A Fractional Model for the Allometric Scaling Laws

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Abstract: From the available literature, the allometric scaling laws generally exist in biology, ecology, etc. These scaling laws obey power law distributions. A possibly better approach to characterize the power law is to utilize fractional derivatives. In this paper, we establish a fractional differential equation model for this allometry by using the Caputo fractional derivatives.

Keywords: Biology, Allometric scaling law, Caputo fractional derivative, Fractional differential equation.

INTRODUCTION

From the literature available, the scaling laws of allometry typically appear in biology, ecology, etc [1-14]. For details, the dependance of a biological variable Y upon body mass M is emblematically characterized by an allometric scaling law (namely, power law) of the form

$$Y = Y_0 M^{\alpha} \tag{1}$$

where α is a scaling exponent and Y_0 is a constant that is determined by the kind of organism.

In [1], West *et al.* proposed that a common mechanism underlies following laws: Living things are sustained by the transport of materials through linear networks that branch to supply all parts of the organism. Under this mechanism axiom, they established a quantitative model that explains the origin and ubiquity of quarter-power scaling and also predicts the essential feature of transport systems, such as mammalian blood vessels, bronchial trees, plant vascular systems and insect tracheal tubes. Their model was derived on the basis of three postulates: (i) a space-filling network that branches hierarchically to supply all parts of the three-dimensional body; (ii) body-size invariant terminal units, such as capillaries or leaf petioles; and (iii) minimization of the energy and time required to distribute resources.

The ubiquitous scaling law (1) commonly exists in nature, but for living things, the scaling factors α behave multiplicity of quarter. For example, metabolic rates B of entire organisms scales as $M^{3/4}$, that is, $B \propto M^{3/4}$; rates of cellular metabolism, heartbeat and maximal population growth scale as $M^{-1/4}$; and times of blood circulation, embryonic growth and development, and life-span scale as $M^{1/4}$ [1-14]. The "$3/4$ scaling property" was specifically and detailed reported for mature seed plants [9]. In those papers, a very

interesting fact is shown: the mass of dry leaves M_L scales to the $3/4$ power of the mass of dry root M_R, the mass of dry leaves M_L scales to the $3/4$ power of the mass of dry stem M_S, which follows that $M_S \propto M_R$. The perfect quarter-power scaling law well reflects life phenomena to some extent [15]. Besides, for early plant ontogeny, a different allometric scaling law was presented by [16]. Allometric scaling law is characterized by fractional power, which lies in the fractal world [17]. One of useful tools which are used to disclose fractal is the fractional calculus [18-21]. In this paper, we derive a fractional differential equation model for the allometry of scaling laws by using the Caputo fractional derivative [19, 21-30].

FRACTIONAL EQUATION MODEL

Before deriving a fractional equation model for allometry of scaling laws, we first introduce several fundamental definitions of fractional calculus.

In general, four fractional derivative definitions, i.e., Grünwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, generalized fractional derivative in the sense of generalized functions, and Caputo's fractional derivative, are mostly used [19, 21-24, 31-34]. The former three definitions are often used by pure mathematicians while the last one is adopted by applied scientists since it is more convenient in engineering applications. Here we only discuss Caputo derivative:

$$_C D_{0,t}^q x(t) = J^{m-q} x^{(m)}(t), q > 0, \tag{2}$$

Where $m = \lceil q \rceil$, i.e., m is the first integer which is not less than q, $x^{(m)}$ is a conventional *m*-th order derivative, J^{β} is the β-th order Riemann-Liouville integral operator which is expressed as follows:

$$J^{\beta} x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} x(\tau) d\tau, \beta > 0 \tag{3}$$

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in which $\Gamma(\beta)$ is the usual Euler function, i.e.,

$$\Gamma(\beta) = \int_0^{+\infty} e^{-t}t^{\beta-1}dt.$$

Although these three fractional derivatives with the same order $q > 0$ are not equivalent, the Grünwald-Letnikov fractional integral $_{GL}D_{0,t}^{-q}$, the Riemann-Liouville fractional integral $_{RL}D_{0,t}^{-q}$, and the Caputo's fractional integral $_C D_{0,t}^{-q}$, with order $q > 0$ have the same expression. Obviously, J^β and $_{RL}D_{0,t}^{-\beta}$ are the same. In order to coincide with (2), here and throughout, denote J^β by $_C D_{0,t}^{-\beta}, \beta > 0$.

Caputo derivative (integral) has following properties:

(i) $_C D_{0,t}^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(-\alpha+\gamma+1)}t^{\gamma-\alpha}$

holds for $\alpha < 0, \gamma > -1$ or $0 \le m \le \alpha < m+1, \gamma > m \in Z^+$;

(ii) $_C D_{0,t}^\alpha d = 0$

for any constant d and $\alpha > 0$.

(iii) The Caputo differential operator is a linear operator, i.e.

$$_C D_{0,t}^\alpha (ax+by)(t) = a \cdot _C D_{0,t}^\alpha x(t)$$
$$+ b \cdot _C D_{0,t}^\alpha y(t).$$

for arbitrary constants a and b;

(iv) $_C D_{0,t}^\alpha \cdot _C D_{0,t}^\beta = _C D_{0,t}^\beta \cdot _C D_{0,t}^\alpha = _C D_{0,t}^{\alpha+\beta}$

holds for $\alpha \cdot \beta > 0$ and $0 < \alpha + \beta \le 1$[31], or $\alpha < 0$ and $\beta < 0$.

On the other hand, the fractional differential equation

$$_C D_{0,t}^\alpha y(t) = f(t)$$

with initial value condition

$$y^{(k)}(0) = y_0^k, k = 0,1,\cdots,\lceil\alpha\rceil-1,$$

is equivalent to a Volterra integral equation

$$y(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} y_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau.$$

Next, we establish a fractional model. Here data come from the Appendix of one paper [12, 35] (the web page can be found from the reference of this paper).

Let also B denotes metabolic rates, M total plant biomass; M_L the mass of dry leaves, M_S the mass of dry stems; M_L the mass of dry leaves, M_R the mass of dry roots. Assume still B and M, M_L

and M_S, M_L and M_R satisfy the following relations,

$$B = a_0 M^\alpha, M_L = b_0 M_S^\beta, M_L = c_0 M_R^\gamma$$

By the method of least square, we find that

$$B = 0.0158M^{0.7595}, M_L = 0.9642M_S^{0.7486},$$

$$M_L = 1.4543M_R^{0.7413}, \tag{4}$$

where the node number are 344, 460 and 171 respectively, see Appendix in which $a_1 = 0$. These relations are in line with prediction, $B \propto M^{0.75}, M_L \propto M_S^{0.75}, M_L \propto M_R^{0.75}$.

Their diagrams are in Fig. (**1**), which is almost the same with that in Enquist-Niklas's work [9].
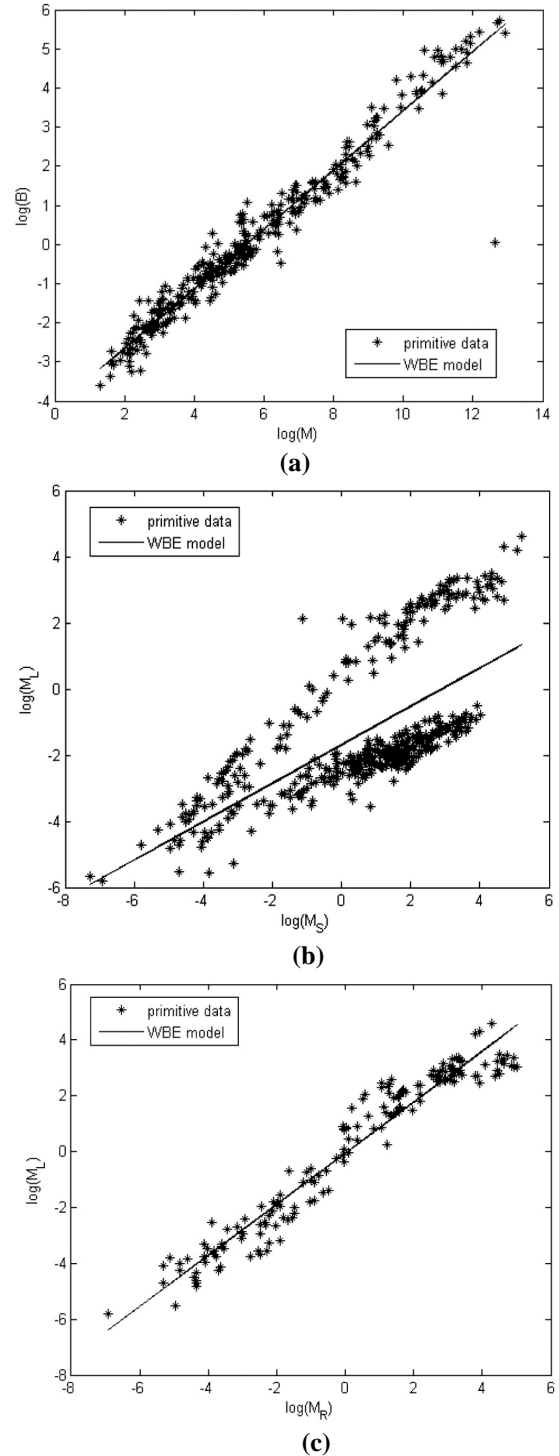

(a)

(b)

(c)

Fig. (1). The log-log linear curve denotes the statistical fit of actual data (original units in kg of dry weight per plant) WBE stands for West-Brown-Enquist model. (**a**) $\log(B)$ versus $\log(M)$; (**b**) $\log(M_L)$ versus $\log(M_S)$; (**c**) $\log(M_L)$ versus $\log(M_R)$.

We also verify a relation between M_S and M_R by the same numerical method,

$$M_S \propto M_R,$$

in which the node number is 171.

It is easy to see that three equations of (4) satisfy the following equations, respectively,

$$\begin{cases} _C D_{0,M}^{0.7595} B(M) = 0.0158 \cdot \Gamma(1.7595), \\ \qquad B(0) = 0; \end{cases} \tag{5}$$

$$\begin{cases} _C D_{0,M_S}^{0.7486} M_L(M_S) = 0.9642 \cdot \Gamma(1.7486), \\ \qquad M_L(0) = 0; \end{cases} \tag{6}$$

$$\begin{cases} _C D_{0,M_R}^{0.7413} M_L(M_R) = 1.4543 \cdot \Gamma(1.7413), \\ \qquad M_L(0) = 0. \end{cases} \tag{7}$$

The right hand sides of equations (5)-(7) are constants. If we assume that the relations between B and M, M_L and M_S, M_L and M_R obey the following law,

$$\begin{cases} _C D_{0,t}^{\alpha} x(t) = f(t), \alpha \in (0,1), \\ \qquad x(0) = 0, \end{cases} \tag{8}$$

in which $f(t) = \Gamma(\alpha + 2)(a_0 + a_1 t)$, it will be somewhat appropriate. Furthermore, one can suppose that

$$f(t) = \Gamma(\alpha + m + 1) \sum_{l=0}^{m} a_l t^l.$$

For case with $f(t) = \Gamma(\alpha + 2)(a_0 + a_1 t)$, (8) can be analytically solved, i.e.,

$$x(t) = a_0(\alpha + 1)t^{\alpha} + a_1 t^{\alpha+1}. \tag{9}$$

For the case with

$$f(t) = \Gamma(\alpha + m + 1) \sum_{l=0}^{m} a_l t^l,$$

the solution to (8) is given by

$$x(t) = \sum_{l=0}^{m} \frac{\Gamma(\alpha + m + 1)}{\Gamma(\alpha + l + 1)} a_l t^l. \tag{10}$$

The existed results coincide with (10) for

$a_1 = a_2 = \cdots = a_m = 0$ in reality. However, (10) is a little tedious if $m \geq 2$. In this paper, we only find unknowns α, a_0 and a_1 in (9) by the method of least square, then the associate fractional model (8) follows. For details, see the Appendix.
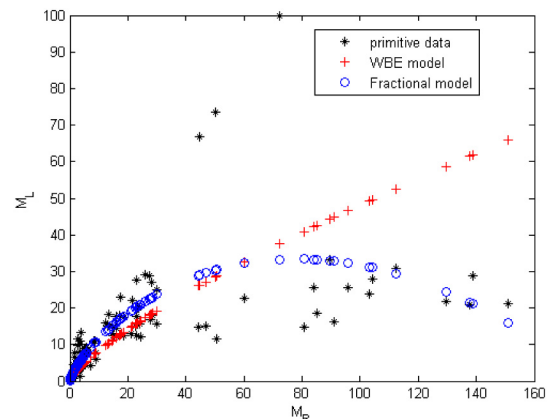
The comparison among the primitive data, WBE model (West-Brown-Enquist model) and fractional model are in Fig. (**2**). From this figure, our fractional model is more appropriate.
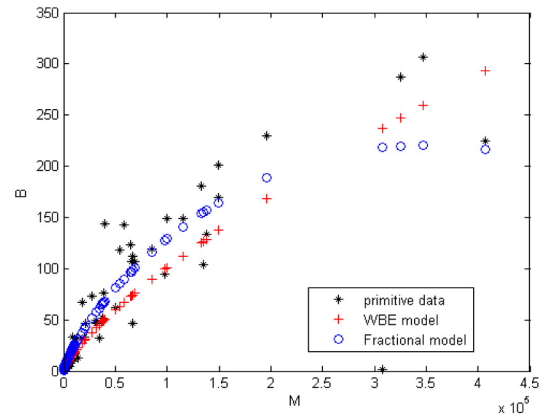
(a)
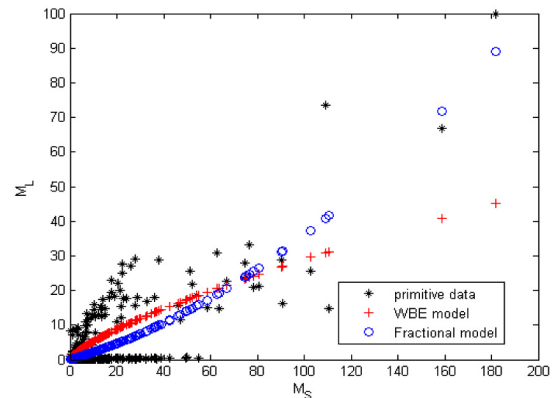
(b)

(c)

Fig. (2). The comparison between WBE and fractional model which are drawn from (5')-(7'). (**a**) B versus M; (**b**) M_L versus M_S; (**c**) M_L versus M_R.

CONCLUSION

In this paper, we establish a fractional model for the allometric scaling laws in biology, ecology. The derived model works well. On one hand, if the scaling exponent α is negative, we can also establish an integral equation model by using the fractional derivatives. On the other hand, scaling laws widely exist in the world, which belong a fractal com-

munity, for example, besides mentioned in biology, ecology, also in fluid, finance, complex networks, etc. To disclose the allometric secrets in fractal community, modeling the allometry is a long-term duty.

APPENDIX

To seek parameters α, a_0 and a_1 of equation (9), we use the method of least square. Set

$$Y = \sum_{l=0}^{n} [x_i - (\alpha+1)a_0 t_i^{\alpha} - a_1 t_i^{\alpha+1}]^2. \qquad (11)$$

By the method of least square, one gets

$$
\begin{cases}
\dfrac{\partial Y}{\partial a_0} = 2\sum_{i=0}^{n}(x_i - (\alpha+1)a_0 t_i^{\alpha+1} - a_1 t_i^{\alpha+1}) \\
\qquad\qquad \cdot(\alpha+1)t_i^{\alpha} = 0, \\
\dfrac{\partial Y}{\partial a_1} = 2\sum_{i=0}^{n}(x_i - (\alpha+1)a_0 t_i^{\alpha+1} - a_1 t_i^{\alpha+1}) \\
\qquad\qquad \cdot t_i^{\alpha+1} = 0, \\
\dfrac{\partial Y}{\partial \alpha} = 2\sum_{i=0}^{n}(x_i - (\alpha+1)a_0 t_i^{\alpha+1} - a_1 t_i^{\alpha+1}) \\
\cdot(-a_0 t_i^{\alpha} - (\alpha+1)a_0 t_i^{\alpha}\ln t_i - a_1 t_i^{\alpha+1}\ln t_i) = 0.
\end{cases} \qquad (12)
$$

The system of equations (12) is nonlinear. We here use the Newton method. The results are listed below,

$$B = 0.0182 M^{0.7816} - 2.28\times10^{-8} M^{1.7816}, \qquad (4'\text{a})$$

$$M_L = 0.3425 M_S^{0.7889} + 0.00606 M_S^{1.7889}, \qquad (4'\text{b})$$

$$M_L = 2.1269 M_R^{0.7621} - 0.011 M_R^{1.7621}, \qquad (4'\text{c})$$

They are just the solutions of following equations, respectively,

$$
\begin{cases}
_C D_{0,M}^{0.7816} B(M) = \Gamma(2.7816) \\
\quad \cdot(0.0182 - 2.28\times10^{-8} M), \\
\qquad B(0) = 0;
\end{cases} \qquad (5')
$$

$$
\begin{cases}
_C D_{0,M_S}^{0.7889} M_L(M_S) = \Gamma(2.7889) \\
\quad \cdot(0.3425 - 0.00606 M_S), \\
\qquad M_L(0) = 0;
\end{cases} \qquad (6')
$$

$$
\begin{cases}
_C D_{0,M_R}^{0.7413} M_L(M_R) = \Gamma(2.27621) \\
\quad \cdot(2.1269 - 0.011 M_R), \\
\qquad M_L(0) = 0.
\end{cases} \qquad (7')
$$

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