Parameter Estimation of Fractional Low Order Time-Frequency Autoregressive Based on Infinite Variance Analysis

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Abstract: The Parameter model analysis algorithms include autoregressive (AR) model, moving average (MA) and autoregressive moving average (ARMA) model. The existing TFAR model is improved, the new fractional low order time-frequency autoregressive (FLO-TFAR) model and the concept of generalized TF-Yule-Walker equation are proposed, fractional low-order covariance is preferred instead of autocorrelation in the model; The parameter estimation of the model is derived, spectrum estimation algorithm based on the FLO-TFAR model is presented, and the steps of the algorithm are summarized. The detailed comparison of the FLO-TFAR spectrum estimation only can realize the frequency spectrum estimation of the higher precision and resolution.

Keywords: Generalized TF-Yule-Walker equation d index, Stable distribution, time-frequency autoregressive, time-frequency spectrum.

1. INTRODUCTION

AR model parameter estimation is the simplest and widely used. The MA and ARMA model can be represented by an infinite order AR model. Hence, the AR model has been widely used in stationary random signal modeling, such as radar and communication, etc. A lot of random signals is non-stationary process in signal processing, therefore, non-stationary process TF-ARMA model concept has been proposed [1, 2]. Michael put forward the time-frequency AR(TFAR) non-stationary process model [3], and the parameter estimation method based on the TF - Yule - Walker equation is given. The TFAR model can accomplish the signal’s high resolution time-frequency spectrum estimation without the cross terms. The improved TFAR model methods have been proposed, such as the LS-TFAR TFAR model parameter estimation algorithm and ML-TFAR model algorithm [4], vector time-frequency AR model algorithm (VTFAR) [5]; at some special occasions, such as biomedical signal, meteorological data, stock price etc., random signal or noise process often have strong pulse characteristics, the variance of the process is not finite, they can be described by $\alpha$ stable distribution [6-8].

The larger error is produced if the AR model based on Gaussian is used in $\alpha$ stable distribution environment, therefore, the AR $\alpha$S process parameter estimation is proposed based on the fractional lower order moment (FLM) in the literature [9, 10]. And the corresponding $\alpha$ spectrum estimation which can realize the frequency spectrum estimation under $\alpha$ stable distribution environment is proposed. The new improved AR model and ARMA model are put forward using the fractional lower order covariance (FLOM) replacing FLM [11, 12], which realize the frequency spectrum estimation of the higher precision and resolution.

The $\alpha$ spectrum estimation only can realize the frequency estimation of the stationary $\alpha$S process, and in view of the time-varying non-stationary process, TFAR non-stationary Gaussian excitation linear AR model method in literature [9-12] will no longer be applicable. Hence, traditional Cohen class time-frequency distribution is improved based on the fractional lower order moment. A fractional lower order Cohen class time-frequency distribution is got [13, 14], using fractional lower-order covariance to replace correlation in the model, to put forward the non-stationary process fractional lower order time-frequency autoregressive (FLO-TFAR) model. The generalized TF-Yule-Walker equation is defined to compute the parameter estimation of the FLO-TFAR model. The FLO-TFAR model time-frequency spectrum estimation is defined, which can realize model time-frequency distribution of the observation signals. Computer simulation shows that the proposed FLO-TFAR model can realize linear approximation of non-stationary $\alpha$S process, can realize high-resolution frequency estimation, it has better performance and fractional lower order Cohen class time-frequency distribution than the existing TFAR model algorithm, and has a certain toughness.

2. STABLE DISTRIBUTION

A. $\alpha$ Stable Distribution

$\alpha$ stable distribution is a kind of generalized Gaussian distribution, the process is not limited in variance and their
probability density function has a serious tail. Its characteristic function can be described as [6-8].

\[
\phi(t) = \exp \left\{ j \mu - \gamma \right\}^\alpha \left[ 1 + j \beta \text{sign}(t) \omega(t, \alpha) \right]
\]

The time-domain waveform of the $StaS$ stable distribution are shown in Fig. (1), its probability density function (PDF) is shown in Fig. (2).

\[\alpha = 0.5\]

Fig. (1). Time-domain waveform of the $StaS$ stable distribution under $\alpha = 0.5, 1.0, 1.5$ and $2.0$

Fig. (3) shows variance waveforms of the $StaS$ stable distribution with successive increase of sample numbers from $\alpha = 0.5, 1.0, 1.5$ to $2.0$. The result shows that variance is not limited when the values of $\alpha$ belong to $0 < \alpha < 2$, variance is convergent when $\alpha = 2$ (Gaussian distribution), where $\gamma = 2\sigma^2 = 2 (\sigma = 1)$.

![Fig. (1)](image1)

![Fig. (2)](image2)

![Fig. (3)](image3)

Fig. (1). Time-domain waveform of the $StaS$ stable distribution under $\alpha = 0.5, 1.0, 1.5$ and $2.0$

Fig. (2). PDF of the $StaS$ stable distribution in different alpha ($\alpha$)

Fig. (3). Variance of the $StaS$ stable distribution with successive increase of sample numbers of different alpha ($\alpha$).

B. Fractional Lower Order Covariation

The covariance of the $StaS$ distribution does not exist because its variance is not limited. Hence, covariance concept is put forward by Miller in 1978, which is similar to the covariance of Gaussian random process. Covariance of two $StaS$ distribution random variables $X$ and $Y$ is defined as:

\[
[X, Y]_\alpha = \int_x x^{\alpha - 1} \mu(ds), \quad 1 < \alpha \leq 2
\]

Where $S$ denotes the unit circle, $< >$ denotes the operation of $X$ sign($z$), the covariance coefficient of $X$ and $Y$ is defined as:

\[
\lambda_{XY} = \frac{[X, Y]_\alpha}{[Y, Y]_\alpha}
\]

If the dispersion coefficient of $Y$ is $\gamma_y$, the covariation and covariation coefficient can be written as:

\[
[X, Y]_\alpha = \frac{E(XY^{\alpha - 1})}{E(Y^{\alpha - 1})} \gamma_y, \quad 1 \leq p < \alpha \leq 2
\]

\[
\lambda_{XY} = \frac{E(XY^{\alpha - 1})}{E(Y^{\alpha - 1})}, \quad 1 \leq p < \alpha \leq 2
\]

According to the definition of covariation coefficient, the covariation coefficient of a real observation sequence $X(n)$ ($n = 0, 1, K N$) can be defined as:

\[
\lambda(m) = \frac{E(X(n)X(n + m)^{\alpha - 1})}{E(X(n + m)^{\alpha - 1})}, \quad 1 \leq p < \alpha \leq 2
\]
\[ \hat{\lambda}(m) = \sum_{n=1}^{N} X(n)[X(n+m)]^{\alpha-1} \text{sign}[X(n+m)], \quad 1 \leq p < \alpha \leq 2 \]  

Where, \( \hat{\lambda}(m) \) is the approximate estimation of \( \lambda(m) \). Compared with (6), a more simplified fractional lower order moments method is used in array signal processing, it can be expressed as:

\[ \hat{\lambda}_{FLOM}(m) = E[X(n)X(n+m)]^{\alpha-1}, \quad 1 \leq p < \alpha \leq 2 \]

\[ \hat{\lambda}_{FLOAC}(m) = \begin{cases} 1 \frac{1}{L_{z} - L_{x} - a/n} \sum_{n=1}^{N} X(n)[X(n+m)]^{\alpha-1} \text{sign}[X(n+m)] \quad & \text{if } X(n) \text{ is real} \\ 1 \frac{1}{L_{z} - L_{x} - a/n} \sum_{n=1}^{N} X(n)[X(n+m)]^{\alpha-2} X'(n+m) \quad & \text{if } X(n) \text{ is complex} \end{cases} \]

\[ 1 \leq p < \alpha \leq 2 \]

Where, \( L_{1} = \max(0,-m) \), \( L_{2} = \min(N - m, N) \).

C. Fractional Lower Order Covariance

Because the fractional lower order covariation and fractional lower order moments provide \( \alpha \) for \( 1 < \alpha \leq 2 \) and the range from 0 to 1 is not defined. Hence, the fractional lower order covariation (FLOC) is given, which can provide \( \alpha \) for \( 0 < \alpha \leq 2 \). The fractional lower order autocovariance (FLOAC) of \( N \) pairs of observations \( X(n) \) \((n = 0, 1, K, N)\) based on the definition of FLOC can be defined as:

\[ R_{\alpha}(m) = E\{X(n)^{c_{\alpha}}X(n+m)^{c_{\alpha}}\}, \quad 0 \leq a \leq \alpha / 2, \quad 0 \leq b \leq \alpha / 2 \]

Where \( 0 < \alpha \leq 2 \), if \( X(n) \) is real, the FLOC is estimated by the sample FLOC \( \hat{R}_{\alpha}(m) \).

\[ \hat{R}_{\alpha}(m) = \frac{1}{L_{z} - L_{x} - a/n} \sum_{n=1}^{N} X(n)^{\alpha-1}[X(n+m)]^{\alpha-1} \text{sign}[X(n)X(n+m)] \]

And if \( X(n) \) is complex, the FLOAC is estimated by the sample FLOAC \( \hat{R}_{\alpha}(m) \)

\[ \hat{R}_{\alpha}(m) = \frac{1}{L_{z} - L_{x} - a/n} \sum_{n=1}^{N} X(n)^{\alpha-2} X'(n)X'(n+m) \]

Where, \( L_{1} = \max(0,-m) \), \( L_{2} = \min(N - m, N) \), * denotes the conjugate operation.

3. THE FLO-TFAR TIME-FREQUENCY ESTIMATION METHOD

A. The Non-Stationary \( \mathcal{S} \alpha \mathcal{S} \) Process TFAR Model

A stationary AR process \( x[n] \) can be defined by

\[ x[n] = -\sum_{i=1}^{M} a_{i} x[n-i] + e[n] = -\sum_{i=1}^{M} a_{i} (\Psi^{i} x)[n] + e(n) \]

Where \( a_{i} \) is the AR model parameters, \( M \) is the AR model order, \( e(n) \) is stationary independent identically distributed (I.I.D) Gaussian process \( (\Psi^{i} x)[n] = x[n-i] \), for example \( (\Psi^{i} x)[n] = x[n-1] \). Because a lot of signals are non-stationary in real-world signal processing, the type (13) is improved and a non-stationary AR process \( x[n] \) is defined by:

\[ x[n] = -\sum_{i=1}^{M} a_{i} x[n-i] + e[n] = -\sum_{i=1}^{M} a_{i} (\Psi^{i} x)[n] + e(n) \]

A new non-stationary TFAR process \( x[n] \) is defined by:

\[ x[n] = -\sum_{i=1}^{M} a_{i} e^{2\pi i n} x[n-i] + e[n] = -\sum_{i=1}^{M} a_{i} (Z^{i} x)[n] + e(n) \]

Where \( a_{i} = \sum_{i=1}^{M} a_{i} e^{2\pi i n}, (Z^{i} x)[n] = e^{2\pi i n} x[n], a_{i,j} \) is the TFAR model extension parameters, \( M \) and \( L \) are the model order, among them \( M \) is the order in time domain and \( L \) is the order in frequency domain. When \( L = 0 \), the TFAR model will degenerate into the AR model. Because \( e(n) \) is a non-stationary process and it’s variance is time-varying, it can be defined as:

\[ \sigma^{2}[n] = \sum_{i=2}^{2L} \sigma_{i} e^{2\pi i n} \]  

According to the type (14),(15) method, we also define a time-frequency non-stationary \( \mathcal{S} \alpha \mathcal{S} \) process TFAR model as:

\[ x[n] = -\sum_{i=1}^{M} a_{i} e^{2\pi i n} X[n-i] + U[n] = -\sum_{i=1}^{M} a_{i} (Z^{i} x)[n] + U(n) \]

Where \( U(n) \) is stationary \( \mathcal{S} \alpha \mathcal{S} \) distribution process, according to the type (16), we define the dispersion coefficient of stationary \( \mathcal{S} \alpha \mathcal{S} \) process as:

\[ \chi[n] = \sum_{i=2}^{2L} \gamma_{i} e^{2\pi i n} \]

B. The generalized TF-Yule-Walker equations and parameter estimation

When the traditional Yule - Walker equations is solved, the TFAR model coefficient is acquired. We can also compute the generalized TF-Yule-Walker equations to get TFAR model parameters \( a_{i,j} \) for non-stationary \( \mathcal{S} \alpha \mathcal{S} \) process. On multiplying (17) by \( X^{c_{\alpha}-1} [n-i'] \) and taking expectation, this will yield

\[ C_{x}[n,i] = -\sum_{i=1}^{M} a_{i} e^{2\pi i n} C_{x}[n-i',i'] + C_{U,x}[n,i] \]
Where \( E[\eta_i^p \eta_j^p] \) is the order covariance of \( \eta_i, \eta_j \) \( (1 < p < \alpha \leq 2) \) is \( p \)-order covariance of \( \eta_i, \eta_j \) \[6\], \( \eta_i^p > 0 \) is the auto-coefficient of \( X[l] \) \( \delta^n \), \( E[X[l]^p X[l-i]^p] = E[X[l]^p X[l-i]^p] \) is the auto-covariance function of \( X[l] \), \( C_{X}[n,l] = E[X[n]] = \frac{E[X[n]]}{\mu} \) is the cross-covariance function of \( X[n] \) and \( U[n] \). \( * \) indicates conjugate, \( \eta_i^p = (\eta_i^*)^p = (\eta_i^p)^* \).

Because the \( X[n] \) and \( U[n] \) are independent of each other, type (19) can be simplified as:

\[
C_{X}[n,l] = -\sum_{i=1}^{\infty} \sum_{l=1}^{\infty} a_i^l e^{j \frac{2\pi nl}{M}} C_{X}[n-i,l-i] \quad i > 0
\]

We assume the observed value of \( X[n] \) is \( [0, N-1] \), both sides of type (20) yield is taken for the length-N discrete Fourier transform (DFT):

\[
\mathcal{X}[i,l] = -\sum_{l=1}^{\infty} \sum_{l=1}^{\infty} a_i^l \mathcal{X}[i-l,l-1] e^{-j \frac{2\pi nl}{M}} \quad i > 0
\]

We call equation (21) for the generalized TF - Yule - Walker equations, when \( p = 2 \), it degenerates into the TF - Yule - Walker equations of the non-stationary Gaussian process. (21) includes \( (2L+1)M \) separated equations and the solved parameters \( a_i^l \) are \( (2L+1)M \), so, the model coefficient \( a_i^l \) is acquired by solving the equations (21).

\[
\lambda_{X}[i,l] \quad \text{is similar to the Cohen- class ambiguity function (AF) of the second order correlation function in the time-frequency distribution in types (22-23), where the auto-correlation is replaced by the auto-covariance. It can be named discrete fractional lower order ambiguity function (FLO-AF), where} \ \mathcal{X}[i,l] \quad \text{is called fractional lower order expected ambiguity function (FLO - EAF). It indicates the statistical auto-covariance of the time shift} \ i \ \text{and frequency shift} \ l \ \text{in the time-frequency domain. In order to solve the model coefficients} \ a_i^l \text{ of (21), we define according to the frequency shift} \ M \times M \text{ Toeplitz matrix} \ \mathcal{\tilde{\lambda}}_i \quad \text{is defined as:}
\]

\[
M \times M \quad \text{Toeplitz matrix} \ \mathcal{\tilde{\lambda}}_i \quad \text{is defined as:}
\]

\[
\lambda_{i} = \mathcal{\tilde{\lambda}}_{i} M
\]
ate the non-stationary signal $x[n]$ by passing normalized stationary $U(n)$ through a time-varying IIR filter. Let $M = 3, L = 2 \ (i = 1,2,3, \ l = 2,1,0,-1,-2)$, the filter parameters $A_{i,l}$ are defined as:

$$A_{i,l} = \begin{bmatrix} a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,0} & a_{2,0} & a_{3,0} \\ a_{1,-1} & a_{2,-1} & a_{3,-1} \\ a_{1,-2} & a_{2,-2} & a_{3,-2} \end{bmatrix}$$

(31)

Fig. (4). The observation signal $x[n]$ in time domain.

The $x[n]$ is a non-stationary process, whose time-domain waveform diagram of the real and imaginary part are shown in Fig. (4). Its evolutionary time-frequency spectrum is computed from the given TF parameters $A_{i,l}$ as shown in Fig. (5).

We assume that parameter estimation of TFAR model algorithm is $\hat{a}_{ij}$ and FLO-TFAR model parameter estimation is $\hat{a}'_{ij}$. The following comparison is made in order to compare parameter estimation of two methods. Let $\alpha = 1.3$, $N = 256$, we estimate the model parameter by performing the TFAR model algorithms and FLO-TFAR model algorithm. After independent 20 times estimates is runned, the averaged parameters are shown in Table 1.

**Table 1. The parameter estimation of TFAR model and FLO-TFAR model.**

<table>
<thead>
<tr>
<th>The Actual Parameters</th>
<th>The Estimated Parameters Based on TFAR Model</th>
<th>The Estimated Parameters Based on FLO-TFAR Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{1,2}$</td>
<td>0.2774 + 0.1809i</td>
<td>0.2399 + 0.1452i</td>
</tr>
<tr>
<td>$a_{1,1}$</td>
<td>0.1762 + 0.1503i</td>
<td>0.1865 + 0.2122i</td>
</tr>
<tr>
<td>$a_{1,0}$</td>
<td>-0.2140 - 0.1005i</td>
<td>-0.2558 - 0.0701i</td>
</tr>
<tr>
<td>$a_{1,-1}$</td>
<td>0.0744 + 0.1909i</td>
<td>0.0869 + 0.2195i</td>
</tr>
<tr>
<td>$a_{1,-2}$</td>
<td>0.1601 + 0.1545i</td>
<td>0.2263 + 0.1847i</td>
</tr>
<tr>
<td>$a_{2,2}$</td>
<td>-0.0145 - 0.0404i</td>
<td>-0.0441 - 0.1057i</td>
</tr>
<tr>
<td>$a_{2,1}$</td>
<td>0.0565 - 0.0074i</td>
<td>0.1344 - 0.0532i</td>
</tr>
<tr>
<td>$a_{2,0}$</td>
<td>0.2221 + 0.1832i</td>
<td>0.2821 + 0.2277i</td>
</tr>
<tr>
<td>$a_{2,-1}$</td>
<td>0.0529 + 0.1038i</td>
<td>0.1480 + 0.1358i</td>
</tr>
<tr>
<td>$a_{2,-2}$</td>
<td>-0.0889 - 0.0562i</td>
<td>-0.1345 - 0.0560i</td>
</tr>
<tr>
<td>$a_{3,2}$</td>
<td>0.0541 + 0.0448i</td>
<td>0.1069 + 0.0281i</td>
</tr>
<tr>
<td>$a_{3,1}$</td>
<td>0.0807 - 0.0163i</td>
<td>0.0672 + 0.0224i</td>
</tr>
<tr>
<td>$a_{3,0}$</td>
<td>-0.1783 - 0.0609i</td>
<td>-0.2319 - 0.0145i</td>
</tr>
<tr>
<td>$a_{3,-1}$</td>
<td>0.1087 - 0.0698i</td>
<td>0.1103 - 0.0191i</td>
</tr>
<tr>
<td>$a_{3,-2}$</td>
<td>-0.0116 + 0.0304i</td>
<td>0.0107 + 0.0506i</td>
</tr>
</tbody>
</table>

Table 1 shows that the parameter estimation of FLO-TFAR model algorithm is closer to the actual value, and TFAR model algorithm has a greater deviation. In order to further compare the performance of the two algorithms under different characteristic index $\alpha$, we measured them by the mean square error estimation precision, the parameter estimation mean square error (MSE) of TFAR and FLO-TFAR model is defined respectively as $MSE$ and $MSE'$.

$$MSE = \sqrt{\sum_{i=1}^{M} \sum_{l=2}^{2} (\hat{a}_{ij} - a_{ij})^2}$$

(32)

$$MSE' = \sqrt{\sum_{i=1}^{M} \sum_{l=2}^{2} (\hat{a}'_{ij} - a_{ij})^2}$$

(33)
The mean square error curve is shown in Fig. (6) when the characteristics of the process index $\alpha$ (Alpha) change from 1.0 to 2.0. The simulation shows that the TFAR model parameters error change from 1 db - 11 db, and the FLO-TFAR model parameter estimation error maintain around -12 db.

Fig. (6). The MSE comparison of the TFAR model and FLO-TFAR model.

B. Time-Frequency Spectrum Estimation Comparison

We respectively used the parameter estimation algorithm based on TFAR model and FLO - TFAR model parameter estimation algorithm to estimate the time-frequency spectrum; simulation spectrum diagram is shown in Figs. (7, 8).

Fig. (7). The TFAR model time-frequency spectrum.

The results show that the estimated spectrum based on TFAR model Fig. (7) is different from the actual spectrum Fig. (5), and the proposed FLO - TFAR model spectrum estimation is very close to the actual time-frequency distribution Fig. (5).

Fig. (8). The FLO-TFAR model time-frequency spectrum.

CONCLUSION

We proposed a FLO-TFAR model spectrum method which can work in $\mathcal{S}_\alpha$ stable distribution environment by combining the stationary process $\alpha$ spectrum with the existing AR model time-frequency algorithm and using fractional low-order covariance instead of the second order autocorrelation matrix. The dimension of the method is extended to 3-D(time, frequency, amplitude). Simulations show that the proposed algorithm has good performance in parameter estimation and spectrum estimation, FLO - TFAR model algorithm has more advantages when $\alpha$ is smaller. Hence, the proposed FLO - TFAR model algorithm has shown better toughness and wider applicability in this paper. In order that the FLO - TFAR model can be applied to more fields, in the next step, it is planned to extend it to the fractional lower order autoregressive moving average (FLO-TFARMA) model, and realize the parameter estimation and spectrum estimation.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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Declared none.

REFERENCES


