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J_2-Gravity Perturbed Motion of Artifical Satellite in Terms of Euler Parameters

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Abstract: In this paper the solution of an artificial satellite motion under the influence of the Earth's gravitational field with axial symmetry of any conic section with zonal harmonics J_2 in terms of Euler parameters is established. Applications of this method enable anyone to predict the motion of artificial satellites.

Keywords: Satellite, euler, conic sections, classical newtonian equations.

1. INTRODUCTION

The applications of the special perturbation methods to the equations of motion in terms of the redundant variables, provide the most powerful and accurate techniques that have been devised recently for satellite ephemeris with respect to any type of perturbing forces [1-3].

It is well known that the solutions of the Classical Newtonian Equations of motion are unstable and not suitable for long-term integration. So, many transformations have emerged in the literature in the recent past, for example the set of Eulerain redundant parameters [4, 5]. These parameters combine between orbit dynamics and rigid body dynamics. So, the equations yielded are the equations of motion in terms of the Eulerian parameters including perturbations that can arise from a potential and perturbations that cannot be derived from a potential. Also, they are valid for any type of orbital motion.

2. FORMULATION OF THE PROBLEM

The equations of motion of an artificial satellite are given generally as

$$\ddot{\vec{x}} + \frac{\mu}{r^3}\vec{x} = \vec{P} \tag{2.1}$$

where $\vec{X}$ is the relative position vector, $r = \|\vec{X}\|$, μ is the Earth's gravitational constant,

P is the all perturbing forces, which equals to $\left(-\frac{\partial V}{\partial \vec{x}} + \vec{P}^*\right)$ and hence $\vec{P}^*$ is the resultant of all non-conservative perturbing forces, and V is the perturbed time-independent potential, which can be expressed as

$$V = \mu \sum_{i=2}^{\infty} R^i J_i (1/r)^{i+1} L_i(x_3/r)$$

where R is the Earth's equatorial radius, J_i is the non-dimensional coefficient of the Earth's potential and L_i is the Legendre polynomial of order i.

As mentioned before, the only forces acting on an artificial satellite are those due to the Earth's oblateness. Therefore, we have

$$\vec{P}^* = 0, \quad V = \frac{3}{2} q_2 x_3^2 r^{-5} - \frac{1}{2} q_2 r^{-5} \tag{2.2}$$

where

$q_2 = \mu R^2 J_2$, $\quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Eq.(2.1) are the basic classical equations for the prediction of artificial satellites and the corresponding one in terms of the Euler parameters with $\tilde{\phi}$ as independent variable (perturbed true anomaly) can be written as [6],

$$y_1' = 0.5(\ w_1 y_4 + w_3 y_2), \tag{2.3.1}$$

$$y_2' = 0.5(\ w_1 y_3 - w_3 y_1), \tag{2.3.2}$$

$$y_3' = 0.5(-w_1 y_2 + w_3 y_4), \tag{2.3.3}$$

$$y_4' = 0.5(-w_1 y_1 - w_3 y_3), \tag{2.3.4}$$

$$y_5' = y_6 \tag{2.3.5}$$

$$y_6' = -y_5 + \frac{1}{y_7} - \frac{P_\xi}{\mu y_5^2 y_7} + \frac{2V}{\mu y_5^2 y_7} - \frac{g_2 y_6}{2 y_7}, \tag{2.3.6}$$

$$y_7' = g_2 = \frac{2 w_1 P_\eta^*}{\mu y_5^3} + \frac{2}{y_5^4 \sqrt{\mu^3 y_7}} \frac{\partial V}{\partial t} - \frac{2 y_6}{\mu y_5^3}\left(\left\langle \frac{\partial V}{\partial \vec{x}}, \vec{x} \right\rangle + 2V\right), \tag{2.3.7}$$

$$y_8' = \frac{1}{y_5^2 \sqrt{\mu y_7}}, \tag{2.3.8}$$

where

$$w_1 = \frac{P_\zeta}{\mu w_3 y_5^3 y_7}, \tag{2.4.1}$$

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$$w_3 = \sqrt{1 - \frac{2V}{\mu y_5^2 y_7}}\,, \tag{2.4.2}$$

$$P_\xi = C_{11} P_1 + C_{12} P_2 + C_{13} P_3\,, \tag{2.5.1}$$

$$C_{11} = y_1^2 - y_2^2 - y_3^2 + y_4^2\,, \tag{2.6.1}$$

$$P_\eta^* = C_{21} P_1^* + C_{22} P_2^* + C_{23} P_3^*\,, \tag{2.5.2}$$

$$C_{12} = 2(y_1 y_2 + y_3 y_4)\,, \tag{2.6.2}$$

$$P_\zeta = C_{31} P_1 + C_{32} P_2 + C_{33} P_3\,, \tag{2.5.3}$$

$$C_{13} = 2(y_1 y_3 - y_2 y_4)\,, \tag{2.6.3}$$

$$C_{21} = 2(y_1 y_2 - y_3 y_4)\,, \tag{2.7.1}$$

$$C_{31} = C_{12} C_{23} - C_{13} C_{22}\,, \tag{2.8.1}$$

$$C_{22} = -y_1^2 + y_2^2 - y_3^2 + y_4^2\,, \tag{2.7.2}$$

$$C_{32} = C_{13} C_{21} - C_{11} C_{23}\,, \tag{2.8.2}$$

$$C_{23} = 2(y_2 y_3 + y_1 y_4)\,, \tag{2.7.3}$$

$$C_{33} = C_{11} C_{22} - C_{12} C_{21} \tag{2.8.3}$$

and

$$x_i = r\, C_{1i} = C_{1i} / y_5\,. \tag{2.9}$$

Here $\langle \vec{a}, \vec{b} \rangle$ is used to denote the scalar product of any two vectors $\vec{a}$ & b, and we denote the differentiation with respect to the independent variable $\tilde{\phi}$ by a prime ('), since

$$(') = \frac{d}{d\tilde{\phi}}(\ )\,.$$

3. RECURRENT POWER SERIES SOLUTION

In this section, recurrent power series solution of Eqs. (2.3) will be derived in the three following steps.

1) Rewriting the equations of motion in terms of y's only.

We can rewrite Eq.(2.2) as

$$V = 1.5\, q_2\, C_{13}^2 - 0.5\, q_2\, y_5^3$$

or

$$= 6 q_2 y_1^2 y_3^2 y_5^3 - 12 q_2 y_1 y_2 y_3 y_4 y_5^2 + 6 q_2 y_2^2 y_4^2 y_5^3 - 0.5 q_2 y_5^3\,. \tag{3.1}$$

Also, from Eq.(2.2) we can get

$$\frac{\partial V}{\partial x_i} = -\frac{15}{2} q_2 C_{1i} C_{13}^2 y_5^4 + \frac{3}{2} q_2 C_{1i} y_5^4\,, \quad \text{i} = 1,2 \tag{3.2.1}$$

$$\frac{\partial V}{\partial x_3} = -\frac{15}{2} q_2 C_{13}^3 y_5^4 + \frac{9}{2} q_2 C_{13} y_5^4\,, \tag{3.2.2}$$

then

$$\left(\left\langle \frac{\partial V}{\partial \vec{x}}, \vec{x} \right\rangle + 2V\right) = -\frac{3}{2} q_2 C_{13}^2 y_5^3 + \frac{1}{2} q_2 y_5^3\,. \tag{3.3}$$

But, as mentioned before in Section 2, we have

$$P_\xi = -C_{11}\frac{\partial V}{\partial x_1} - C_{12}\frac{\partial V}{\partial x_2} - C_{13}\frac{\partial V}{\partial x_3} = \frac{9}{2} q_2 C_{13}^2 y_5^4 - \frac{3}{2} q_2 y_5^4\,, \tag{3.4.1}$$

$$P_\eta^* = 0\,, \tag{3.4.2}$$

$$P_\zeta = -C_{31}\frac{\partial V}{\partial x_1} - C_{32}\frac{\partial V}{\partial x_2} - C_{33}\frac{\partial V}{\partial x_3} = -3 q_2 C_{13} C_{33} y_5^4 \tag{3.4.3}$$

Since,

$$(1 \pm \lambda)^{-n} = 1 \mp n\lambda + \frac{n(n+1)}{2!}\lambda^2 \mp \frac{n(n+1)(n+2)}{3!}\lambda^3 + \dots\,,$$

then from Eq.(2.4.2) and using the above equation we get

$$\frac{1}{w_3} = \left[1 - \frac{2V}{\mu y_5^2 y_7}\right]^{-1/2} = 1 + \frac{V}{\mu y_5^2 y_7}\,,$$

or

$$\begin{aligned} \frac{1}{w_3} = {} & 1 + 6 Q_2 y_1^2 y_3^2 y_5 y_9 - 12 Q_2 y_1 y_2 y_3 y_4 y_5 y_9 + \\ & 6 Q_2 y_2^2 y_4^2 y_5 y_9 - 0.5 Q_2 y_5 y_9\,, \end{aligned} \tag{3.5}$$

where

$$Q_2 = q_2 / \mu \tag{3.6}$$

$$y_9 = 1 / y_7 \tag{3.7}$$

then we can rewrite Eq.(2.4.1) by using Eq.(3.4.3) and Eq.(3.5) as

$$\begin{aligned} w_1 = {} & 6 Q_2 y_1^5 y_3 y_5 y_9 - 6 Q_2 y_1^4 y_2 y_4 y_5 y_9 + 12 Q_2 y_1^3 y_2^2 y_3 y_5 y_9 - \\ & 12 Q_2 y_1^2 y_2^3 y_4 y_5 y_9 + 6 Q_2 y_1 y_2^4 y_3 y_5 y_9 - 6 Q_2 y_2^5 y_4 y_5 y_9 - \\ & 6 Q_2 y_1 y_3^5 y_5 y_9 + 6 Q_2 y_2 y_3^4 y_4 y_5 y_9 - 12 Q_2 y_1 y_3^3 y_4^2 y_5 y_9 + \\ & 12 Q_2 y_2 y_3^2 y_4^3 y_5 y_9 - 6 Q_2 y_1 y_3 y_4^4 y_5 y_9 + 6 Q_2 y_2 y_4^5 y_5 y_9 + \\ & 36 Q_2^2 y_1^7 y_3^3 y_5^2 y_9^2 - 108 Q_2^2 y_1^6 y_2 y_3^2 y_4 y_5^2 y_9^2 + \\ & 72 Q_2^2 y_1^5 y_2^2 y_3^3 y_5^2 y_9^2 - 216 Q_2^2 y_1^4 y_2^3 y_3^2 y_4 y_5^2 y_9^2 + \\ & 36 Q_2^2 y_1^3 y_2^4 y_3^3 y_5^2 y_9^2 - 108 Q_2^2 y_1^2 y_2^5 y_3^2 y_4 y_5^2 y_9^2 - \\ & 36 Q_2^2 y_1^3 y_3^7 y_5^2 y_9^2 + 108 Q_2^2 y_1^2 y_2^5 y_3^2 y_4 y_5^2 y_9^2 - \\ & 72 Q_2^2 y_1^3 y_3^5 y_4^2 y_5^2 y_9^2 + 216 Q_2^2 y_1^2 y_2 y_3^4 y_4^3 y_5^2 y_9^2 - \\ & 36 Q_2^2 y_1^3 y_3^3 y_4^4 y_5^2 y_9^2 + 108 Q_2^2 y_1^2 y_2 y_3^2 y_4^5 y_5^2 y_9^2 + \\ & 108 Q_2^2 y_1^5 y_2^2 y_3 y_4^2 y_5^2 y_9^2 + 216 Q_2^2 y_1^3 y_2^4 y_3 y_4^2 y_5^2 y_9^2 + \\ & 108 Q_2^2 y_1 y_2^6 y_3 y_4^2 y_5^2 y_9^2 - 108 Q_2^2 y_1 y_2^2 y_3^5 y_4^2 y_5^2 y_9^2 - \\ & 216 Q_2^2 y_1 y_2^2 y_3^3 y_4^4 y_5^2 y_9^2 - 108 Q_2^2 y_1 y_2^2 y_3 y_4^6 y_5^2 y_9^2 - \\ & 36 Q_2^2 y_1^4 y_2^3 y_4^3 y_5^2 y_9^2 - 72 Q_2^2 y_1^2 y_2^5 y_4^3 y_5^2 y_9^2 - \\ & 36\, Q_2^2 y_2^7 y_4^3 y_5^2 y_9^2 + 36 Q_2^2 y_2^3 y_3^4 y_4^3 y_5^2 y_9^2 + \\ & 72 Q_2^2 y_2^3 y_3^2 y_4^5 y_5^2 y_9^2 + 36\, Q_2^2 y_2^3 y_4^7 y_5^2 y_9^2 - \\ & 3 Q_2^2 y_1^5 y_3 y_5^2 y_9^2 + 3 Q_2^2 y_1^4 y_2 y_4 y_5^2 y_9^2 - \\ & 6 Q_2^2 y_1^3 y_2^2 y_3 y_5^2 y_9^2 + 6 Q_2^2 y_1^2 y_2^3 y_4 y_5^2 y_9^2 - \\ & 3 Q_2^2 y_1 y_2^4 y_3 y_5^2 y_9^2 + 3 Q_2^2 y_2^5 y_4 y_5^2 y_9^2 + \\ & 3 Q_2^2 y_1 y_3^5 y_5^2 y_9^2 - 3 Q_2^2 y_2 y_3^4 y_4 y_5^2 y_9^2 + \\ & 6 Q_2^2 y_1 y_3^3 y_4^2 y_5^2 y_9^2 - 6 Q_2^2 y_2 y_3^2 y_4^3 y_5^2 y_9^2 + \\ & 3 Q_2^2 y_1 y_3 y_4^4 y_5^2 y_9^2 - 3 Q_2^2 y_2 y_4^5 y_5^2 y_9^2\,, \end{aligned} \tag{3.8}$$

where

$$Q_2^2 = (q_2/\mu)^2 \tag{3.9}$$

$$y_9^2 = (1/y_7)^2 . \tag{3.10}$$

Since, $(1 \pm \lambda)^n = 1 \pm n\lambda + \frac{n(n-1)}{2!}\lambda^2 \pm \frac{n(n-1)(n-2)}{3!}\lambda^3 + \ldots$,

then from Eq.(2.4.2) and using the above equation we get

$$w_3 = \left[1 - \frac{2V}{\mu y_5^2 y_7}\right]^{1/2} = 1 - \frac{V}{\mu y_5^2 y_7},$$

then

$$w_3 = 1 - 6Q_2 y_1^2 y_3^2 y_5 y_9 + 12 Q_2 y_1 y_2 y_3 y_4 y_5 y_9 - 6Q_2 y_2^2 y_4^2 y_5 y_9 + \frac{1}{2} Q_2 y_5 y_9 . \tag{3.11}$$

Now, from Eq.(3.8) and Eq.(3.11) we can rewrite Eqs.(2.3.1), (2.3.2), (2.3.3) and (2.3.4) as

$$\begin{aligned}
y_1' = {} & 3Q_2 y_1^5 y_3 y_4 y_5 y_9 - 3Q_2 y_1^4 y_2 y_4^2 y_5 y_9 + \\
& 6Q_2 y_1^3 y_2^2 y_3 y_4 y_5 y_9 - 6Q_2 y_1^2 y_2^3 y_4^2 y_5 y_9 + \\
& 3Q_2 y_1 y_2^4 y_3 y_4 y_5 y_9 - 3Q_2 y_2^5 y_4^2 y_5 y_9 - \\
& 3Q_2 y_1 y_3^5 y_4 y_5 y_9 + 3Q_2 y_2 y_3^4 y_4^2 y_5 y_9 - \\
& 6Q_2 y_1 y_3^3 y_4^3 y_5 y_9 + 6Q_2 y_2 y_3^2 y_4^4 y_5 y_9 - \\
& 3Q_2 y_1 y_3 y_4^5 y_5 y_9 + 3Q_2 y_2 y_4^6 y_5 y_9 - \\
& 3Q_2 y_1^2 y_2 y_3^2 y_5 y_9 + 6Q_2 y_1 y_2^2 y_3 y_4 y_5 y_9 - \\
& 3Q_2 y_2^3 y_4^2 y_5 y_9 + 0.25 Q_2 y_2 y_5 y_9 + \\
& 18Q_2^2 y_1^7 y_3^3 y_4 y_5^2 y_9^2 - 54Q_2^2 y_1^6 y_2 y_3^2 y_4^2 y_5^2 y_9^2 + \\
& 36Q_2^2 y_1^5 y_2^2 y_3^3 y_4 y_5^2 y_9^2 - 108Q_2^2 y_1^4 y_2^3 y_3^2 y_4^2 y_5^2 y_9^2 + \\
& 18Q_2^2 y_1^3 y_2^4 y_3^3 y_4 y_5^2 y_9^2 - 54Q_2^2 y_1^2 y_2^5 y_3^2 y_4^2 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^3 y_3^7 y_4 y_5^2 y_9^2 + 54Q_2^2 y_1^2 y_2^5 y_3^2 y_4^2 y_5^2 y_9^2 - \\
& 36Q_2^2 y_1^3 y_3^5 y_4^3 y_5^2 y_9^2 + 108Q_2^2 y_1^2 y_2 y_3^4 y_4^4 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^3 y_3^3 y_4^5 y_5^2 y_9^2 + 54Q_2^2 y_1^2 y_2 y_3^2 y_4^6 y_5^2 y_9^2 + \\
& 54Q_2^2 y_1^5 y_2^2 y_3 y_4^3 y_5^2 y_9^2 + 108Q_2^2 y_1^3 y_2^4 y_3 y_4^3 y_5^2 y_9^2 + \\
& 54Q_2^2 y_1 y_2^6 y_3 y_4^3 y_5^2 y_9^2 - 54Q_2^2 y_1 y_2^2 y_3^5 y_4^3 y_5^2 y_9^2 - \\
& 108Q_2^2 y_1 y_2^2 y_3^3 y_4^5 y_5^2 y_9^2 - 54Q_2^2 y_1 y_2^2 y_3 y_4^7 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^4 y_2^3 y_4^4 y_5^2 y_9^2 - 36Q_2^2 y_1^2 y_2^5 y_4^4 y_5^2 y_9^2 - \\
& 18Q_2^2 y_2^7 y_4^4 y_5^2 y_9^2 + 18Q_2^2 y_2^3 y_3^4 y_4^4 y_5^2 y_9^2 + \\
& 36Q_2^2 y_2^3 y_3^2 y_4^6 y_5^2 y_9^2 + 18 Q_2^2 y_2^3 y_4^8 y_5^2 y_9^2 - \\
& 1.5Q_2^2 y_1^5 y_3 y_4 y_5^2 y_9^2 + 1.5Q_2^2 y_1^4 y_2 y_4^2 y_5^2 y_9^2 - \\
& 3Q_2^2 y_1^3 y_2^2 y_3 y_4 y_5^2 y_9^2 + 3Q_2^2 y_1^2 y_2^3 y_4^2 y_5^2 y_9^2 - \\
& 1.5Q_2^2 y_1 y_2^4 y_3 y_4 y_5^2 y_9^2 + 1.5Q_2^2 y_2^5 y_4^2 y_5^2 y_9^2 + \\
& 1.5Q_2^2 y_1 y_3^5 y_4 y_5^2 y_9^2 - 1.5Q_2^2 y_2 y_3^4 y_4^2 y_5^2 y_9^2 + \\
& 3Q_2^2 y_1 y_3^3 y_4^3 y_5^2 y_9^2 - 3Q_2^2 y_2 y_3^2 y_4^4 y_5^2 y_9^2 + \\
& 1.5Q_2^2 y_1 y_3 y_4^5 y_5^2 y_9^2 - 1.5Q_2^2 y_2 y_4^6 y_5^2 y_9^2 + 0.5 y_2 ,
\end{aligned} \tag{3.12.1}$$

$$\begin{aligned}
y_2' = {} & 3Q_2 y_1^5 y_3^2 y_5 y_9 - 3Q_2 y_1^4 y_2 y_3 y_4 y_5 y_9 + \\
& 6Q_2 y_1^3 y_2^2 y_3^2 y_5 y_9 - 6Q_2 y_1^2 y_2^3 y_3 y_4 y_5 y_9 + \\
& 3Q_2 y_1 y_2^4 y_3^2 y_5 y_9 - 3Q_2 y_2^5 y_3 y_4 y_5 y_9 - \\
& 3Q_2 y_1 y_3^6 y_5 y_9 + 3Q_2 y_2 y_3^5 y_4 y_5 y_9 - \\
& 6Q_2 y_1 y_3^4 y_4^2 y_5 y_9 + 6Q_2 y_2 y_3^3 y_4^3 y_5 y_9 - \\
& 3Q_2 y_1 y_3^2 y_4^4 y_5 y_9 + 3Q_2 y_2 y_3 y_4^5 y_5 y_9 + \\
& 3Q_2 y_1^3 y_3^2 y_5 y_9 - 6Q_2 y_1^2 y_2 y_3 y_4 y_5 y_9 + \\
& 3Q_2 y_1 y_2^2 y_4^2 y_5 y_9 - 0.25 Q_2 y_1 y_5 y_9 + \\
& 18Q_2^2 y_1^7 y_3^4 y_5^2 y_9^2 - 54Q_2^2 y_1^6 y_2 y_3^3 y_4 y_5^2 y_9^2 + \\
& 36Q_2^2 y_1^5 y_2^2 y_3^4 y_5^2 y_9^2 - 108Q_2^2 y_1^4 y_2^3 y_3^3 y_4 y_5^2 y_9^2 + \\
& 18Q_2^2 y_1^3 y_2^4 y_3^4 y_5^2 y_9^2 - 54Q_2^2 y_1^2 y_2^5 y_3^3 y_4 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^3 y_3^8 y_5^2 y_9^2 + 54Q_2^2 y_1^2 y_2 y_3^7 y_4 y_5^2 y_9^2 - \\
& 36Q_2^2 y_1^3 y_3^6 y_4^2 y_5^2 y_9^2 + 108Q_2^2 y_1^2 y_2 y_3^5 y_4^3 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^3 y_3^4 y_4^4 y_5^2 y_9^2 + 54Q_2^2 y_1^2 y_2 y_3^3 y_4^5 y_5^2 y_9^2 + \\
& 54Q_2^2 y_1^5 y_2^2 y_3^2 y_4^2 y_5^2 y_9^2 + 108Q_2^2 y_1^3 y_2^4 y_3^2 y_4^2 y_5^2 y_9^2 + \\
& 54Q_2^2 y_1 y_2^6 y_3^2 y_4^2 y_5^2 y_9^2 - 54Q_2^2 y_1 y_2^2 y_3^6 y_4^2 y_5^2 y_9^2 - \\
& 108Q_2^2 y_1 y_2^2 y_3^4 y_4^4 y_5^2 y_9^2 - 54Q_2^2 y_1 y_2^2 y_3^2 y_4^6 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^4 y_2^3 y_3 y_4^3 y_5^2 y_9^2 - 36Q_2^2 y_1^2 y_2^5 y_3 y_4^3 y_5^2 y_9^2 - \\
& 18Q_2^2 y_2^7 y_3 y_4^3 y_5^2 y_9^2 + 18Q_2^2 y_2^3 y_3^5 y_4^3 y_5^2 y_9^2 + \\
& 36Q_2^2 y_2^3 y_3^3 y_4^5 y_5^2 y_9^2 + 18 Q_2^2 y_2^3 y_3 y_4^7 y_5^2 y_9^2 - \\
& 1.5Q_2^2 y_1^5 y_3^2 y_5^2 y_9^2 + 1.5Q_2^2 y_1^4 y_2 y_3 y_4 y_5^2 y_9^2 - \\
& 3Q_2^2 y_1^3 y_2^2 y_3^2 y_5^2 y_9^2 + 3Q_2^2 y_1^2 y_2^3 y_3 y_4 y_5^2 y_9^2 - \\
& 1.5Q_2^2 y_1 y_2^4 y_3^2 y_5^2 y_9^2 + 1.5Q_2^2 y_2^5 y_3 y_4 y_5^2 y_9^2 + \\
& 1.5Q_2^2 y_1 y_3^6 y_5^2 y_9^2 - 1.5Q_2^2 y_2 y_3^5 y_4 y_5^2 y_9^2 + \\
& 3Q_2^2 y_1 y_3^4 y_4^2 y_5^2 y_9^2 - 3Q_2^2 y_2 y_3^3 y_4^3 y_5^2 y_9^2 + \\
& 1.5Q_2^2 y_1 y_3^2 y_4^4 y_5^2 y_9^2 - 1.5Q_2^2 y_2 y_3 y_4^5 y_5^2 y_9^2 - 0.5 y_1 ,
\end{aligned} \tag{3.12.2}$$

$$\begin{aligned}
y_3' = {} & -3Q_2 y_1^5 y_2 y_3 y_5 y_9 + 3Q_2 y_1^4 y_2^2 y_4 y_5 y_9 - \\
& 6Q_2 y_1^3 y_2^3 y_3 y_5 y_9 + 6Q_2 y_1^2 y_2^4 y_4 y_5 y_9 - \\
& 3Q_2 y_1 y_2^5 y_3 y_5 y_9 + 3Q_2 y_2^6 y_4 y_5 y_9 + \\
& 3Q_2 y_1 y_2 y_3^5 y_5 y_9 - 3Q_2 y_2^2 y_3^4 y_4 y_5 y_9 + \\
& 6Q_2 y_1 y_2 y_3^3 y_4^2 y_5 y_9 - 6Q_2 y_2^2 y_3^2 y_4^3 y_5 y_9 + \\
& 3Q_2 y_1 y_2 y_3 y_4^4 y_5 y_9 - 3Q_2 y_2^2 y_4^5 y_5 y_9 - \\
& 3Q_2 y_1^2 y_3^2 y_4 y_5 y_9 + 6Q_2 y_1 y_2 y_3 y_4^2 y_5 y_9 - \\
& 3Q_2 y_2^2 y_4^3 y_5 y_9 + 0.25 Q_2 y_4 y_5 y_9 - \\
& 18Q_2^2 y_1^7 y_2 y_3^3 y_5^2 y_9^2 + 54Q_2^2 y_1^6 y_2^2 y_3^2 y_4 y_5^2 y_9^2 - \\
& 36Q_2^2 y_1^5 y_2^3 y_3^3 y_5^2 y_9^2 + 108Q_2^2 y_1^4 y_2^4 y_3^2 y_4 y_5^2 y_9^2 - \\
& 18Q_2^2 y_1^3 y_2^5 y_3^3 y_5^2 y_9^2 + 54Q_2^2 y_1^2 y_2^6 y_3^2 y_4 y_5^2 y_9^2 +
\end{aligned} \tag{3.12.3}$$

$$
\begin{aligned}
&18Q_2^2\,y_1^3\,y_2\,y_3^7\,y_5^2\,y_9^2-54Q_2^2\,y_1^2\,y_2^2\,y_3^6\,y_4\,y_5^2\,y_9^2+\\
&36Q_2^2\,y_1^3\,y_2\,y_3^5\,y_4^2\,y_5^2\,y_9^2-108Q_2^2\,y_1^2\,y_2^2\,y_3^4\,y_4^3\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1^3\,y_2\,y_3^3\,y_4^4\,y_5^2\,y_9^2-54Q_2^2\,y_1^2\,y_2^2\,y_3^2\,y_4^5\,y_5^2\,y_9^2-\\
&54Q_2^2\,y_1^5\,y_2^3\,y_3\,y_4^2\,y_5^2\,y_9^2-108Q_2^2\,y_1^3\,y_2^5\,y_3\,y_4^2\,y_5^2\,y_9^2-\\
&54Q_2^2\,y_1\,y_2^7\,y_3\,y_4^2\,y_5^2\,y_9^2+54Q_2^2\,y_1\,y_2^3\,y_3^5\,y_4^2\,y_5^2\,y_9^2+\\
&108Q_2^2\,y_1\,y_2^3\,y_3^3\,y_4^4\,y_5^2\,y_9^2+54Q_2^2\,y_1\,y_2^3\,y_3\,y_4^6\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1^4\,y_2^4\,y_4^3\,y_5^2\,y_9^2+36Q_2^2\,y_1^2\,y_2^6\,y_4^3\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_2^8\,y_4^3\,y_5^2\,y_9^2-18Q_2^2\,y_2^4\,y_3^4\,y_4^3\,y_5^2\,y_9^2-\\
&36Q_2^2\,y_2^4\,y_3^2\,y_4^5\,y_5^2\,y_9^2-18Q_2^2\,y_2^4\,y_4^7\,y_5^2\,y_9^2+\\
&1.5Q_2^2\,y_1^5\,y_2\,y_3\,y_5^2\,y_9^2-1.5Q_2^2\,y_1^4\,y_2^2\,y_4\,y_5^2\,y_9^2+\\
&3Q_2^2\,y_1^3\,y_2^3\,y_3\,y_5^2\,y_9^2-3Q_2^2\,y_1^2\,y_2^4\,y_4\,y_5^2\,y_9^2+\\
&1.5Q_2^2\,y_1\,y_2^5\,y_3\,y_5^2\,y_9^2-1.5Q_2^2\,y_2^6\,y_4\,y_5^2\,y_9^2-\\
&1.5Q_2^2\,y_1\,y_2\,y_3^5\,y_5^2\,y_9^2+1.5Q_2^2\,y_2^2\,y_3^4\,y_4\,y_5^2\,y_9^2-\\
&3Q_2^2\,y_1\,y_2\,y_3^3\,y_4^2\,y_5^2\,y_9^2+3Q_2^2\,y_2^2\,y_3^2\,y_4^3\,y_5^2\,y_9^2-\\
&1.5Q_2^2\,y_1\,y_2\,y_3\,y_4^4\,y_5^2\,y_9^2+1.5Q_2^2\,y_2^2\,y_4^5\,y_5^2\,y_9^2+0.5y_4\,,
\end{aligned}
$$

$$
\begin{aligned}
y_4' =& -3Q_2\,y_1^6\,y_3\,y_5\,y_9+3Q_2\,y_1^5\,y_2\,y_4\,y_5\,y_9-\\
&6Q_2\,y_1^4\,y_2^2\,y_3\,y_5\,y_9+6Q_2\,y_1^3\,y_2^3\,y_4\,y_5\,y_9-\\
&3Q_2\,y_1^2\,y_2^4\,y_3\,y_5\,y_9+3Q_2\,y_1\,y_2^5\,y_4\,y_5\,y_9+\\
&3Q_2\,y_1^2\,y_3^5\,y_5\,y_9-3Q_2\,y_1\,y_2\,y_3^4\,y_4\,y_5\,y_9+\\
&6Q_2\,y_1^2\,y_3^3\,y_4^2\,y_5\,y_9-6Q_2\,y_1\,y_2\,y_3^2\,y_4^3\,y_5\,y_9+\\
&3Q_2\,y_1^2\,y_3\,y_4^4\,y_5\,y_9-3Q_2\,y_1\,y_2\,y_4^5\,y_5\,y_9+\\
&3Q_2\,y_1^2\,y_3^3\,y_5\,y_9-6Q_2\,y_1\,y_2\,y_3^2\,y_4\,y_5\,y_9+\\
&3Q_2\,y_2^2\,y_3\,y_4^2\,y_5\,y_9-0.25Q_2\,y_3\,y_5\,y_9-\\
&18Q_2^2\,y_1^8\,y_3^3\,y_5^2\,y_9^2+54Q_2^2\,y_1^7\,y_2\,y_3^2\,y_4\,y_5^2\,y_9^2-\\
&36Q_2^2\,y_1^6\,y_2^2\,y_3^3\,y_5^2\,y_9^2+108Q_2^2\,y_1^5\,y_2^3\,y_3^2\,y_4\,y_5^2\,y_9^2-\\
&18Q_2^2\,y_1^4\,y_2^4\,y_3^3\,y_5^2\,y_9^2+54Q_2^2\,y_1^3\,y_2^5\,y_3^2\,y_4\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1^4\,y_3^7\,y_5^2\,y_9^2-54Q_2^2\,y_1^3\,y_2\,y_3^6\,y_4\,y_5^2\,y_9^2+\\
&36Q_2^2\,y_1^4\,y_3^5\,y_4^2\,y_5^2\,y_9^2-108Q_2^2\,y_1^3\,y_2\,y_3^4\,y_4^3\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1^4\,y_3^3\,y_4^4\,y_5^2\,y_9^2-54Q_2^2\,y_1^3\,y_2\,y_3^2\,y_4^5\,y_5^2\,y_9^2-\\
&54Q_2^2\,y_1^6\,y_2^2\,y_3\,y_4^2\,y_5^2\,y_9^2-108Q_2^2\,y_1^4\,y_2^4\,y_3\,y_4^2\,y_5^2\,y_9^2-\\
&54Q_2^2\,y_1^2\,y_2^6\,y_3\,y_4^2\,y_5^2\,y_9^2+54Q_2^2\,y_1^2\,y_2^2\,y_3^5\,y_4^2\,y_5^2\,y_9^2+\\
&108Q_2^2\,y_1^2\,y_2^2\,y_3^3\,y_4^4\,y_5^2\,y_9^2+54Q_2^2\,y_1^2\,y_2^2\,y_3\,y_4^6\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1^5\,y_2^3\,y_4^3\,y_5^2\,y_9^2+36Q_2^2\,y_1^3\,y_2^5\,y_4^3\,y_5^2\,y_9^2+\\
&18Q_2^2\,y_1\,y_2^7\,y_4^3\,y_5^2\,y_9^2-18Q_2^2\,y_1\,y_2^3\,y_3^4\,y_4^3\,y_5^2\,y_9^2-\\
&36Q_2^2\,y_1\,y_2^3\,y_3^2\,y_4^5\,y_5^2\,y_9^2-18Q_2^2\,y_1\,y_2^3\,y_4^7\,y_5^2\,y_9^2+\\
&1.5Q_2^2\,y_1^6\,y_3\,y_5^2\,y_9^2-1.5Q_2^2\,y_1^5\,y_2\,y_4\,y_5^2\,y_9^2+\\
&3Q_2^2\,y_1^4\,y_2^2\,y_3\,y_5^2\,y_9^2-3Q_2^2\,y_1^3\,y_2^3\,y_4\,y_5^2\,y_9^2+\\
&1.5Q_2^2\,y_1^2\,y_2^4\,y_3\,y_5^2\,y_9^2-1.5Q_2^2\,y_1\,y_2^5\,y_4\,y_5^2\,y_9^2-\\
&1.5Q_2^2\,y_1^2\,y_3^5\,y_5^2\,y_9^2+1.5Q_2^2\,y_1\,y_2\,y_3^4\,y_4\,y_5^2\,y_9^2-\\
&3Q_2^2\,y_1^2\,y_3^3\,y_4^2\,y_5^2\,y_9^2+3Q_2^2\,y_1\,y_2\,y_3^2\,y_4^3\,y_5^2\,y_9^2-\\
&1.5Q_2^2\,y_1^2\,y_3\,y_4^4\,y_5^2\,y_9^2+1.5Q_2^2\,y_1\,y_2\,y_4^5\,y_5^2\,y_9^2-0.5y_3\,,
\end{aligned}
\tag{3.12.4}
$$

Also, from Eq.(3.8) and Eqs.(3.1, 3.3 and 3.4) we can rewrite Eqs.(2.3.5, 2.3.6, 2.3.7 and 2.3.8) as

$$y_5' = y_6 \tag{3.12.5}$$

$$
\begin{aligned}
y_6' =& -y_5+y_9-6Q_2\,y_1^2\,y_3^2\,y_5^2\,y_9+12Q_2\,y_1\,y_2\,y_3\,y_4\,y_5^2\,y_9-\\
&6Q_2\,y_2^2\,y_4^2\,y_5^2\,y_9+0.5Q_2\,y_5^2\,y_9-6Q_2\,y_1^2\,y_3^2\,y_6^2\,y_9+\\
&12Q_2\,y_1\,y_2\,y_3\,y_4\,y_6^2\,y_9-6Q_2\,y_2^2\,y_4^2\,y_6^2\,y_9+0.5Q_2\,y_6^2\,y_9\,,
\end{aligned}
\tag{3.12.6}
$$

$$y_7' = 12Q_2\,y_1^2\,y_3^2\,y_6-24Q_2\,y_1\,y_2\,y_3\,y_4\,y_6+12Q_2\,y_2^2\,y_4^2\,y_6-Q_2\,y_6\,, \tag{3.12.7}$$

$$y_8' = \mu^{-1/2}\,y_{10}\,r_2\,, \tag{3.12.8}$$

where

$$y_{10} = 1/y_7^{1/2} \tag{3.13}$$

$$r_2 = r^2 \tag{3.14}$$

2) Substituting with new variables to reduce the order of Eqs.(3.12). So, we let

$y_{ii} = y_i\,y_i$, i = 1(1)4; $y_{i9} = y_i\,y_9$, i = 5, 6; $t_{11} = y_{59}\,y_{59}$,

$e_i = y_i\,y_{59}$, i = 1(1)5; $f_{ij} = y_i\,y_j$, i =1,2, j =3,4; $r_3 = y_{10}\,r_2$;

$e_6 = y_6\,y_{69}$; $u_i = y_{i-2,i}\,y_{59}$, i = 3,4;

$t_i = f_{13} - 3\,f_{24}$, i = 1(1)3; $t_4 = 3\,f_{13} - f_{24}$; $z_{ii} = y_{ii}\,y_{ii}$, i = 1(1)4;

$d_1 = f_{13}\,f_{13}$; $d_2 = f_{13}\,f_{24}$; $d_3 = f_{24}\,f_{24}$;

$d_{1i} = y_i\,t_{11}$, i = 1(1)4; $b_i = y_i\,f_{24}$, i = 1(1)4;

also,

$z_1 = z_{11} + 2\,z_{12} + z_{22} - z_{33} - 2\,z_{34} - z_{44}$; $d_4 = 6\,d_1 - 12\,d_2 + 6\,d_3 - 0.5$;

$z_2 = 1.5\,z_{11} + 3\,z_{12} + 1.5\,z_{22} - 1.5\,z_{33} - 3\,z_{34} - 1.5\,z_{44}$;

$w_i = b_i\,u_4$, i = 1(1)4; $a_i = y_i\,z_1$, i = 1(1)4; $u_2 = u_3\,t_2$;

$u_{2i} = y_i\,u_2$, i = 1(1)5; $t_{13} = z_{13}\,t_3$; $t_{24} = z_{24}\,t_4$;

$t_5 = t_{13} + t_{24}$; $t_{15} = z_1\,t_5$; $t_{12} = t_1\,z_2$;

$t_{18} = 18\,t_5 - t_{12}$; $g = u_3 - u_4$; $g_i = a_i\,g$, i = 1(1)4;

$b_{1i} = d_{1i}\,t_{18}$, i = 1(1)4; $d_7 = y_6\,d_4$; and d_i = ei d_{i-2} , i = 5,6.

Using the previous substitutions in Eqs.(3.12), we get the following first order differential set in eight unknowns

$$y_1' = 3Q_2\,g_4-3Q_2\,u_{22}-3Q_2\,w_2+0.25Q_2\,e_2+Q_2^2\,b_{14}+0.5y_2\,, \tag{3.15.1}$$

$$y_2' = 3Q_2\,g_3+3Q_2\,u_{21}+3Q_2\,w_1-0.25Q_2\,e_1+Q_2^2\,b_{13}-0.5y_1\,, \tag{3.15.2}$$

$$y_3' = -3Q_2\, g_2 - 3Q_2\, u_{24} - 3Q_2\, w_4 + 0.25\, Q_2\, e_4 + Q_2^2\, b_{12} + 0.5\, y_4\,, \quad (3.15.3)$$

$$y_4' = -3Q_2\, g_1 + 3Q_2\, u_{23} + 3Q_2\, w_3 - 0.25\, Q_2\, e_3 - Q_2^2\, b_{11} - 0.5\, y_3\,, \quad (3.15.4)$$

$$y_5' = y_6\,, \quad (3.15.5)$$

$$y_6' = -y_5 + y_9 - 6Q_2\, u_{25} - 6Q_2\, d_5 + 0.5Q_2^2\, e_5 - Q_2\, d_6\,, \quad (3.15.6)$$

$$y_7' = 2Q_2\, d_7\,, \quad (3.15.7)$$

$$y_8' = \mu^{-1/2}\, r_3\,. \quad (3.15.8)$$

Let us define the eight Taylor expansions as follows

$$h_i = \sum_{n=1}^{\infty} H_i^{(n)}\, \tilde{\phi}^{n-1} \quad ;\ \mathrm{i} = 1, 2, \ldots, 8 \quad (3.16)$$

and

$$h_i' = \sum_{n=1}^{\infty} n\, H_i^{(n+1)}\, \tilde{\phi}^{n-1} \quad (3.17)$$

where we have used small letters (h) for the unknown variables and capital letters (H) for the coefficients in their Taylor series expansion.

Now, let us define the product of two infinite power series. If a and b are two infinite power series such that

$$a = \sum_{i=1}^{\infty} A^{(i)}\, S^{i-1}\ ,\quad b = \sum_{i=1}^{\infty} B^{(i)}\, S^{i-1}\ .$$

Then, it is easy to show that

$$c = a\,b = \sum_{n=1}^{\infty} C^{(n)}\, S^{n-1}\ ,$$

where

$$C^{(n)} = \sum_{i=1}^{n} A^{(i)}\, B^{(n-i+1)}\ .$$

Substituting Eqs.(3.16) and (3.17) into Eqs.(3.15), and using the rule for the product of two power series. Equating coefficients of equal powers of $\tilde{\phi}$ in both sides of each of the resulting equations, we get the coefficients of the following recurrence formulae:

$$n\,Y_1^{n+1} = 3Q_2\, G_4 - 3Q_2\, U_{22} - 3Q_2\, W_2 + 0.25\, Q_2\, E_2 + Q_2^2\, B_{14} + 0.5\, Y_2\,, \quad (3.18.1)$$

$$n\,Y_2^{n+1} = 3Q_2\, G_3 + 3Q_2\, U_{21} + 3Q_2\, W_1 - 0.25\, Q_2\, E_1 + Q_2^2\, B_{13} - 0.5\, Y_1\,, \quad (3.18.2)$$

$$n\,Y_3^{n+1} = -3Q_2\, G_2 - 3Q_2\, U_{24} - 3Q_2\, W_4 + 0.25\, Q_2\, E_4 + Q_2^2\, B_{12} + 0.5\, Y_4\,, \quad (3.18.3)$$

$$n\,Y_4^{n+1} = -3Q_2\, G_1 + 3Q_2\, U_{23} + 3Q_2\, W_3 - 0.25\, Q_2\, E_3 - Q_2^2\, B_{11} - 0.5\, Y_3\,, \quad (3.18.4)$$

$$n\,Y_5^{n+1} = Y_6\,, \quad (3.18.5)$$

$$n\,Y_6^{n+1} = -Y_5 + Y_9 - 6Q_2\, U_{25} - 6Q_2\, D_5 + 0.5Q_2^2\, E_5 - Q_2\, D_6\,, \quad (3.18.6)$$

$$n\,Y_7^{n+1} = 2Q_2\, D_7\,, \quad (3.18.7)$$

$$n\,Y_8^{n+1} = \mu^{-1/2}\, R_3\,, \quad (3.18.8)$$

where

$$Y_{ii} = \sum_{i=1}^{n} Y_i^{(i)}\, Y_i^{(n-i+1)}\,,\ i = 1(1)4;\quad Y_{i9} = \sum_{i=1}^{n} Y_i^{(i)}\, Y_9^{(n-i+1)}\,,\ i = 5, 6;$$

$$T_{11} = \sum_{i=1}^{n} Y_{59}^{(i)}\, Y_{59}^{(n-i+1)};\quad E_i = \sum_{i=1}^{n} Y_i^{(i)}\, Y_{59}^{(n-i+1)}\,,\ i = 1(1)5;$$

$$F_{ij} = \sum_{i=1}^{n} Y_i^{(i)}\, Y_j^{(n-i+1)}\,,\ i = 1, 2,\ j = 3, 4;\qquad R_3 = \sum_{i=1}^{n} Y_{10}^{(i)}\, Y_2^{(n-i+1)};$$

$$E_6 = \sum_{i=1}^{n} Y_6^{(i)}\, Y_{69}^{(n-i+1)};\quad T_i = F_{13} - 3F_{24}\,,\ i = 1(1)3;$$

$$T_4 = 3F_{13} - F_{24};\quad Z_{ii} = \sum_{i=1}^{n} Y_{ii}^{(i)}\, Y_{ii}^{(n-i+1)}\,,\ i = 1(1)4;$$

$$D_1 = \sum_{i=1}^{n} F_{13}^{(i)}\, F_{13}^{(n-i+1)};\quad D_2 = \sum_{i=1}^{n} F_{13}^{(i)}\, F_{24}^{(n-i+1)};$$

$$D_3 = \sum_{i=1}^{n} F_{24}^{(i)}\, F_{24}^{(n-i+1)};\quad D_{1i} = \sum_{i=1}^{n} Y_i^{(i)}\, T_{11}^{(n-i+1)}\,,\ i = 1(1)4;$$

$$B_i = \sum_{i=1}^{n} Y_i^{(i)}\, Y_{24}^{(n-i+1)}\,,\ i = 1(1)4;\quad Z_1 = Z_{11} + 2Z_{12} + Z_{22} - Z_{33} - 2Z_{34} - Z_{44};$$

$$D_4 = 6\,D_1 - 12\,D_2 + 6\,D_3 - 0.5;$$

$$Z_2 = 1.5\,Z_{11} + 3Z_{12} + 1.5\,Z_{22} - 1.5\,Z_{33} - 3Z_{34} - 1.5\,Z_{44};$$

$$W_i = \sum_{i=1}^{n} B_i^{(i)}\, U_4^{(n-i+1)}\,,\ i = 1(1)4;\quad A_i = \sum_{i=1}^{n} Y_i^{(i)}\, Z_1^{(n-i+1)}\,,\ i = 1(1)4;$$

$$U_2 = \sum_{i=1}^{n} U_3^{(i)}\, T_2^{(n-i+1)};\quad U_{2i} = \sum_{i=1}^{n} Y_i^{(i)}\, U_2^{(n-i+1)}\,,\ i = 1(1)5;$$

$$T_{13} = \sum_{i=1}^{n} Z_{13}^{(i)}\, T_3^{(n-i+1)};\quad T_{24} = \sum_{i=1}^{n} Z_{24}^{(i)}\, T_4^{(n-i+1)};$$

$$T_5 = T_{13} + T_{24};\quad T_{15} = \sum_{i=1}^{n} Z_1^{(i)}\, T_5^{(n-i+1)};$$

$$T_{12} = \sum_{i=1}^{n} T_1^{(i)}\, Z_2^{(n-i+1)};\quad T_{18} = 18\,T_5 - T_{12};\quad G = U_3 - U_4;$$

$$G_i = \sum_{i=1}^{n} A_i^{(i)} G^{(n-i+1)},\ i = 1(1)4;\ B_{1i} = \sum_{i=1}^{n} D_{1i}^{(i)} T_{18}^{(n-i+1)},\ i = 1(1)4;$$

$$D_7 = \sum_{i=1}^{n} Y_6^{(i)} D_4^{(n-i+1)};\ \text{and}\ D_i = \sum_{i=1}^{n} E_i^{(i)} D_{i-2}^{(n-i+1)},\ i = 5, 6.$$

4. COMPUTATIONAL DEVELOPMENTS

In this section, the computational developments of the formulations of Section 3 are considered.

4.1. Computation Of The Initial Values

Knowing the position and velocity vectors $\vec{x}_0$ and $\vec{x}_0^{\bullet}$ at the instant time (t = 0), we can obtain the initial values of the y's as follows:

1) $r_0 = \sqrt{x_{01}^2 + x_{02}^2 + x_{03}^2}$,

2) $y_5 = 1/r_0$

3) $\dot{r}_0 = y_5 (x_{01}\dot{x}_{01} + x_{02}\dot{x}_{02} + x_{03}\dot{x}_{03})$,

4) $V_0 = 1.5\mu R^2 J_2 y_5^5 x_{03} - 0.5\mu R^2 J_2 y_5^3$,

5) $y_7 = (\dot{x}_{01} + \dot{x}_{02} + \dot{x}_{03} - \dot{r}_0^2 + 2V_0)/(\mu y_5^2)$,

6) $y_6 = -\dot{r}_0 / \sqrt{\mu y_7}$,

7) $C_{1i} = -x_{0i} y_5,\ i = 1(1)3$,

8) $p = (\dot{x}_{01} + \dot{x}_{02} + \dot{x}_{03} - \dot{r}_0^2)/(\mu y_5^2)$,

9) $C_{2i} = (\dot{x}_{01}/y_5 - \dot{r}_0 x_{0i})/\sqrt{\mu p}$,

10) $C_{31} = C_{12} C_{23} - C_{13} C_{22}$,

11) $C_{32} = C_{13} C_{21} - C_{11} C_{23}$,

12) $C_{33} = C_{11} C_{22} - C_{12} C_{21}$,

13) $y_4 = 0.5\sqrt{1 + C_{11} + C_{22} + C_{33}}$,

14) $y_1 = (C_{23} - C_{32})/(4y_4)$,

15) $y_2 = (C_{31} - C_{13})/(4y_4)$,

16) $y_3 = (C_{12} - C_{21})/(4y_4)$,

17) $y_8 = t$.

4.2. Computation Of The Step Size

The related equation between the step size Δt of the time t and the step size $\Delta\tilde{\phi}$ of perturbed true anomaly is

$$\Delta\tilde{\phi} = \Delta t\ y_5^2 \sqrt{\mu y_7}.$$

4.3. Computation Of Accuracy Check

The accuracy of the computed values of the y's at any time could be checked using the relation

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1.$$

5. RESULTS

We'll take as the numerical example, the Indian satellite RS-1 at about 300 Km height [7] which remained in its orbit for 371 days. Its initial position and velocity components are

$\vec{X}$ = (1626.742, 6268.094, -1776.018) Km,

$\vec{x}_0^{\bullet}$ = (-5.920522, 0.239214, -5.15883) Km/sec

at epoch 20 July 1980, where its one orbital revolution is elapsed 1.588352085 hrs. Since the adopted physical constant are

R = 6378.135 Km, μ = 398600.8 Km^3/sec^2,

and the Earth's zonal harmonic coefficient J_2 equal 1.0826157×10^{-3}.

Using the above values to compute the position and velocity components, i.e., the six elements, and the accuracy check at any time, we'll obtain the following table. Table **1** shows the variation in the elements (a, e, i, Ω, ω) and the check relation every 10 days for two hundred and fifty days.

From the table we can conclude that, from 0 up to 110 mean solar days, there is obviously decay in the two elements (a, e) but the other elements (i, Ω, ω) are hardly changed. This is expected because the only force affecting the motion of artificial satellites is the Earth's gravitational field. This force slightly affect the elements (i, Ω, ω) where these elements are strongly affected by other forces such as drag, solar radiation pressure, etc.

CONCLUSION

From the table we can conclude that, from 0 up to 110 mean solar days, there is obviously decay in the two elements (a, e) but the other elements (i, Ω, ω) are hardly changed. This is expected because the only force affecting the motion of artificial satellites is the Earth's gravitational field. This force slightly affect the elements (i, Ω, ω) where these elements are strongly affected by other forces such as drag, solar radiation pressure, etc.

Also, the table shows that the accuracy check is always near to zero, i.e., the predictions of the components of position and velocity (the elements) of the artificial satellite are good.

And after 110 mean solar days up to 259, the accuracy check is not good because it is greater than one and consequently there are contradictory results from the elements when compared with the results of [7]. This is also expected because there is only one force affecting the motion of the artificial satellite which is the potential of the Earth's gravitational field with axial symmetry of the zonal harmonics J_2.

To obtain more accurate prediction of the motion of artificial satellite we have to include other forces, like drag force, and the potential of the Earth's gravitational field (with axial symmetry up to the zonal harmonics J_{36}) or more.

Table 1. **The Results.**

Rev. No.	Mean Solar Day	a (Km)	e	i	Ω	ω	Check Relation
1	0	6991.697292	0.044142927	44° 41′ 12″.46	239° 21′ 13″.38	174° 42′ 30″.18	1.0000000
152	10.026	6991.695583	0.044142616	41′ 12″.46	21′ 03″.37	42′ 30″.69	0.999999997
303	20.052	6991.693801	0.044142369	41′ 12″.45	21′ 13″.37	42′ 30″.45	0.999999994
454	30.046	6991.691766	0.044142360	41′ 12″.39	21′ 13″.34	42′ 27″.46	0.999999983
605	40.039	6991.689229	0.044142915	41′ 12″.25	21′ 13″.26	42′ 18″.07	0.999999953
756	50.033	6991.685934	0.044144558	41′ 11″.97	21′ 13″.09	41′ 56″.61	0.999999891
907	60.026	6991.681716	0.044148076	41′ 11″.45	21′ 12″.80	41′ 14″.84	0.999999786
1058	70.019	6991.676619	0.044154597	41′ 10″.60	21′ 12″.32	40′ 01″.33	0.999999627
1209	80.013	6991.671045	0.044165685	41′ 09″.30	21′ 11″.57	38′ 00″.71	0.999999410
1360	90.006	6991.665913	0.044183446	41′ 07″.42	21′ 10″.49	34′ 52″.90	0.999999139
1511	100.000	6991.662861	0.044210651	41′ 04″.80	21′ 08″.99	30′ 12″.30	0.999998829
1662	109.993	6991.664454	0.044250865	41′ 01″.28	21′ 06″.96	23′ 27″.08	0.999998517
1813	119.986	6991.674421	0.044308594	40′ 56″.69	21′ 04″.30	13′ 58″.61	0.999998258
1964	129.980	6991.697919	0.044389434	40′ 50″.83	21′ 00″.90	01′ 01″.23	0.999998136
2115	139.973	6991.741814	0.044500226	40′ 43″.52	20′ 56″.64	173° 43′ 42″.50	0.999998268
2266	149.966	6991.814981	0.044649213	40′ 34″.56	20′ 51″.40	21′ 04″.10	0.999998813
2417	159.960	6991.928626	0.044846183	40′ 23″.79	20′ 45″.08	172° 52′ 03″.71	0.999999974
2568	169.953	6992.096597	0.045102581	40′ 11″.04	20′ 37″.58	15′ 37″.93	1.000002006
2719	179.947	6992.335672	0.045431552	39′ 56″.18	20′ 28″.81	171° 30′ 46″.35	1.000005222
2870	189.940	6992.665761	0.045847881	39′ 39″.11	20′ 18″.71	170° 36′ 36″.85	1.000009995
3021	199.934	6993.109947	0.046367760	39′ 19″.79	20′ 07″.25	169° 32′ 31″.60	1.000016759
3172	209.927	6993.694273	0.047008322	38′ 58″.23	19′ 54″.45	168° 18′ 13″.30	1.000025998
3323	219.920	6994.447165	0.047786910	38′ 34″.53	19′ 40″.37	166° 53′ 50″.49	1.000038236
3474	229.914	6995.398418	0.048720095	38′ 08″.83	19′ 25″.10	165° 20′ 00″.78	1.000054010
3625	239.907	6996.577769	0.049822566	37′ 41″.35	19′ 08″.79	163° 37′ 50″.55	1.000073829
3776	249.900	6998.013240	0.051106166	37′ 12″.34	18′ 51″.61	161° 48′ 50″.36	1.000098132

CONFLICT OF INTEREST

None declared.

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None declared.

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