eq. (60) gives the ordinary growth index relation of the Standard Cosmological Model. It is clear from eq. (59) that the latter suggests a dependence of the growth index on the scale which is contained in the corrective term Q and that this dependence can be safely neglected when $Q \to 0$ (see Fig. (**2**)). In the most general case one can resort to the limit $aH < k < 10^{-3}hMpc^{-1}$ in which eq. (59) is a good approximation, and non-linear effects on the matter power spectrum can be neglected.

By studying numerically eq. (59) one obtains the evolution of the growth index in term of the scale factor. Assuming, for simplicity, the initial condition $s(a_{ls})=1$ at the last scattering surface as in the case of matter domination, the results are summarized in Fig. (**1**), which displays the evolution of the growth index in R^n gravity and in the ΛCDM model.

In the case of $\Omega_m = \Omega_{bar} : 0.04$, one can observe a strong disagreement between the expected rate of the growth index and the behavior induced by power-law fourth order gravity models. This negative result is evident in the predicted value of $s(a_{z=0.15})$, which has been observationally estimated by the analysis of the correlation function for 220,000 galaxies in the 2dFGRS dataset at the survey effective depth $z=0.15$. The observational result suggests $s=0.58 \pm 0.11$ [118], while our model gives $s(a_{z=0.15})$:$0.117\ (k=0.01), 0.117(k=0.001), 0.122\ (k=0.0002)$. Although this result seems frustrating with respect to the underlying idea of discarding the dark components in the cosmological dynamics, it does not give substantial improvement in the case of R^n gravity model plus dark matter. In fact, it is possible to show that, even in this case, the growth index prediction is far from being in agreement with the ΛCDM model and again, at the observational scale $z=0.15$, there is not enough growth of perturbations to match the observed large cale structure. In this case one obtains $s(a_{z=0.15})$:0.29(k=0.01), 0.29(k=0.001), 0.31 (k=0.0002), *i.e.*, values which are substantially increased with respect to the previous case but still very far from the experimental estimate. No significantly different results are obtained if one varies the power n (of course, for $n \to 1$, one recovers the standard behavior if a cosmological constant is added to the model). These results seem to suggest that an extended gravity model incorporating a simple power-law of the Ricci scalar, although cosmologically relevant at late times, is not a viable description of the cosmic evolution at all scales. Such a scheme seems too simple to account for the entire cosmological phenomenology. In [117] a gravity Lagrangian considering an exponential correction to the Ricci scalar $f(R)=R+A\exp(-BR)$ (with A, B constants) produces more interesting results and exhibits a grow factor rate in agreement with the observational results at least in the dark matter case. To corroborate this point of view, one has to consider that when $f(R)$ is chosen starting from observational data in an inverse approach as in [78], the reconstructed Lagrangian is a non-trivial polynomial in the Ricci scalar. This result suggests that the whole cosmological phenomenology can be accounted only by a suitable non-trivial function of the Ricci scalar rather than a simple power-law. The results obtained in the study of the matter power spectra for simple R^n gravity do not invalidate the general approach.

6. DARK MATTER AS CURVATURE

The results obtained at cosmological scales motivate further analysis of $f(R)$ theories from the phenomenological point of view. One wonders whether the curvature fluid which works as dark energy could also play the role of effective dark matter, providing an opportunity to reproduce the observed astrophysical phenomenology using only visible matter (see for a discussion [71]). It is well known that, in the low energy limit, higher order gravity implies a modified gravitational potential, which will play a fundamental role in our discussion. By considering a spherical mass distribution with mass m and solving the vacuum field equations for a Schwarzschild-like metric, one obtains the modified gravitational potential of the theory $f(R) = f_0 R^n$ [119]

$$\Phi(r) = -\frac{Gm}{2r}\left[1+\left(\frac{r}{r_c}\right)^{\beta}\right], \tag{61}$$

where

$$\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2}. \tag{62}$$

This potential corrects the ordinary Newtonian potential with a power-law term. The correction becomes important on scales larger than r_c and the value of this threshold constant depends essentially on the mass of the system. The corrected potential (61) reduces to the standard Newtonian potential $\Phi \propto 1/r$ for $n = 1$, as follows from the inspection of eq. (62).

The result (61) deserves some comments. As discussed in detail in [119], we have assumed spherical symmetry in the the weak-field approximation of the field equations, which leads to a corrected Newtonian potential due to the strong non-linearity of higher order gravity. Note that Birkhoff's theorem does not hold, in general, in $f(R)$ gravity, and that spherically symmetric solutions different from the Schwarzschild one exist in these ETGs [120].

The generalization of eq. (61) to extended sources is achieved by dividing the latter into infinitesimal mass elements and integrating the potentials generated by these individual elements. An integral over the mass density of the system is calculated, taking care of possible symmetries of the mass distribution [119]. Once the gravitational potential has been computed, one can evaluate the rotation curve $v_c^2(r)$ and compare it with the astronomical data. For extended systems, one typically must resort to numerical techniques, but the main effect may be illustrated by the rotation curve for the point-like situation, that is,

$$v_c^2(r) = \frac{Gm}{2r}\left[1+(1-\beta)\left(\frac{r}{r_c}\right)^{\beta}\right]. \tag{63}$$

In comparison with the Newtonian result $v_c^2 = Gm/r$, the corrected rotation curve is modified by the addition of the second term on the right hand side of eq. (63). For $0 < \beta < 1$, the corrected rotation curve is higher than the Newtonian one. Since measurements of the rotation curves of spiral galaxies signal circular velocities larger than predicted by the observed luminous mass and Newtonian potential, the above result suggests the possibility that the modified gravitational potential of fourth order gravity may fill the gap between theory and observations without the need for additional dark matter.

The corrected rotation curve vanishes asymptotically as in the Newtonian case, while it is usually claimed that observed rotation curves are flat (*i.e.*, asymptotically constant). Actually, observations do not probe v_c up to infinite radii but only show a flat rotation curve (within the uncertainties) up to the last measured point. The possibility that v_c goes to zero at infinity is by no means excluded. In order to check observationally this result, we have considered a sample of low surface brightness (LSB) galaxies with well measured HI and Hα rotation curves extending far beyond the visible edge of the system. LSB galaxies are known to be ideal candidates to test dark matter models because of their high gas content, which allows the rotation curves to be well measured and corrected for possible systematic errors by comparing 21 cm HI line emission with optical Hα and $[NII]$ data. Moreover, these galaxies are supposed to be dominated by dark matter, so fitting their rotation curves without this elusive component would support ETGs as alternatives to dark matter.

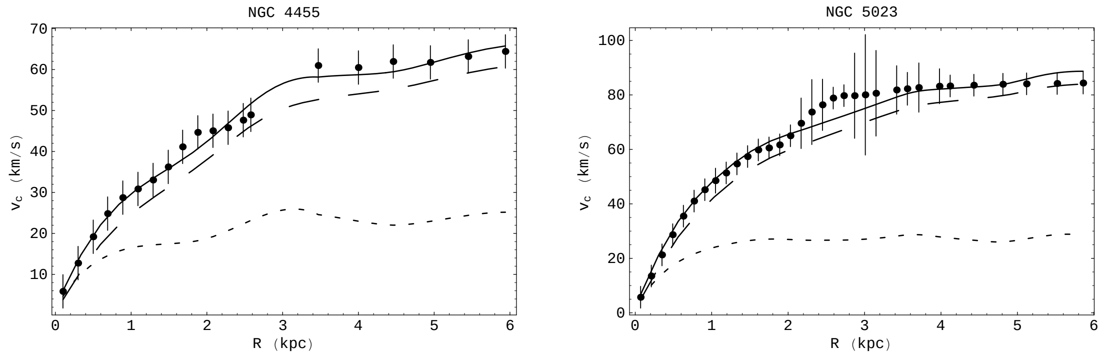

Fig. (3). Best-fit theoretical rotation curve superimposed to the data for the LSB galaxy NGC 4455 (left) and NGC 5023 (right). To better show the effect of the correction to the Newtonian gravitational potential, we report the total rotation curve $v_c(r)$ (solid line), the Newtonian one (short dashed) and the correction term (long dashed).

Our sample contains fifteen LSB galaxies with data on the rotation curve, the surface mass density of the gas component and R-band disk photometry extracted from a larger sample selected by de Blok & Bosma [121]. We assume that stars are distributed in an infinitely thin and circularly symmetric disk with surface density $\Sigma(r) = \Upsilon_{\star} I_0 \exp(-r/r_d)$, where the central surface luminosity I_0 and the disk scale length r_d are obtained from fitting to the stellar photometry. The gas surface density has been obtained by interpolating the data over the range probed by HI measurements and extrapolated outside this range. When fitting the theoretical rotation curve, there are three quantities to be determined, namely the stellar mass-to-light (M/L) ratio $\Upsilon_{\star}$, and the theory parameters (β, r_c). It is worth stressing that, while fit results for different galaxies should provide the same value of β, r_c is related to one of the integration constants in the field equations. As such, this quantity is not universal and its value must be determined on a galaxy by galaxy basis. However, it is expected that galaxies with similar mass distributions have similar values of r_c so that the scatter in r_c must reflect the scatter in the circular velocities. In order to match the model with the data, we perform a likelihood analysis for each galaxy, using as fitting parameters β, $\log r_c$ (with r_c in kpc) and the gas mass fraction[6] f_g. As it is evident from the results of the different fits, the experimental data are successfully fitted by the model [119]. In particular, from a purely phenomenological point of view and leaving aside for the moment other viability criteria, from the best fit range $\beta = 0.80 \pm 0.08$, one can conclude that R^n gravity with $2.3 < n < 5.3$ (best fit value $n = 3.2$ which overlaps well the above-mentioned range of n fitting SNeIa Hubble diagram) can be a good candidate for solving the missing matter problem in LSB galaxies without dark matter.

At this point, one wonders whether a link may be found between R^n gravity and the standard approach based on dark matter haloes since both theories fit equally well the same data. As a matter of fact, it is possible to define an *effective dark matter halo* by imposing that its rotation curve equals the correction term to the Newtonian curve induced by R^n gravity (Fig. **3**). Mathematically, one can split the total rotation curve derived from R^n gravity as $v_c^2(r) = v_{c,N}^2(r) + v_{c,corr}^2(r)$, where the second term is the correction. Considering, for simplicity a spherical halo embedding a thin exponential disk, we may also write the total rotation curve as $v_c^2(r) = v_{c,disk}^2(r) + v_{c,DM}^2(r)$ with $v_{c,disk}^2(r)$ the Newtonian disk rotation curve and $v_{c,DM}^2(r) = GM_{DM}(r)/r$ the dark matter one, $M_{DM}(r)$ being its mass distribution. Equating the two expressions yields

$$M_{DM}(\eta) = M_{vir}\left(\frac{\eta}{\eta_{vir}}\right)\frac{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}I_0(\eta) - V_d(\eta)}{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}I_0(\eta_{vir}) - V_d(\eta_{vir})} \tag{64}$$

with $\eta = r/r_d$ and $\Sigma_0 = \Upsilon_{\star} i_0$, $V_d(\eta) = I_0(\eta/2)K_0(\eta/2) \times I_1(\eta/2)K_1(\eta/2)$.[7] Moreover,

$$I_0(\eta,\beta) = \int_0^\infty F_0(\eta,\eta',\beta)k^{3-\beta}\eta'^{\frac{\beta-1}{2}}\mathrm{e}^{-\eta'}d\eta', \tag{65}$$

where F_0 depends only on the geometry of the system and the subscript "vir" indicates virial quantities. Eq. (64) defines the mass profile of an effective spherically symmetric dark matter halo whose ordinary rotation curve provides the part of the corrected disk rotation curve resulting from the addition of the curvature correction to the gravitational potential. Clearly, from a phenomenological point of view there is no way to distinguish this dark halo model from R^n gravity.

---

[6]This is related to the M/L ratio by $\Upsilon_{\star} = [(1-f_g)M_g]/(f_g L_d)$, where w $M_g = 1.4M_{HI}$ is the gas (HI + He) mass, and $M_d = \Upsilon_{\star} L_d$ and $L_d = 2\pi I_0 r_d^2$ are the disk total mass and luminosity, respectively.

[7]Here I_l and K_l, with $l=1,2$ are the Bessel functions of first and second type, respectively.

Having assumed spherical symmetry for the mass distribution, it is straightforward to compute the mass density for the effective dark halo as $\rho_{DM}(r) = (1/4\pi r^2) dM_{DM}/dr$. The most interesting feature of the density profile is its asymptotic behavior quantified by the logarithmic slope $\alpha_{DM} = d\ln\rho_{DM}/d\ln r$, which can be computed only numerically as a function of η for fixed values of β (or n). As expected, α_{DM} depends explicitly on β, while (r_c, Σ_0, r_d) enter indirectly through η_{vir}. The asymptotic values at the centre and at infinity (α_0 and α_∞, respectively) are of particular interest. α_0 almost vanishes and, in the innermost regions, the density is approximately constant. Indeed, $\alpha_0 = 0$ is the value corresponding to models with an isothermal sphere as the inner core. It is well known that galactic rotation curves are typically best-fitted by cored dark halo models. Moreover, the outer asymptotic slope lies between -3 and -2, values typical of most dark halo models in the literature. In particular, for $\beta = 0.80$ one finds $(\alpha_0, \alpha_\infty) = (-0.002, -2.41)$, values which are quite similar to those in the Burkert model, $(0, -3)$. This empirical model has been proposed to fit the LSB and dwarf galaxies rotation curves. The values of $(\alpha_0, \alpha_\infty)$ found for the best-fit effective dark halo therefore suggest a possible theoretical motivation for Burkert-like models. By construction, the properties of the effective dark matter halo are closely related to the disk properties, hence some correlation between the dark halo and the disk parameters is expected. In this regard, exploiting the relation between the virial mass and the disk parameters, one obtains the relation for the Newtonian virial velocity $V_{vir} = GM_{vir}/r_{vir}$

$$M_d \propto \frac{(3/4\pi\delta_{th}\Omega_m\rho_{crit})^{\frac{1-\beta}{4}} r_d^{\frac{1+\beta}{2}} \eta_c^\beta}{2^{\beta-6}(1-\beta)G^{\frac{5-\beta}{4}}} \frac{V_{vir}^{\frac{5-\beta}{2}}}{I_0(V_{vir},\beta)}. \tag{66}$$

We have checked numerically that eq. (66) may be well approximated by $M_d \propto V_{vir}^a$. This relation has the same formal structure of the baryonic Tully-Fisher (BTF) relation $M_b \propto V_{flat}^a$ where M_b is the total (gas plus stars) baryonic mass and V_{flat} is the circular velocity on the flat part of the observed rotation curve. In order to test whether the BTF can be explained by the effective dark matter halo proposed, we should look for a relation between V_{vir} and V_{flat}. Such a relation cannot be derived analytically because the estimate of V_{flat} depends on the peculiarities of the observed rotation curve, such as how far it extends, and the uncertainties on the outermost points. For given values of the disk parameters, we simulated theoretical rotation curves for some values of r_c and measured V_{flat} finally choosing the fiducial value for r_c that gives a value of V_{flat} as close as possible to the measured one. Inserting the relation thus found between V_{flat} and V_{vir} into eq. (66) and averaging over different simulations, we finally obtain

$$\log M_b = (2.88 \pm 0.04)\log V_{flat} + (4.14 \pm 0.09), \tag{67}$$

while a direct fit to the observed data gives [122]

$$\log M_b = (2.98 \pm 0.29)\log V_{flat} + (3.37 \pm 0.13). \tag{68}$$

The slope of the predicted and observed BTF are in good agreement, lending further support to our approach. The zero point is markedly different from the predicted one, being significantly larger than the observed one. However, both relations fit the data with a similar scatter. A discrepancy in the zero point can be due to our approximate treatment of the effective halo which does not take into account the gas component. Neglecting this term, we should increase the effective halo mass and hence V_{vir} which affects the relation with V_{flat} leading to a higher than observed zero point. Indeed, the larger M_g/M_d, the more the points deviate from our predicted BTF thus confirming our hypothesis. Given this caveat, we can conclude with some confidence that R^n gravity offers a theoretical foundation even for the empirically found BTF relation.

Although the results outlined here pertain to the simplistic choice $f(R) = f_0 R^n$ of fourth order gravity, they are nevertheless interesting. The incompatibility of this model with the correct matter power spectrum suggests that a more complicated Lagrangian is needed to reproduce the entire dark sector phenomenology at all scales, but it has been shown that ETGs allow one to approach important issues in cosmological and astrophysical phenomenology. We have seen that ETGs can reproduce the SNeIa Hubble diagram without dark matter and predict the age of the universe. The modification of the gravitational potential arising in higher order gravity could constitute a fundamental ingredient in interpreting the flatness of the rotation curves of LSB galaxies. Furthermore, if one considers the model parameters selected by the fit of the observational data of LSB rotation curves, it is possible to construct a phenomenological analog of the dark matter halo with shape similar to that of the Burkert model. Since the latter has been empirically introduced to account for the dark matter distribution in LSB and dwarf galaxies, this result provides a theoretical motivation of the Burkert model.

By investigating the relation between dark halo and disk parameters, a relation has been deduced between M_d and V_{flat}, which reproduces the baryonic Tully-Fisher law. Exploiting the relation between the virial mass and the disk parameters, one can obtain a relation for the virial velocity which can be satisfactorily approximated as $M_d \propto V_{vir}^a$. This result is also intriguing because it provides a theoretical interpretation of another phenomenological relation. Although not definitive, these phenomenological aspects of $f(R)$ point to a potentially interesting avenue of research and support the quest for a unified view of the dark side of the universe.

7. MASSIVE SCALAR MODES OF *f*(*R*) GRAVITATIONAL WAVES

As we have seen, a pragmatic point of view could be to "reconstruct" the suitable theory of gravity starting from data. The main issues of this "inverse" approach is matching consistently observations at different scales and taking into account wide classes of gravitational theories where "ad hoc" hypotheses are avoided. In principle, as discussed in the previous section, the most popular dark energy cosmological models can be achieved by considering $f(R)$ gravity without considering unknown ingredients. The main issue to achieve such a goal is to have at disposal suitable datasets at every redshift. In particular, this philosophy can be taken into account also for the cosmological stochastic background of gravitational waves (GW) which, together with CMBR, would carry, if detected, a huge amount of information on the early stages of the Universe evolution. In this section we discuss the cosmological background of gravitational waves (GWs) in generic $f(R)$ theories. The achievement of detecting massive modes or selecting $f(R)$-signatures in the stochastic background could be the final way to retain or rule out such theories with respect to GR. GWs are perturbations $h_{\mu\nu}$ of the metric which transform as 3-tensors. In GR, the equations ruling the propagation of GWs in the transverse-traceless gauge are

$$\Box h_i^j = 0, \tag{69}$$

where Latin indexes run from 1 to 3. We want to derive the analog of eq. (69) for a generic $f(R)$ theory described by the action (2). The linearized theory in vacuo ($\mathcal{S}^{(m)} = 0$) is considered below, so that

$$\mathcal{S} = \frac{1}{2k}\int d^4x\sqrt{-g}\, f(R). \tag{70}$$

Using a conformal transformation, the scalar degree of freedom $f'(R)$ of metric $f(R)$ gravity appears as the conformal factor in

$$\tilde{g}_{\mu\nu} = e^{2\Phi} g_{\mu\nu} \qquad e^{2\Phi} = f'(R). \tag{71}$$

The conformally equivalent Einstein-Hilbert action is

$$\tilde{\mathcal{S}} = \frac{1}{2k}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} + \mathcal{L}\left(\Phi,\Phi_{;\mu}\right)\right], \tag{72}$$

where $\mathcal{L}\left(\Phi,\Phi_{;\mu}\right)$ is the scalar field Lagrangian obtained using the relation

$$\tilde{R} = e^{-2\Phi}\left(R - 6\Box\Phi - 6\Phi_{;\delta}\Phi^{;\delta}\right) \tag{73}$$

between the Ricci curvatures of the conformally related metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. The equation for the gravitational waves is now

$$\tilde{\Box}\tilde{h}_i^j = 0\,, \tag{74}$$

where

$$\tilde{\Box} = e^{-2\Phi}\left(\Box + 2\Phi^{;\lambda}\partial_{;\lambda}\right). \tag{75}$$

Since scalar and tensor modes are decoupled, we have

$$\tilde{h}_i^j = \tilde{g}^{lj}\delta\tilde{g}_{il} = e^{-2\Phi}g^{lj}e^{2\Phi}\delta g_{il} = h_i^j\,, \tag{76}$$

which means that h_i^j is conformally invariant. As a consequence, the plane wave amplitudes $h_i^j(t) = h(t)e_i^j\exp(ik_m x^m)$, where e_i^j is the polarization tensor, are the same in both metrics, a fact that is important in the following.

In a FLRW background, eq. (74) becomes

$$\ddot{h} + \left(3H + 2\dot{\Phi}\right)\dot{h} + k^2a^{-2}h = 0 \tag{77}$$

where k is the wave number and h is the amplitude. The solutions of this equation are linear combinations of Bessel functions. Several primordial mechanisms generating GWs are possible. In principle, one could seek for contributions due to all known high-energy processes in the early phases of the cosmic history.

Here we consider the background of GWs generated during inflation, which is strictly related to the dynamics of the cosmological model. In particular, one can assume that the main contribution to this background comes from the amplification of vacuum fluctuations at the transition between the inflationary phase and the radiation era. However, we can assume that the GWs generated as zero-point fluctuations during inflation undergo adiabatically damped ($\sim 1/a$) oscillations until they reach the Hubble radius H^{-1}. This is the particle horizon for the growth of perturbations. Any previous fluctuation is smoothed away by the inflationary expansion. The GWs freeze out for $a/k \gg H^{-1}$ and re-enter the horizon after reheating. The re-entry in the Friedmann era depends on the spatial scale of the GWs. After re-entry, GWs are in principle detectable due the Sachs-Wolfe effect that they induce on the CMB temperature anisotropy $\Delta T/T$ at decoupling. If Φ is the inflaton field, then $\dot{\Phi} = H$ during inflation. By using the conformal time η defined by $d\eta = dt/a$, eq. (77) becomes

$$h'' + 2\frac{\chi'}{\chi}h' + k^2h = 0\,, \tag{78}$$

where $\chi = ae^{\Phi}$ and a prime now denotes differentiation with respect to η. Inside the radius H^{-1}, it is $k\eta \gg 1$. Since there are no gravitons in the initial vacuum state, only negative-frequency modes appear and the solution of eq. (78) is

$$h = k^{1/2}\sqrt{2/\pi}\,\frac{1}{aH}C\exp(-ik\eta), \tag{79}$$

where C is the amplitude parameter. At the first horizon crossing $aH = k$, the averaged amplitude $A_h = (k/2\pi)^{3/2}|h|$ of the perturbation is

$$A_h = \frac{C}{2\pi^2}. \tag{80}$$

When the scale a/k becomes larger than the Hubble radius H^{-1}, the growing mode freezes (see Fig. **4**). It can be shown that the upper limit $\Delta T/T \sim A_h$ is valid since other effects can contribute to the background anisotropy. From this consideration, it is clear that the only relevant quantity is the initial amplitude C in eq. (79), which is conserved until re-entry. This amplitude depends on the fundamental mechanism that generates the perturbations. Inflation produces perturbations as zero-point energy fluctuations, a mechanism which depends on the gravitational interaction and $(\Delta T/T)$ further constrains the theory of gravity. Considering a single graviton in the form of a monochromatic wave, its zero-point amplitude is obtained from the canonical commutation relation

$$[h(t,x),\pi_h(t,y)] = i\delta^3(x-y) \tag{81}$$

at fixed time t, where the amplitude h is the field and π_h is the conjugate momentum operator. The Lagrangian for the h-quantity is

$$\tilde{\mathcal{L}} = \frac{1}{2}\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}h_{;\mu}h_{;\nu} \tag{82}$$

in the conformally rescaled FLRW metric $\tilde{g}_{\mu\nu}$, where the amplitude h is conformally invariant. This Lagrangian leads to

$$\pi_h = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{h}} = e^{2\Phi}a^3\dot{h} \tag{83}$$

and eq. (81) becomes

$$[h(t,x),\dot{h}(y,y)] = i\frac{\delta^3(x-y)}{a^3e^{2\Phi}}. \tag{84}$$

The fields h and $\dot{h}$ can be expanded in terms of creation and annihilation operators. The commutation relations in conformal time are

$$[hh'^* - h^*h'] = \frac{i(2\pi)^3}{a^3e^{2\Phi}}. \tag{85}$$

Eqs. (79) and (80) yield $C = \sqrt{2}\pi^2He^{-\Phi}$, where H and Φ are calculated at the first horizon crossing and, using $e^{2\Phi} = f'(R)$, the relation

$$A_h = \frac{H}{\sqrt{2f'(R)}} \tag{86}$$

is found to hold for a generic $f(R)$ theory. This result deserves some discussion. Clearly, the GW amplitude produced during inflation depends on the theory of gravity which, if different from GR, contains extra degrees of freedom which could be probed by the Sachs-Wolfe effect. This effect could be combined with other constraints on the GW background if ETGs are probed independently at other scales [123, 124].

We are by now familiar with the trace of the field equations

$$3\Box f'(R) + Rf'(R) - 2f(R) = 0, \tag{87}$$

and, using the identifications [125]

$$\Phi \to f'(R) \qquad and \qquad \frac{dV}{d\Phi} \to \frac{2f(R) - Rf'(R)}{3} \tag{88}$$

the Klein-Gordon equation for the effective scalar field Φ

$$\Box\Phi = \frac{dV}{d\Phi} \tag{89}$$

follows. Linearizing around a constant curvature background corresponding to $\Phi = \Phi_0$, assuming that V has a minimum at Φ_0 [124], and expanding as in

$$V \simeq \frac{1}{2}\alpha\delta\Phi^2\,, \qquad \frac{dV}{d\Phi} \simeq m^2\delta\Phi];, \tag{90}$$

where the constant m has the dimensions of a mass, yields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$

$$\Phi = \Phi_0 + \delta\Phi\,,$$

to first order in $h_{\mu\nu}$ and $\delta\Phi$. If $\tilde{R}_{\mu\nu\rho\sigma}$, $\tilde{R}_{\mu\nu}$, and $\tilde{R}$ are the linearized quantities corresponding to $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$, and R, then the linearized field equations are

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{R} = \partial_\mu\partial_\nu h_f - \eta_{\mu\nu}\Box h_f\,, \qquad \Box h_f = m^2 h_f\,, \tag{91}$$

where

$$h_f \equiv \frac{\delta\Phi}{\Phi_0}. \tag{92}$$

The curvature tensor $\tilde{R}_{\mu\nu\rho\sigma}$ and eqs. (91) are left invariant under the gauge transformations

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\varepsilon_{\nu)}\,, \qquad \delta\Phi \to \delta\Phi' = \delta\Phi\,. \tag{93}$$

By introducing

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + \eta_{\mu\nu}h_f \tag{94}$$

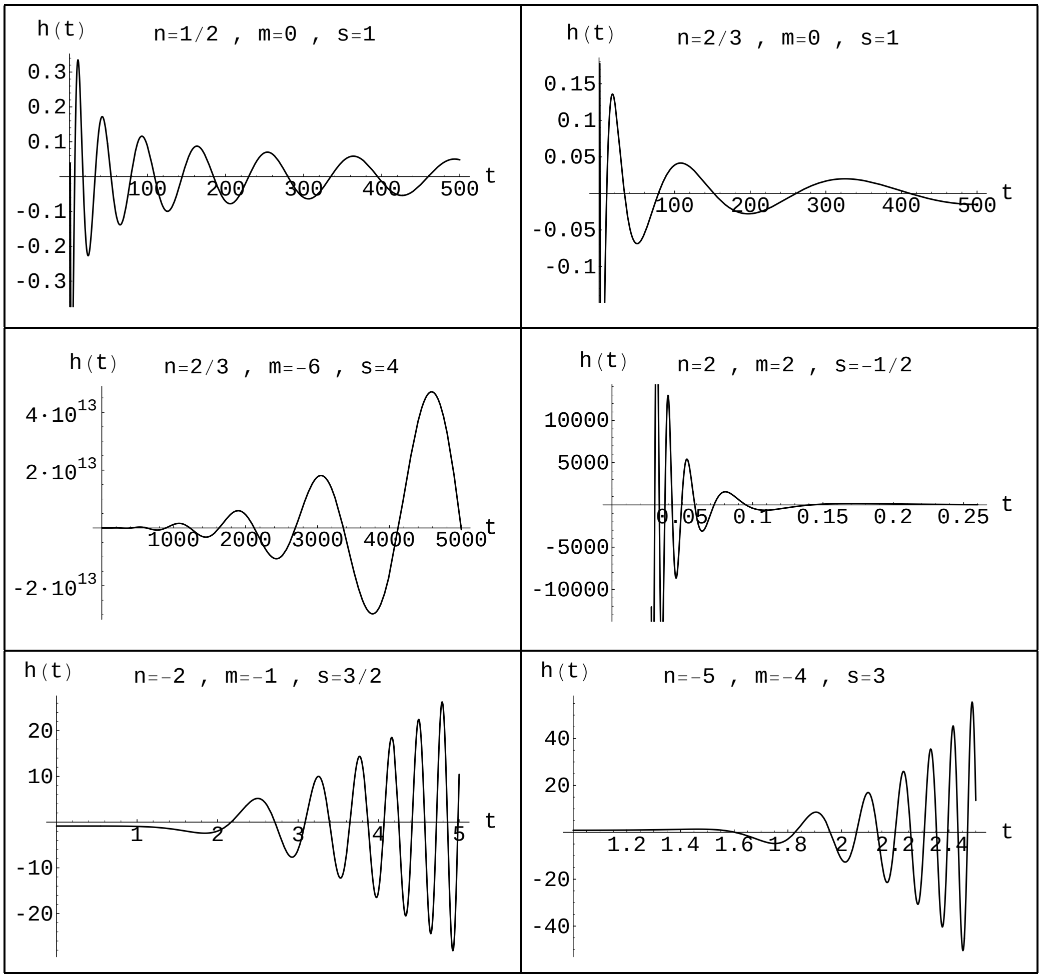

Fig. (4). The evolution of the GW amplitude for a few power-law choices of the scale factor $a(t) : t^s$, the scalar field $\phi : t^m$, and the function $f(R) : R^n$. The horizontal (time) and vertical (amplitude) scales depend on the cosmological background providing a signature of the model.

and considering the gauge vector ε^{μ} given by

$$\Box\varepsilon_{\nu} = \partial^{\mu}\overline{h}_{\mu\nu} \,, \tag{95}$$

the Lorenz gauge

$$\partial^{\mu}\overline{h}_{\mu\nu} = 0 \tag{96}$$

can be chosen. In this gauge the field equations assume the form

$$\Box\overline{h}_{\mu\nu} = 0 \,, \tag{97}$$

$$\Box h_f = m^2 h_f \,. \tag{98}$$

The solutions of eqs. (97) and (98) are the plane waves

$$\overline{h}_{\mu\nu} = A_{\mu\nu}(\vec{p})\exp(ip^{\alpha}x_{\alpha}) + c.c. \,, \tag{99}$$

$$h_f = a(\vec{p})\exp(iq^{\alpha}x_{\alpha}) + c.c. \,, \tag{100}$$

with

$$k^{\alpha} \equiv (\omega, \vec{p}) \qquad \omega = p \equiv |\vec{p}\,|$$
$$q^{\alpha} \equiv (\omega_m, \vec{p}) \qquad \omega_m = \sqrt{m^2 + p^2} \,. \tag{101}$$

Eqs. (97) and (98) are the wave equation for standard GR and its gravitational wave solutions, respectively, whereas eqs. (98) and (100) are the wave equation and its solution for the massive scalar mode of $f(R)$ gravity (cf. [125, 126]). The dispersion relation for the modes of the massive field h_f is non-linear. "Ordinary" (*i.e.*, GR) tensor modes $\overline{h}_{\mu\nu}$ propagate at the speed of light c, but the dispersion law (the second of eqs. (101) for the scalar modes h_f is that of a massive field wave packet [125, 126]. The group velocity of a wave packet of h_f centered on $\vec{p}$ is

$$\vec{v}_G = \frac{\vec{p}}{\omega} , \tag{102}$$

which is the velocity of a massive particle with mass m and momentum $\vec{p}$. The second of eqs. (101) in conjunction with eq. (102) yields

$$v_G = \frac{\sqrt{\omega^2 - m^2}}{\omega} \tag{103}$$

and a wave packet propagates at constant speed if

$$m = \sqrt{(1 - v_G^2)}\,\omega. \tag{104}$$

The Lorenz gauge is preserved by gauge trasformations with $\Box\varepsilon_\nu = 0$; this gauge imposes the transversality condition $k^\mu A_{\mu\nu} = 0$ for the tensor modes, but not for the field $h_{\mu\nu}$ which contains a scalar mode, as seen from eq. (94), or

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \eta_{\mu\nu} h_f. \tag{105}$$

Were the scalar mode massless, one could impose that

$$\Box\varepsilon^\mu = 0\,, \qquad \partial_\mu \varepsilon^\mu = -\frac{\bar{h}}{2} + h_f\,, \tag{106}$$

thus obtaining a transversal "total" field. However, as is clear from the previous sections, we are dealing with a massive scalar mode and transversality is impossible. By applying d'Alembert's operator to the second of eqs. (106) and using eqs. (97) and (98), it follows that

$$\Box\varepsilon^\mu = m^2 h_f\,, \tag{107}$$

in contrast with the first of eqs. (106). Similarly, it is shown that a linear relation between the tensorial modes $\bar{h}_{\mu\nu}$ and the massive scalar h_f cannot exist. Thus, a gauge in wich $h_{\mu\nu}$ is purely spatial cannot be chosen, *i.e.*, it is impossible to impose $h_{\mu 0} = 0$, see eq. (105). However, the traceless gauge condition can be imposed on $\bar{h}_{\mu\nu}$,

$$\Box\varepsilon^\mu = 0\,, \qquad \partial_\mu \varepsilon^\mu = -\frac{\bar{h}}{2}\,, \tag{108}$$

implying that

$$\partial^\mu \bar{h}_{\mu\nu} = 0. \tag{109}$$

The gauge transformations

$$\Box\varepsilon^\mu = 0\,, \qquad \partial_\mu \varepsilon^\mu = 0\,, \tag{110}$$

preserve the gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$, $\bar{h} = 0$. By choosing $\vec{p}$ along the z-direction, a gauge can be chosen in which only A_{11}, A_{22}, and $A_{12} = A_{21}$ are different from zero, with the condition $\bar{h} = 0$ providing $A_{11} = -A_{22}$. The substitution of these equations into eq. (105) then yields

$$h_{\mu\nu}(t,z) = A^+(t-z)e_{\mu\nu}^{(+)} + A^\times(t-z)e_{\mu\nu}^{(\times)} + h_f(t - v_G z)\eta_{\mu\nu}\,. \tag{111}$$

The term $A^+(t-z)e_{\mu\nu}^{(+)} + A^\times(t-z)e_{\mu\nu}^{(\times)}$ describes the two standard polarizations of tensor gravitational waves familiar from GR, while the term $h_f(t - v_G z)\eta_{\mu\nu}$ is the massive scalar field characteristic of $f(R)$ gravity. As expected, the scalar $f'(R)$ generates a third massive polarization for gravitational waves which is absent in GR.

8. CONCLUSIONS

Let us emphasize once more that we regard $f(R)$ gravity theories not as definitive theories, but rather as toy models and proofs of principle that modifying gravity at large scales can explain the observed acceleration of the universe without the need to advocate exotic dark energy. This hope has stimulated a very intense activity among theoreticians ([70] and references therein).

To summarize the status of modified gravity, let us note that metric $f(R)$ gravity models exist that pass all the observational and theoretical constraints (see, e.g., the Starobinsky model [109] $f(R) = R + \lambda R_0 \left[\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right]$). The viable models require the chameleon mechanism in order to pass the weak-field limit tests.

All metric $f(R)$ theories must satisfy the condition $f''(R) \geq 0$ to avoid the Dolgov-Kawasaki local instability. This is a condition on short-wavelength modes. The stability condition (41) is valid for arbitrary wavelengths, but is restricted to de Sitter space (which is, anyway, an adiabatic approximation for slowly expanding FLRW spaces). An important open problem is whether curvature singularities appear, in general, in relativistic strong field stars.

As far as the Palatini formalism is concerned, the central idea of this version of modified gravity is to regard the torsion-free connection $\Gamma^\mu_{\alpha\beta}$ as a quantity independent of the spacetime metric $g_{\mu\nu}$. The Palatini formulation of the standard Hilbert-Einstein theory is equivalent to the purely metric theory, as a consequence of the fact that the field equations for the connection give the Levi-Civita connection of the metric $g_{\mu\nu}$. Therefore, there is no reason to impose the Palatini variational principle, instead of the metric variational principle, in the Einstein-Hilbert theory. However, the situation is difefrent in ETGs containing non-linear functions of the curvature invariants, such as $f(R)$, or non-minimally coupled scalars. In these cases, the Palatini and the metric variational principle yield different field equations and different physics [59, 66]. The relevance of the Palatini approach for cosmological applications has been amply demonstrated [43, 44, 67, 68]. However, Palatini $f(R)$ theories could have some problems due to the fact that they could contain non-dynamical scalar field and the initial value problem could be ill-posed. In any case, when the trace of the matter energy-momentum tensor vanishes identically or it is a constant, and when it can be recast in a perfect-fluid form, the Cauchy problem results well-formulated and well-posed.

Metric-affine gravity has not been developed in sufficient detail to assess its viability according to all the criteria presented here, and its cosmological consequences are essentially unexplored.

It seems fair to say that $f(R)$ theories of gravity can help to progress in our understanding of the peculiarities of GR in the wider landscape of relativistic theories of gravity. Furthermore, these theories point out important aspects of generalizations of GR, and, from a phenomenological point of view, constitute viable alternatives to dark energy models in explaining the cosmic acceleration, and to dark matter in reproducing dynamical features as the galactic rotation curves or the halo of clusters of galaxies [127]. Finally, it is possible to "tune" the stochastic background of GWs and this occurrence could constitute a further cosmological test capable of confirming or ruling out ETGs once data from interferometers, like VIRGO, LIGO and LISA, will be available (see [128] for a discussion on this topic). At present, no definite prediction sets $f(R)$ theories apart from dark energy and other models once and for all, but it is hoped that progress will me made in this direction.

ACKNOWLEDGEMENTS

SC and MD acknowledge V. Cardone, M. Francaviglia, A. Troisi and S. Vignolo for discussions and some common results used in this review paper. VF acknowledges financial support from Bishop's University and the Natural Sciences and Engineering Research Council of Canada (NSERC).

REFERENCES

[1] Buchbinder IL, Odintsov SD, Shapiro IL. Effective action in quantum gravity. Bristol: IOP Publishing 1992.

[2] Bondi H. Cosmology. Cambridge: Cambridge University Press 1952.

[3] Brans CH, Dicke RH. Mach's principle and a relativistic theory of gravitation. Phys Rev 1961; 124: 925 .

[4] Capozziello S, deRitis R, Rubano C, Scudellaro P. Noether symmetries in cosmology. Riv Nuovo Cimento 1996; 4: 19.

[5] Sciama DW. On the origin of the inertia. Mon Not R Ast Soc 1953; 113: 34.

[6] Birrell ND, Davies PCW. Quantum fields in curved space. UK: Cambridge University Press 1982.

[7] Vilkovisky G. Effective action in quantum gravity. Class Quantum Gravity 1992; 9: 895.

[8] Gasperini M, Veneziano G. O(d, d)-covariant string cosmology. Phys Lett B 1992; 277: 256 .

[9] Magnano G, Ferraris M, Francaviglia M. Nonlinear gravitational Lagrangians. Gen Relativity Gravitation 1987; 19: 465.

[10] Barrow J, Ottewill AC. The stability of general relativistic cosmological theory. J Phys A Math Gen 1983; 16: 2757.

[11] Starobinsky AA. A new type of isotropic cosmological models without singularity. Phys Lett B 1980; 91: 99.

[12] Duruisseau JP, Kerner R. The effective gravitational Lagrangian and the energy-momentum tensor in the inflationary universe. Gen Relativity Gravitation 1983; 15: 797.

[13] La D, Steinhardt PJ. Erratum: extended inflationary cosmology. Phys Rev Lett 1989; 62: 376.

[14] Teyssandier P, Tourrenc Ph. The Cauchy problem for the R+R2 theories of gravity without torsion. J Math Phys 1983; 24: 2793.

[15] Maeda K. Towards the Einstein-Hilbert action *via* conformal transformation. Phys Rev D 1989; 39: 3159.

[16] Wands D. Extended gravity theories and the Einstein--Hilbert action. Class Quantum Gravity 1994; 11: 269.

[17] Capozziello S, de Ritis R, Marino AA. Recovering the effective cosmological constant in extended gravity theories. Gen Relativity Gravitation 1998; 30: 1247.

[18] Gottlöber S, Schmidt H-J, Starobinsky AA. Sixth-order gravity and conformal transformations. Class Quantum Gravity 1990; 7: 893

[19] Ruzmaikina TV, Ruzmaikin AA. Quadratic corrections to the lagrangian density of the gravitational field and the singularity. J Exp Theor Phys 1969; 30: 372.

[20] Amendola L, Battaglia-Mayer A, Capozziello S, *et al.* Generalized sixth-order gravity and inflation. Class Quantum Gravity 1993; 10: L43.

[21] Battaglia-Mayer A, Schmidt H-J. The de Sitter spacetime as attractor solution in eighth-order gravity. Class Quantum Gravity 1993; 10: 2441.

[22] Schmidt H-J. Variational derivatives of arbitrarily high order and multi-inflation cosmological models. Class Quantum Grav 1990; 7: 1023.

[23] Capozziello S, Nojiri S, Odintsov SD. Dark energy: the equation of state description versus scalar-tensor or modified gravity. Phys Lett B 2006; 634: 93.

[24] Amendola L, Capozziello S, Litterio M, Occhionero F. Coupling first-order phase transitions to curvature-squared inflation. Phys Rev D1992; 45: 417.

[25] Perlmutter S, Aldering G, Goldhaber G, *et al.* Measurements of omega and lambda from 42 high-redshift supernovae. Astrophys J 1999; 517: 565; Knop RA, Aldering G, Amanullah R, *et al.* New Constraints on Ω_M , Ω_Λ , and w from an independent set of 11 high-redshift supernovae observed with the hubble space telescope. Astrophys J 2003; 598: 102.

[26] Riess AG, Filippenko AV, Challis P, *et al.* Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astrophys J 1998; 116: 1009; Tonry J L, Schmidt B P, Barris B, *et al.* Cosmological results from high-z supernovae. Astrophys J 2003; 594: 1.

[27] De Bernardis P, Bock JJ, Bond JR, *et al.* Flat Universe from high-resolution maps of the cosmic microwave background radiation. Nature 2000; 404: 955.

[28] Stompor R, Abroe M, Ade P, *et al.* Cosmological implications of the maxima-1 high-resolution cosmic microwave background anisotropy measurement. Astrophys J 2001; 561: L7.

[29] Spergel DN, Verde L, Peiris HV, *et al.* First-year wilkinson microwave anisotropy probe (WMAP) observations: determination of cosmological parameters. Astrophys J Suppl 2003; 148: 175 .

[30] Hinshaw G, Barnes C, Bennett CL, *et al.* First-year wilkinson microwave anisotropy probe (wmap) observations: data processing methods and systematic error limits. Astrophys J Suppl 2003; 148: 135.

[31] Spergel DN, Verde L, Peiris HV, *et al.* Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations: implications for cosmology. Astrophys J Suppl 2007; 170: 377.

[32] Riess AG, Strolger LG, Tonry J, *et al.* Type Ia supernova discoveries at z > 1 from the hubble space telescope: evidence for past deceleration and constraints on dark energy evolution. Astrophys J 2004; 607: 665.

[33] Sahni V, Starobinski AA. The case for a positive cosmological Λ-Term. Int J Mod Phys D 2000; 9: 373.

[34] Padmanabhan T. Cosmological constant-the weight of the vacuum. Phys Rep 2003; 380: 235.

[35] Copeland E J, Sami M, Tsujikawa S. Dynamics of dark energy. Int J Mod Phys D 2006; 15: 1753.

[36] Kamenshchik A, Moschella U, Pasquier V. An alternative to quintessence. Phys Lett B 2001; 511: 265 .

[37] Padmanabhan T. Accelerated expansion of the universe driven by tachyonic matter. Phys Rev D 2002; 66: 021301.

[38] Bassett BA, Kunz M, Parkinson D, Ungarelli C. Condensate cosmology: dark energy from dark matter. Phys Rev D 2003; 68: 043504.

[39] Cardone VF, Troisi A, Capozziello S. Unified dark energy models: A phenomenological approach. Phys Rev D 2004; 69: 083517; Capozziello S, Cardone VF, Elizalde E, Nojiri S, Odintsov SD. Observational constraints on dark energy with generalized equations of state. Phys Rev D 2006; 73: 043512.

[40] Lue A, Scoccimarro R, Starkman G. Probing Newton's constant on vast scales: Dvali-Gabadadze-Porrati gravity, cosmic acceleration, and large scale structure. Phys Rev D 2004; 69: 044005.

[41] Freese K , Lewis M. Cardassian expansion: a model in which the universe is flat, matter dominated, and accelerating. Phys Lett B 2002; 540: 1.

[42] Dvali GR, Gabadadze G, Porrati M. 4D gravity on a brane in 5D Minkowski space. Phys Lett B 2000; 485: 208.

[43] Capozziello S, Curvature quintessence. Int J Mod Phys D 2002; 11: 483; Nojiri S, Odintsov SD. Modified gravity with negative and positive powers of curvature: Unification of inflation and cosmic acceleration. Phys Rev D 2003; 68: 123512; Carroll SM, Duvvuri V, Trodden M, Turner MS. Is cosmic speed-up due to new gravitational physics? Phys Rev D 2004; 70: 043528; Allemandi G, Borowiec A, Francaviglia M. Accelerated cosmological models in first-order nonlinear gravity. Phys Rev D 2004; 70: 103503.

[44] Nojiri S, Odintsov SD. Introduction to modified gravity and gravitational alternative for dark energy. Int J Geometric Methods Mod Phys 2007; 4: 115.

[45] Capozziello S, Cardone VF, Piedipalumbo E, Sereno M , Troisi A. Matching torsion Λ-term with observations. Int J Mod Phys D 2003; 12: 381.

[46] Capozziello S, Cardone VF, Carloni S, Troisi A. Curvature quintessence matched with observational data. Int J Mod Phys D 2003; 12: 1969.

[47] Li B, Barrow JD. Cosmology of $f(R)$ gravity in the metric variational approach. Phys Rev D 2007; 75: 084010.

[48] Chen G, Ratra B. Cosmological constraints from compact radio source angular size versus redshift data. Astrophys J 2003; 582: 586; Podariu S, Daly RA, Mory MP, Ratra B. Radio galaxy redshift-angular size data constraints on dark energy. Astrophys J 2003; 584: 577.

[49] Will CM. Theory and experiment in gravitational physics. 2nd ed. UK: Cambridge University Press 1993.

[50] Stelle K. Classical gravity with higher derivatives. Gen Relativity Gravitation 1978; 9: 353.

[51] Sanders RH. Cosmology with modified Newtonian dynamics (MOND). Mon Not R Astron Soc 1998; 296:1009.

[52] Mannheim PD, Kazanas D. Exact vacuum solution to conformal Weyl gravity and galactic rotation curves. Astrophys J 1989; 342: 635.

[53] Anderson JD , Laing PA, Lau E L, *et al* Study of the anomalous acceleration of Pioneer 10 and 11. Phys Rev D 2002; 65: 082004.

[54] Quant I, Schmidt H-J. The Newtonian limit of fourth and higher order gravity. Astron Nachr 1991; 312: 97.

[55] Schneider P, Ehlers J, Falco EE. Gravitational Lenses. Springer-Verlag; Berlin 1992.

[56] Krauss LM, White M. Grand unification, gravitational waves, and the cosmic microwave background anisotropy. Astrophys J 1992; 397: 357.

[57] Ferraris M, Francaviglia M, Magnano G. Do non-linear metric theories of gravitation really exist? Class Quantum Gravity 1988; 5: L95.

[58] Sokolowski LM. Physical versions of non-linear gravity theories and positivity of energy. Class Quantum Gravity 1989; 6: 2045.

[59] Magnano G, Sokolowski LM. Physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field. Phys Rev D 1994; 50: 5039.

[60] Dicke RH. Mach S´. Principle and Invariance under transformation of units. Phys Rev 1962; 125: 2163.

[61] Damour T, Esposito-Farese G. Tensor-multi-scalar theories of gravitation. Class Quantum Gravity 1992; 9: 2093.

[62] Faraoni V. Cosmology in scalar-tensor. Gravity. Dordrecht: Kluwer Academic 2004.

[63] Einstein A. Einheitliche Feldtheorie yon Gravitation und Elektrizit. Sitzung-ber Preuss Akad Wiss Phys Math K1 1925; 22: 414.

[64] Buchdahl HA. Quadratic Lagrangians and Palatini's device. J Phys A 1979, 12 (8): 1229.

[65] Ferraris M, Francaviglia M, Reina C. Variational formulation of general relativity from 1915 to 1925 Palatini S′ method discovered by Einstein in 1925. Gen Relativity Gravitation 1982; 14: 243.

[66] Ferraris M, Francaviglia M, Volovich I. The universality of vacuum Einstein equations with cosmological constant. Class Quantum Gravity 1994; 11: 1505.

[67] Vollick DN. 1/R curvature corrections as the source of the cosmological acceleration. Phys Rev D 2003; 68: 063510.

[68] Li B, Chu MC. CMB and matter power spectra of early $f(R)$ cosmology in the Palatini formulation. Phys Rev D 2006; 74: 104010.

[69] Allemandi G, Capone M, Capozziello S, Francaviglia M. Conformal aspects of the palatini approach in extended theories of gravity. Gen Relativity Gravitation 2006; 38: 33.

[70] Sotiriou TP, Faraoni V. $f(R)$ Theories of gravity. Preprint available from arXiv:0805.1726, to appear in Rev Mod Phys.

[71] Nojiri S, Odintsov SD. Problems of modern theoretical physics, a volume in honour of Prof. I L Buchbinder in the occasion of his 60th birthday, p.266-285, TSPU Publishing, Tomsk, available from arXiv:0807.0685; Straumann N. Problems with modified theories of gravity, as alternatives to dark energy. Available from: arXiv:0809.5148; Schmidt H.-J Fourth order gravity: Equations, history, and application to cosmology. Int J Geometric Methods Phys 2007; 4: 209; Faraoni V. $f(R)$ gravity: successes and challenges. Available from: arXiv:0810.2602; Capozziello S, Francaviglia M. Extended theories of gravity and their cosmological and astrophysical applications. Gel Relativity Gravity 2008; 40: 357.

[72] Wald RM. General relativity. Chicago: Chicago University Press 1984.

[73] de Souza JCC, Faraoni V. The phase-space view of $f(R)$ gravity. Class Quantum Grav 2007; 24: 3637; Faraoni V. Phase space geometry in scalar-tensor cosmology. Ann Phys (NY) 2005; 317: 366.

[74] Exirifard Q, Sheikh-Jabbari MM. Lovelock gravity at the crossroads of palatini and metric formulations. Phys Lett B 2008; 661: 158.

[75] Sotiriou TP. Constraining $f(R)$ gravity in the Palatini formalism. Class Quantum Grav 2006; 23: 5117; Sotiriou TP, Liberati S. Metric-affine $f(R)$ theories of gravity. Ann Phys (NY) 2007; 322: 935.

[76] Whitt B. Fourth-order gravity as general relativity plus matter. Phys Lett B 1984; 145: 176; Barrow JD, Cotsakis S. Inflation and the conformal structure of higher-order gravity theories. Phys Lett B 1988; 214: 515; Chiba T. 1/R gravity and scalar-tensor gravity. Phys Lett B 2003; 575: 1.

[77] O'Hanlon J. Intermediate-range gravity: a generally covariant model. Phys Rev Lett 1972; 29: 137.

[78] Amendola L, Polarski D, Tsujikawa S. Are $f(R)$ dark energy models cosmologically viable? Phys Rev Lett 2007; 98: 131302; Capozziello S, Nojiri S, Odintsov SD, Troisi A. Cosmological viability of -gravity as an ideal fluid and its compatibility with a matter dominated phase. Phys Lett B 2006; 639: 135; Amendola L, Gannouji R, Polarski D, Tsujikawa S. Conditions for the cosmological viability of $f(R)$ dark energy models. Phys Rev D 2007; 75: 083504; Nojiri S, Odintsov SD. Modified $f(R)$ gravity consistent with realistic cosmology: from a matter dominated epoch to a dark energy universe. Phys Rev D 2006; 74: 086005; Brookfield AW, van de Bruck C, Hall LMH. Viability of $f(R)$ theories with additional powers of curvature. Phys Rev D 2006; 74: 064028.

[79] Capozziello S, Cardone VF, Troisi A. Reconciling dark energy models with $f(R)$ theories. Phys Rev D 2005; 71: 043503; Nojiri S, Odintsov SD. Modified gravity as an alternative for ΛCDM cosmology. J Phys A 2006; 40: 6725; Modified gravity and its reconstruction from the universe expansion history. J Phys Conf Ser 2007; 66: 012005; Cruz-Dombriz A, Dobado A. $f(R)$ gravity without a cosmological constant. Phys Rev D 2006; 74: 087501; Fay S, Nesseris S, Perivolaropoulos L. Can $f(R)$ modified gravity theories mimic a CDM cosmology? Phys Rev D 2007; 76: 063504.

[80] Dolgov AD, Kawasaki M. Can modified gravity explain accelerated cosmic expansion? Phys Lett B 2003; 573: 1.

[81] Faraoni V. Solar system experiments do not yet veto modified gravity models. Phys Rev D 2006; 74: 104017.

[82] Cognola G, Zerbini S. One-loop *f*(*R*) gravitational modified models. J Phys A 2006; 39: 6245; Cognola G, Elizalde E, Nojiri S , Odintsov SD, Zerbini S. One-loop *f*(*R*) gravity in de Sitter universe. JCAP 2005; 0502: 010; Cognola G, Gastaldi M, Zerbini S. On the stability of a class of modified gravitational models. Int J Theor Phys 2008; 47: 898.

[83] Faraoni V. Erratum: solar system experiments do not yet veto modified gravity models [Phys. Rev. D 74, 023529 (2006)]. Phys Rev D 2007; 75: 067302 .

[84] Sotiriou TP. Curvature scalar instability in *f*(*R*) gravity. Phys Lett B 2007; 645: 389.

[85] Faraoni V. Negative energy and stability in scalar-tensor gravity. Phys Rev D 2004; 70: 044037; Modified gravity and the stability of de Sitter space. Phys Rev D 2005; 72: 061501; Faraoni V, Nadeau S. Stability of modified gravity models. Phys Rev D 2005; 72: 124005.

[86] Bardeen JM. Gauge-invariant cosmological perturbations. Phys Rev D 1980; 22: 1882; Ellis GFR, Bruni M. Covariant and gauge-invariant approach to cosmological density fluctuations. Phys Rev D 1989; 40: 1804; Ellis GFR, Hwang J-C, Bruni M. Covariant and gauge-independent perfect-fluid Robertson-Walker perturbations. Phys Rev D 1989; 40: 1819; Ellis GFR, Bruni M, Hwang J-C. Density-gradient-vorticity relation in perfect-fluid Robertson-Walker perturbations. Phys Rev D 1990; 42: 1035.

[87] Hwang J-C, Noh H. Cosmological perturbations in generalized gravity theories. Phys Rev D 1996; 54:1460.

[88] Abdalla MCB, Nojiri S, Odintsov SD. Consistent modified gravity: dark energy, acceleration and the absence of cosmic doomsday . Class Quant Gravity 2005; 22: L35; Briscese F, Elizalde E, Nojiri S, Odintsov SD. Phantom scalar dark energy as modified gravity: Understanding the origin of the Big Rip singularity. Phys Lett B 2007; 646:105.

[89] Frolov AV. Singularity problem with *f*(*R*) models for dark energy. Phys Rev Lett 2008; 101: 061103; Appleby SA, Battye RA. Do consistent F(R)F(R) models mimic general relativity plus Λ ? Phys Lett B 2007; 654: 7; Kobayashi T, Maeda K. Relativistic stars in *f*(*R*) gravity, and absence thereof. Phys Rev D 2008; 78: 064019; Babichev E, Langlois D. Relativistic stars in *f*(*R*) gravity. Available from arXiv:0904.1382; Miranda V, Joras SE, Waga I, Quartin M. Viable singularity-free f(r) gravity without a cosmological constant. Phys Rev Lett 2009; 102: 221101; Thongkool I, Sami M, Rai CS. How delicate are the *f*(*R*) gravity models with disappearing cosmological constant? Available from: arXiv:0908.1693; Appleby S, Battye R, Starobinsky AA. Cosmology and extragalactic astrophysics. High energy Phys Ther. Available from arXiv: 0909. 1737.

[90] Bamba K, Nojiri S, Odintsov SD. The future of the universe in modified gravitational theories: approaching a finite-time future singularity. J Cosmol Astropart Phys 2008; 0810: 045; Nojiri S, Odintsov SD. Future evolution and finite-time singularities in *f*(*R*) gravity unifying inflation and cosmic acceleration. Phys Rev D 2008; 78: 046006; Capozziello S, De Laurentis M, Nojiri S, Odintsov SD. Classifying and avoiding singularities in the alternative gravity dark energy models. Phys Rev D 2009; 79: 124007.

[91] De Felice A, Hindmarsh M, Trodden M. Ghosts, instabilities, and superluminal propagation in modified gravity models. J Cosmol Astropart Phys 2006; 08: 005; Calcagni G , de Carlos B, De Felice A. Ghost conditions for Gauss-Bonnet cosmologies. Nucl Phys B 2006; 752: 404.

[92] Comelli D. Born-Infeld-type gravity. Phys Rev D 2005; 72: 064018.

[93] Navarro I, Van Acoleyen K. Consistent long distance modification of gravity from inverse powers of the curvature. J Cosmol Astropart Phys 2006, 0603: 008.

[94] Erickcek A , Smith T L, Kamionkowski M. Solar system tests do rule out 1/R gravity. Phys Rev D 2006; 74: 121501(R).

[95] Olmo G J. Limit to general relativity in *f*(*R*) theories of gravity. Phys Rev D 2007; 75: 023511.

[96] Chiba T, Smith TL, Erickcek AL. Solar system constraints to general *f(R)* gravity. Rev D 2007; 75: 124014.

[97] Bertotti B, Iess L, Tortora P. A test of general relativity using radio links with the Cassini spacecraft. Nature 2003; 425: 374.

[98] Faulkner T, Tegmark M, Bunn EF, Mao Y. Constraining *f*(*R*) gravity as a scalar-tensor theory. Phys Rev D 2007; 76: 063505.

[99] Khoury J, Weltman A. Chameleon Fields: Awaiting Surprises for Tests of Gravity in Space. Phys Rev Lett 2004; 93: 171104; Khoury J, Weltman A. Chameleon cosmology. Phys Rev D 2004; 69: 044026.

[100] Song Y-S, Hu W, Sawicki I. Large scale structure of *f*(*R*) gravity. Phys Rev D 2007; 75: 044004; Hu W, Sawicki I. Models of *f*(*R*) cosmic acceleration that evade solar system tests. Phys Rev D 2007; 76: 104043; de la Cruz-Dombriz A, Dobado A. *f*(*R*) gravity without a cosmological constant. Phys Rev D 2006; 74: 087501; Faulkner T, Tegmark M, Bunn EF, Mao Y. Constraining *f*(*R*) gravity as a scalar-tensor theory. Phys Rev D 2007; 76: 063505; Starobinsky A A. Disappearing cosmological constant in *f*(*R*) gravity JETP Lett 2007; 86: 157; Sokolowski L M. Metric gravity theories and cosmology: II. Stability of a ground state in *f*(*R*) theories. Class Quantum Gravity 2007; 24: 3713.

[101] Carloni S , Ananda KN, Dunsby PKS, Abdelwahab MES. Unifying the study of background dynamics and perturbations in *f*(*R*)-gravity. available from: arXiv:0812.2211 [astro-ph].

[102] Ananda KN, Carloni S, Dunsby PKS. The evolution of cosmological gravitational waves in *f*(*R*) gravity. Phys Rev D 2008; 77: 024033.

[103] Carloni S, Ananda KN, Dunsby PKS, Troisi A. The evolution of density perturbations in *f*(*R*) gravity. Phys Rev D 2008; 77:024024.

[104] Noakes DR. The initial value formulation of higher derivative gravity. J Math Phys 1983; 24: 1846.

[105] Cocke WJ, Cohen JM. Cauchy problem in the scalar-tensor gravitational theory. J Math Phys 1968; 9: 971.

[106] Salgado M. The Cauchy problem of scalar Ðtensor theories of gravity. Class Quantum Gravity 2006; 23: 4719.

[107] Salgado M, Martinez-del Rio D, Alcubierre M, Nunez D. Hyperbolicity of scalar-tensor theories of gravity. Phys Rev D 2008; 77: 104010.

[108] Solín. Partial Differential equations and the finite element method. Wiley: New York 2006.

[109] Lanahan-Tremblay N, Faraoni V. The Cauchy problem of *f*(*R*) gravity. Class Quantum Gravity 2007; 24: 5667.

[110] Capozziello S, Vignolo S. On the well formulation of the Initial Value Problem of metric--affine *f*(*R*)-gravity. Int Jou Geometric Methods Mod Phys 2009; 6: 985.

[111] Capozziello S, Vignolo S. A comment on the Cauchy problem of *f*(*R*) gravity. Class Quantum Gravity 2009; 26: 168001; Faraoni V. Reply to A comment on 'The Cauchy problem of *f*(*R*) gravity. Class Quantum Gravity 2009; 26: 168002; Olmo G J, Singh P. Covariant effective action for loop quantum cosmology a la palatini. J Cosmol Astropart Phys 2009; 0901: 030.

[112] Faraoni V, Lanahan-Tremblay N. The Cauchy problem of *f*(*R*) gravity. Phys Rev D 2008; 78: 064017.

[113] Barausse E, Sotiriou TP, Miller J. Curvature singularities, tidal forces and the viability of Palatini *f*(*R*) gravity. Class Quantum Gravity 2008; 25: 062001.

[114] Iglesias A, Kaloper N, Padilla A, Park M. How (not) to use the Palatini formulation of scalar-tensor gravity? Phys Rev D 2007; 76: 104001.

[115] Faraoni V. Palatini *f*(*R*) gravity as a fixed point. Phys Lett B 2008; 665: 135.

[116] Daly RA, Djorgovsky SG. Direct determination of the kinematics of the universe and properties of the dark energy as functions of redshift. Astrophys J 2004; 612: 652 .

[117] Zhang P. Testing gravity against the early time integrated Sachs-Wolfe effect. Phys Rev D 2006; 73: 123504.

[118] Lahav O, Bridle SL, Percival WJ, *et al*. The 2dF Galaxy Redshift Survey: the amplitudes of fluctuations in the 2dFGRS and the CMB, and implications for galaxy biasing. Mon Not R Astron Soc 2002; 333: 961.

[119] Capozziello S, Cardone VF, Troisi A. Low surface brightness galaxy rotation curves in the low energy limit of Rn gravity: no need for dark matter? Mon Not R Astron Soc 2007; 375: 1423.

[120] Capozziello S, Stabile A, Troisi A. Spherically symmetric solutions in *f*(*R*) gravity via the Noether symmetry approac. Class Quantum Gravity 2007; 24: 2153.

[121] de Blok WJG, Bosma A. High-resolution rotation curves of low surface brightness galaxies. Astron Astrophys 2002; 385: 816.

[122] McGaugh SS. The baryonic tully-fisher relation of galaxies with extended rotation curves and the stellar mass of rotating galaxies. Astrophys J 2005; 632: 859.
[123] Capozziello S, Corda C, De Laurentis M. Stochastic background of gravitational waves tuned by $f(R)$-gravity. Mod Phys Lett A 2007; 22: 1097.
[124] Capozziello S, De Laurentis M, Francaviglia M. Higher-order gravity and the cosmological background of gravitational waves. Astropart Phys 2008; 29 (2): 125.
[125] Capozziello S, Corda C, DeLaurentis M. Massive gravitational waves from $f(R)$ theories of gravity: Potential detection with LISA. Phys Lett B 2008; 699: 255.
[126] Bellucci S, Capozziello S, De Laurentis M, Faraoni V. Position and frequency shifts induced by massive modes of the gravitational wave background in alternative gravity. Phys Rev D 2009; 79: 104004.
[127] Capozziello S, De Filippis E, Salzano V. Modelling clusters of galaxies by $f(R)$ gravity. Mon Not R Astron Soc 2009; 394: 947.
[128] Corda C. Interferometric detection of gravitational waves: the definitive test for general relativity. Int Jou Mod Phys D 2009; 18(14): 2275-82.

Received: September 23, 2009 Revised: September 28, 2009 Accepted: September 28, 2009