Notes on Variational Minimizing Solutions for the 2-Fixed Center Problems

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Abstract In the paper [1] of Shen Hong, JMAA 306 (2005) 761-766, the proof of Theorem 1.1 had a gap, here we give a complete proof.

Keywords: Two fixed center problem, variational minimizing methods.

1. INTRODUCTION

The motion of a point mass, moving in the gravitational field of two fixed attracting centers, is an old problem first posed by Euler [2-4] in 18th century, as an intermediate step towards the solution of the famous 3-body problem. Euler integrated the equations of motion for the 2-dimensional case, i. e., the case where the point mass moves on a plane containing the two attracting centers. The problem has been used in the calculation of satellite trajectories in the gravitational field of the Earth (Alexeev [5], Marchal [6, 7]), for some recent papers, we can refer [8-10]. The Least Action Principle of Fermat-Maupertuis is the most basic principle in our nature, so in this paper, we try to use it to study the 2-fixed center problems. For Newtonian 2 and 3 body problems, we refer to [9, 11, 12].

Assume two particles P_1, $P_2 \in R^n$ with masses $1-\mu > 0$ and $\mu > 0$ are fixed at x-axis, the origin of the inertial systems is located at the center of $\overrightarrow{P_1P_2}$ and $|\overrightarrow{P_1P_2}| = 1$.

Assume the particle $q \in R^2$ with mass $m_3 > 0$ is moving under the Newtonian gravitational force of $P_1 = (-\frac{1}{2}, 0)$ and $P_2 = (\frac{1}{2}, 0)$.

The equation of motion for m_3 is

$$\ddot{q} = \frac{\partial u}{\partial q} \tag{1}$$

where $q = (x_1, x_2)$.

$$U(q) = \frac{1-\mu}{|q-P_1|} + \frac{\mu}{|q-P_2|} \tag{2}$$

We define Lagrangian action

$$f(q) = \int_0^T \left(\frac{1}{2}|q|^2 + U(q)\right) dt, \tag{3}$$

where

$$q \in A = \left\{ \begin{array}{c} q = (x_1, x_2) \\ x_i \in W^{1,2}(R/TZ, R), \\ q(t) \neq P_1, P_2 \forall t \in R, \\ q(t+T/2) = -q(t) \text{ or } q(-t) = -q(t) \end{array} \right\} \tag{4}$$

and

$$W^{1,2}(R/T \bullet Z, R) = \left\{ \begin{array}{c} x \mid x, \dot{x} \in L^2(R, R) \\ x(t+T) = x(t) \end{array} \right\}. \tag{5}$$

We have the following result:

Theorem 1.1. *Let* μ =1/2, *then the global minimizers of* $f(q)$ on the closure $\overline{\Lambda}$ of Λ are just the origin.

2. PROOF OF THEOREM 1.1

Lemma 2.1. (Palais [13]). *Let* σ *an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for* $\forall \sigma \in G$,

$$f(\sigma \circ x) = f(x), \tag{1}$$

where $f: H \to R$. Let

$$\text{Fix} = \{x \in H \mid \sigma \circ x = x, \forall \sigma \in G\} \tag{2}$$

Then the critical point of f in Fix is also a critical point of f in H.

By Lemma 2.1, we have

Lemma 2.2. *Let* μ =1/2, *then the critical point of* f(q) *in* Λ *is a noncollision T -periodic solution for* (1).

In order to prove Theorem 1.1, we need some famous inequalities.

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Lemma 2.3. (Poincare-Wirtinger [14]). Let $q \in W^{1,2}(R/TZ, R^n)$ and $\int_0^T q(t)\,dt = 0$; *then*

$$(i)\quad \int_0^T |q(t)|^2 \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |a(t)|^2\,dt; \tag{3}$$

(ii) inequality (3) takes the equality if and only if

$$q(t) = \cos\frac{2\pi}{T}t + \beta\sin\frac{2\pi}{T}t, \quad \alpha,\beta \in R^n. \tag{4}$$

Lemma 2.4. (Jensen [14]).

(1°) *Assume* ϕ *is a convex function on* $[r, R]$, $-\infty \leq r \leq R \leq +\infty$, $\hat{f}$ *and p are integrable functions on* $[c, d]$, $-\infty \leq c \leq d \leq +\infty$, $r \leq \hat{f}(x) \leq R$, $\hat{p}(x) \geq 0$, $\forall x \in [c, d]$ and $\int_c^d \hat{p}(x)\,dx > 0$, *then*

$$\phi\left(\frac{\int_c^d \hat{p}(x)\hat{f}(x)\,dx}{\int_c^d \hat{p}(x)\,dx}\right) \leq \frac{\int_c^d \hat{p}(x)\phi(\hat{f}(x))\,dx}{\int_c^d \hat{p}(x)\,dx} \tag{5}$$

(2°) Inequality (5) takes the equality if and only if

$$\hat{f}(x) = \text{const}. \tag{6}$$

Lemma 2.5. ([12]) For $m_i > 0$, $\alpha > 0$, we have

$$\sum_{1\leq i\leq l,\, l+1\leq j\leq N}\left(\frac{m_i m_j}{|q_i, q_j|^\alpha}\right) \geq \left(\sum_{1\leq i\leq l,\, l+1\leq j\leq N} m_i m_j\right)^{1+\frac{\alpha}{2}} \cdot \left(\sum_{1\leq i\leq l,\, l+1\leq j\leq N} m_i m_j |q_i, q_j|2\right)^{-\frac{\alpha}{2}} \tag{7}$$

and the above inequality takes the equality if and only if

$$|q_i(t) - q_j(t)| = \lambda(t) > 0, \quad 1 \leq i \leq l,\ l+1 \leq j \leq N, \tag{8}$$

Now we prove Theorem 1.1.

$q\left(t + \frac{T}{2}\right) = -q(t)$ or $q(t) = -q(-t)$ implies $\int_0^T q_i(t)\,dt = 0$, so by Poincare-Wirtinger inequality, we have

$$f(q) \geq \frac{1}{2}\left(\frac{2\pi}{T}\right)^2 \int_0^T |q|^2\,dt + \frac{1}{2}\int_0^T |q - P_1|^{-1}\,dt + \frac{1}{2}\int_0^T |q - P_2|^{-1}\,dt. \tag{9}$$

By (7), we have

$$f(q) \geq \frac{1}{2}\left(\frac{2\pi^2}{T^2}\right)\int_0^T |q|^2\,dt + 2^{1/2}\int_0^T [|q - P_1|^2 + |q - P_2|^2]dt. \tag{10}$$

By Jensen's inequality we have

$$\begin{aligned} f(q) &\geq \left(\frac{2\pi^2}{T^2}\right)\int_0^T |q|^2\,dt + 2^{\frac{1}{2}}T^{3/2}[\int_0^T (|q - P_1|^2 + |q - P_2|^2)dt]^{-1/2} \\ &= \left(\frac{2\pi^2}{T^2}\right)\int_0^T [|q - P_1|^2 + |q - P_2|^2]dt \\ &\quad + 2^{\frac{1}{2}}.T^{3/2}.\left\{\int_0^T [|q - P_1|^2 + |q - P_2|^2]dt\right\}^{-\frac{1}{2}} \\ &\quad - \left(\frac{2\pi^2}{T^2}\right).\left[\int_0^T q.P_1 + \int_0^T |P_1|^2 - 2\int_0^T q.P_2 + \int_0^T |P_2|^2\right] \\ &= \varphi(s) = \left(\frac{2\pi^2}{T^2}\right)s^2 + 2^{\frac{1}{2}}.T^{3/2}.s^{-1} \\ &\quad \left(\frac{2\pi^2}{T^2}\right)(T/2) \geq \inf\{\varphi(s), s > 0\}, \end{aligned} \tag{11}$$

where

$$s^2 = \int_0^T [|q - P_1|^2 + |q - P_2|^2]dt \tag{12}$$

We notice that $\varphi(s)$ is a strictly convex smooth function on $s > 0$ and $\varphi(s) \to +\infty$ as $s \to 0^+$ and $s \to +\infty$, so $\varphi(s)$ attains its infimum at some $s_0 > 0$.

We notice that the inequality (11) take the equalities if and only if Poincare Writinger's inequality and (7) and Jensen's inequality take the equalities simultaneously, hence we have

$$q(t) = \alpha\cos\frac{2\pi}{T}t + \beta\sin\frac{2\pi}{T}t, \alpha,\beta \in R^n, \tag{13}$$

$$|q(t) - P_1| = |q(t) - P_2|, \tag{14}$$

$$|q(t) - P_1|^2 + |q(t) - P_2|^2 = const, \tag{15}$$

By (14) and (15) we have

$$|q(t) - P_1|^2 = |q(t) - P_2|^2 = \text{const} \tag{16}$$

Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$. Then

$$\begin{aligned} |q(t) - P_1|^2 &= (a_1\cos\frac{2\pi}{T}t + a_2\sin\frac{2\pi}{T}t + \frac{1}{2})^2 \\ &\quad + (b_1\cos\frac{2\pi}{T}t + b_2\sin\frac{2\pi}{T}t)^2 = \text{const} \end{aligned} \tag{17}$$

Let $t = 0$ and $t = T/2$ we have Then

$$(a_1 + \frac{1}{2})^2 + b_1^2 = (-a_1 + \frac{1}{2})^2 + (-b_1)^2 \tag{18}$$

Then

$$\alpha_1 = 0 \tag{19}$$

Let $t = T/4$, $\frac{3}{4}T$, we have

$$\left(\frac{1}{2} + a_2\right)^2 + b_2^2 = \left(\frac{1}{2} - a_2\right)^2 + (-b_2)^2, \tag{20}$$

Hence

$$a_2 = 0 \tag{21}$$

By $a_1 = a_2 = 0$ and (17), we have

$$\left|b_1\cos\frac{2\pi}{T}t + b_2\sin\frac{2\pi}{T}t\right|^2 = \text{const} \tag{22}$$

Let $t = 0$, $\frac{T}{4}$, we have

$$b_1^2 = b_2^2 \tag{23}$$

Hence by (22) and (23) we have

$$b_1^2 + b_1 b_2\sin\frac{4\pi}{T}t = \text{const} \tag{24}$$

$$b_1 = b_2 = 0 \tag{25}$$

So

$$q(t) \equiv 0 \tag{26}$$

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