Open Access

Two-Sided Bounds on the Displacement $y(t)$ and the Velocity $\dot{y}(t)$ of the Vibration Problem $M\ddot{y} + B\dot{y} + Ky = 0, y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0$ with Application of the Differential Calculus of Norms

L. Kohaupt*

Beuth University of Technology Berlin, Department of Mathematics, Luxemburger Str. 10, D-13353 Berlin, Germany

Abstract: If the vibration problem $M\ddot{y}+B\dot{y}+Ky=0, y(t_0)=y_0, \dot{y}(t_0)=\dot{y}_0$, is cast into state-space form $\dot{x}=Ax, x(t_0)=x_0$, so far only two-sided bounds on $x(t)$ could be derived, but not on the quantities $y(t)$ and $\dot{y}(t)$. By means of new methods, this gap is now filled by deriving two-sided bounds on $y(t)$ and $\dot{y}(t)$; they have the same shape as those for $x(t)$. The best constants in the upper bounds are computed by the differential calculus of norms developed by the author in earlier work. As opposed to this, the lower bounds cannot be determined in the same way since $\| y(t)\|_2$ and $\| \dot{y}(t)\|_2$ have kinks at their local minima (like $|t|^{1/2}$ at $t=0$). The best lower bounds are therefore determined through their local minima. The obtained results cannot be obtained by the methods used so far.

Keywords: Initial value problem; vibration problem; state-space description; two-sided bounds; displacement; velocity; differential calculus of norms.

1. INTRODUCTION

The vibration problem $M\ddot{y}+B\dot{y}+Ky=0, y(t_0)= y_0, \dot{y}(t_0)=\dot{y}_0$, can be solved by writing it in the state-space form $\dot{x}=Ax, x(t_0)=x_0$. For this initial value problem, in [1, 2], so far we have derived two-sided bounds on $x(t)$ of various forms to describe the asymptotic behavior of $x(t)$, from which it would be easy to obtain also upper bounds on the displacement vector $y(t)$ and the velocity vector $\dot{y}(t)$. But, lower bounds on these quantities would not be obtainable.

In this paper, as the *main new point*, we fill this gap and derive two-sided bounds on $y(t)$ and $\dot{y}(t)$ of the same type as in [1, 2] for $x(t)$. The best constants in the upper bounds are computed by the differential calculus of norms developed by the author in earlier work, the lower bounds by their local minima.

The paper is structured as follows.

In Section 2, the initial value problem $\dot{x}=Ax, x(t_0)=x_0$, is formulated for a general square matrix A. In Section 3, the two-sided bounds on $x(t)$ in [2] are derived in a new way that allows one to carry over the results from $x(t)$ to $y(t)$ and $\dot{y}(t)$, in a simple manner. In Section 4, the state-space description $\dot{x}=Ax, x(t_0)=x_0$, of $M\ddot{y}+B\dot{y}+Ky=0, y(t_0)=y_0, \dot{y}(t_0)=\dot{y}_0$, is given where M, B, $K \in IR^{n\times n}$ are the mass, damping, and stiffness matrices, as the case may be, and where $y(t)\in IR^n$ is the displacement vector. In Sections 5, 6, and 7, two-sided bounds of the same types as for $x(t)$ are derived for $y(t)$, $\dot{y}(t)$, and even for $x_S(t)$, (where $S \subset \{1,\cdots,m=2n\}$ is an index set), respectively. We mention that, in Sections 3 and 5 - 7, first the case of a diagonalizable matrix A is studied and then the case of a general square matrix. In Section 8, applications follow. Here, the optimal constants in the upper bounds are computed by the differential calculus of norms. In Section 9, conclusions are drawn. In Section 10, we comment on the References, and in Section 11, an outlook on future research is given.

2. THE INITIAL VALUE PROBLEM $\dot{x} = Ax, x(t_0) = x_0$

Let $A\in \mathbb{C}^{m\times m}$ and $x_0 \in \mathbb{C}^m$. When we assume $A\in IR^{m\times m}$, then it is implicitly assumed that also $x_0 \in IR^m$. We consider the initial value problem

$$\dot{x}=Ax, x(t_0)=x_0, \tag{1}$$

first without any reference to a vibration problem. Later on, in Section 8, matrix A will be the *system matrix* of a specific vibration model.

3. TWO-SIDED BOUNDS ON $x(t)$

In this section, we derive *in a new way* two-sided bounds on the solution $x(t)$ of $\dot{x}=Ax, x(t_0)=x_0$, obtained already in [1, 2]. The reason for this is that the new method allows us to carry over the results in a simple manner to the displacement vector $y(t)$ and the velocity vector $\dot{y}(t)$ of the vibration problem $M\ddot{y}+B\dot{y}+Ky=0, y(t_0)=y_0, \dot{y}(t_0)=\dot{y}_0$, which is our actual goal. One even obtains, in this way, two-

*Address correspondence to this author at the Beuth University of Technology Berlin, Department of Mathematics, Luxemburger Str. 10, D-13353 Berlin, Germany; Tel: +49-30-4504-2970; Fax: +49-30-4504-2011; E-mail: kohaupt@beuth-hochschule.de

sided bounds on $x_S(t)$, where $S \subset \{1,\cdots,m=2n\}$ is an index set. First, the case of a diagonalizable matrix A is studied and then the case of a general square matrix.

3.1. Diagonalizable Matrix A

a. Hypothese on A

First, we formulate some hypotheses on matrix A.

(H1) $A \in IR^{m\times m}$,

(H2) $T^{-1}AT = J = diag(\lambda_k)_{k=1,\cdots,m}$, where $\lambda_k = \lambda_k(A), k=1,\cdots,m$ are the eigenvalues of A,

(H3) $\lambda_i = \lambda_i(A) \neq 0, i=1,\cdots,m$,

(H4) $\lambda_i \neq \lambda_j, i \neq j, i,j=1,\cdots,m$,

(HS) $m=2n$ and the eigenvectors $p_1,\cdots,p_n;\bar{p}_1,\cdots,\bar{p}_n$ form a basis of $\mathbb{C}^m$.

Remark: Let $(H1)$ be fulfilled and let $Ap=\lambda p$. Then, we have $A\bar{p}=\bar{\lambda}\bar{p}$, where the bar denotes the complex conjugate. So, together with (λ,p), also $(\bar{\lambda},\bar{p})$ is a solution of the eigenvalue problem $Ap=\lambda p$. But, if λ and p would be real, then p and $\bar{p}$ would not be linearly independent. This situation cannot happen when hypothesis (HS) is supposed.

Remark: In the sequel, when the special hypothesis (HS) is chosen, we do this in order to be specific in the construction of a solution basis. Other cases such as the model from [3], Fig. (**1**) with $A \in IR^{3\times 3}$ can be handled in a similar manner, however. Therefore, if (HS) is supposed, this is done without loss of generality. Another reason to assume (HS) is that it is adapted to the examples in Section 8.

b. Representation of the Basis $x_k^{(r)}(t), x_k^{(i)}(t), k=1,\cdots,n$

Under the hypotheses $(H1),(H2)$, and (HS), from [4], we obtain the following *real basis functions* for the ODE $\dot{x}=Ax$:

$$
\begin{aligned}
x_k^{(r)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\cos\lambda_k^{(i)}(t-t_0)p_k^{(r)} - \sin\lambda_k^{(i)}(t-t_0)p_k^{(i)}\right],\\
x_k^{(i)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\sin\lambda_k^{(i)}(t-t_0)p_k^{(r)} + \cos\lambda_k^{(i)}(t-t_0)p_k^{(i)}\right],
\end{aligned}
\tag{2}
$$

$k=1,\cdots,n$, where

$$
\begin{aligned}
\lambda_k &= \lambda_k^{(r)} + i\lambda_k^{(i)} = Re\lambda_k + iIm\lambda_k,\\
p_k &= p_k^{(r)} + ip_k^{(i)} = Re p_k + iIm p_k,
\end{aligned}
$$

$k=1,\cdots,m=2n$ are the decompositions of λ_k and p_k into their real and imaginary parts. As in [4], the indices are chosen such that $\lambda_{n+k}=\bar{\lambda}_k, p_{n+k}=\bar{p}_k, k=1,\cdots,n$.

c. New Derivation of the Two-Sided Bounds on $x(t)$ by $e^{\nu_0(t-t_0)}$

First, we prove the following lemma.

Lemma 1:

Let the conditions $(H1),(H2)$ and (HS) be fulfilled and $p_k^{(r)} = Re\{p_k\}, p_k^{(i)} = Im\{p_k\}, k=1,\cdots,n$. Then, the vectors $p_k^{(r)}\, p_k^{(i)}, k=1,\cdots,n$ are linearly independent.

Proof: Let

$$\sum_{k=1}^{n}[\gamma_k^{(r)}p_k^{(r)} + \gamma_k^{(i)}p_k^{(i)}] = 0.$$

Then,

$$\sum_{k=1}^{n}[\gamma_k^{(r)}\frac{p_k+\bar{p}_k}{2} + \gamma_k^{(i)}\frac{p_k-\bar{p}_k}{2i}] = 0.$$

This entails

$$\sum_{k=1}^{n}[(\frac{\gamma_k^{(r)}}{2}+\frac{\gamma_k^{(i)}}{2i})p_k + (\frac{\gamma_k^{(r)}}{2}-\frac{\gamma_k^{(i)}}{2i})\bar{p}_k] = 0.$$

Now, $p_1,\cdots,p_n;\bar{p}_1,\cdots,\bar{p}_n$ are linearly independent and therefore,

$$(\frac{\gamma_k^{(r)}}{2}+\frac{\gamma_k^{(i)}}{2i}) = 0, \quad (\frac{\gamma_k^{(r)}}{2}-\frac{\gamma_k^{(i)}}{2i}) = 0, \quad k=1,\cdots,n.$$

This delivers

$$\gamma_k^{(r)} = 0, \quad \gamma_k^{(i)} = 0, \quad k=1,\cdots,n$$

Let, $u_k^*, k=1,\cdots,m=2n$ be the eigenvectors of A^* corresponding to the eigenvalues $\bar{\lambda}_k, k=1,\cdots,m=2n$. Under $(H1),(H2)$, and (HS), the solution $x(t)$ of (1) has the form

$$x(t) = \sum_{k=1}^{m=2n} c_{1k}p_k e^{\lambda_k(t-t_0)} = \sum_{k=1}^{n}[c_{1k}p_k e^{\lambda_k(t-t_0)} + c_{2k}\bar{p}_k e^{\bar{\lambda}_k(t-t_0)}]$$

with uniquely determined coefficients $c_{1k}, k=1,\cdots,m=2n$. Using the relations

$$c_{2k} = c_{1,n+k} = \bar{c}_{1k},\ k=1,\cdots,n, \tag{3}$$

(see [4], Section 3.1 for the last relation),then according to [2], the *spectral abscissa of A with respect to the initial vector $x_0 \in IR^n$* is given by

$$
\begin{aligned}
\nu_0 := \nu_{x_0}[A] &:= \max_{k=1,\cdots,m=2n}\{\lambda_k^{(r)}(A) | x_0 \not\perp u_k^*\}\\
&= \max_{k=1,\cdots,m=2n}\{\lambda_k^{(r)}(A) | c_{1k} \neq 0\}\\
&= \max_{k=1,\cdots,n}\{\lambda_k^{(r)}(A) | c_{1k} \neq 0\}\\
&= \max_{k=1,\cdots,n}\{\lambda_k^{(r)}(A) | x_0 \not\perp u_k^*\}
\end{aligned}
\tag{4}
$$

Index Sets

In the sequel, we need the following index sets:

$$J_{\nu_0} := \{k_0 \in IN \mid 1 \le k_0 \le n \; and \; \lambda_{k_0}^{(r)}(A) = \nu_0\} \tag{5}$$

and

$$\begin{aligned} J_{\nu_0}^- &:= \{1, \cdots, n\} \setminus J_{\nu_0} \\ &= \{k_0^- \in IN \mid 1 \le k_0^- \le n \; and \; \lambda_{k_0^-}^{(r)}(A) < \nu_0\}. \end{aligned} \tag{6}$$

Starting Point: Appropriate Representation of *x(t)*

We have

$$x(t) = \sum_{k=1}^{n} [c_k^{(r)} x_k^{(r)}(t) + c_k^{(i)} x_k^{(i)}(t)]$$

with

$$c_k^{(r)} = 2Re c_{1k}, \quad c_k^{(i)} = -2Im c_{1k}, \quad k = 1, \cdots, n$$

(cf. [4]). Thus, due to (2),

$$x(t) = \sum_{k=1}^{n} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \tag{7}$$

with

$$\begin{aligned} f_k(t) &:= c_k^{(r)} [\cos \lambda_k^{(i)}(t-t_0) p_k^{(r)} - \sin \lambda_k^{(i)}(t-t_0) p_k^{(i)}] \\ &+ c_k^{(i)} [\sin \lambda_k^{(i)}(t-t_0) p_k^{(r)} + \cos \lambda_k^{(i)}(t-t_0) p_k^{(i)}], \end{aligned} \tag{8}$$

$k = 1, \cdots, n$.

Estimate from above

From (7), (8), one has immediately

$$\| x(t) \| \le X_1 e^{\nu_0 (t-t_0)}, t \ge t_0, \tag{9}$$

where $\| \cdot \|$ is any vector norm.

Estimate from below

From (7), one obtains

$$\begin{aligned} \| x(t) \| &= \| \sum_{k=1}^{n} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \| \\ &\ge \| \sum_{k \in J_{\nu_0}} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \| - \| \sum_{k \in J_{\nu_0^-}} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \| \\ &= \| \sum_{k \in J_{\nu_0}} f_k(t) \| e^{\nu_0 (t-t_0)} - \| \sum_{k \in J_{\nu_0^-}} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \|, \quad t \ge t_0 \end{aligned} \tag{10}$$

Now,

$$\sum_{k \in J_{\nu_0}} f_k(t) \ne 0, \quad t \ge t_0, \tag{11}$$

since $p_k^{(r)}, p_k^{(i)}, k = 1, \cdots, n$ are linearly independent according to the above Lemma 1. Further, the functions $f_k(t), t \ge t_0, k = 1, \cdots, n$ are periodic with period $2\pi / \lambda_k^{(i)}, k = 1, \cdots, n$ if $\lambda_k^{(i)} \ne 0, k = 1, \cdots, n$. This entails

$$\| \sum_{k \in J_{\nu_0}} f_k(t) \| \ge \inf_{t \ge t_0} \| \sum_{k \in J_{\nu_0}} f_k(t) \| =: X_{\nu_0} > 0, \quad t \ge t_0, \tag{12}$$

which remains valid also if $\lambda_k^{(i)} = 0$ for some or all $k \in J_{\nu_0}$. With this,

$$\begin{aligned} \| \sum_{k \in J_{\nu_0^-}} e^{\lambda_k^{(r)}(t-t_0)} f_k(t) \| &= \| \sum_{k \in J_{\nu_0^-}} e^{(\lambda_k^{(r)} - \nu_0)(t-t_0)} f_k(t) \| e^{\nu_0 (t-t_0)} \\ &\le \frac{X_{\nu_0}}{2} e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \end{aligned} \tag{13}$$

for sufficiently large $t_1 \ge t_0$ since $\lambda_k^{(r)} - \nu_0 < 0, k \in J_{\nu_0^-}$. From (10), (12), (13), we infer

$$\| x(t) \| \ge \frac{X_{\nu_0}}{2} e^{\nu_0 (t-t_0)} = X_0 e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{14}$$

with

$$X_0 := \frac{X_{\nu_0}}{2} > 0, \tag{15}$$

for sufficiently large t_1 .

Two-sided bound on *x(t)* by $e^{\nu_0 (t-t_0)}$

Summarizing, we obtain

Theorem 2: (Two-sided bound on $x(t)$ by $e^{\nu_0 (t-t_0)}$)

Let the hypotheses $(H1), (H2)$, and (HS) be fulfilled. Then, there exist constants $X_0 > 0$ and $X_1 > 0$ such that

$$X_0 e^{\nu_0 (t-t_0)} \le \| x(t) \| \le X_1 e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{16}$$

for sufficiently large t_1 , where the upper bound holds for $t_1 = t_0$. If $x(t) \ne 0, t \ge t_0$, then $t_1 = t_0$.

Proof: It remains to prove the last assertion. For this, let $x(t) \ne 0, t \ge t_0$. Then,

$$(\min_{t_0 \le t \le t_1} \frac{\| x(t) \|}{e^{\nu_0 (t-t_0)}}) e^{\nu_0 (t-t_0)} \le \| x(t) \|, \quad t_0 \le t \le t_1.$$

d. Two-sided bound on $x(t)$ by $\| \psi(t) \|$

Let

$$A^* u_k^* = \bar{\lambda}_k u_k^*, \quad k = 1, \cdots, m = 2n$$

and

$$\psi_k(t) := (x_0, u_k^*) e^{Re \lambda_k (t-t_0)} = (x_0, u_k^*) e^{\lambda_k^{(r)}(t-t_0)}, \quad t \ge t_0, \tag{17}$$

$k = 1, \cdots, n$ as well as

$$\psi(t) := [\psi_1(t), \cdots, \psi_n(t)]^T. \tag{18}$$

Herewith, we get

Theorem 3: (Two-sided bound on $x(t)$ by $\| \psi(t) \|$)

Let the hypotheses $(H1),(H2)$, and (HS) be fulfilled. Then, there exist constants $\xi_0 > 0$ and $\xi_1 > 0$ such that

$$\xi_0 \, \| \psi(t) \| \le \| x(t) \| \le \xi_1 \, \| \psi(t) \|, \quad t \ge t_0. \tag{19}$$

Proof: This follows from [2], Theorem 11 and the equivalence of norms in finite-dimensional spaces.

e. Determination of the Constants $c_k, k = 1,\cdots,m = 2n$ without Hypothesis (H4), i.e., without $\lambda_i \ne \lambda_j, i \ne j, i,j = 1,\cdots,m$

We start with the representation

$$x(t) = \sum_{k=1}^{m=2n} c_k p_k e^{\lambda_k (t-t_0)}. \tag{20}$$

Using the initial condition $x(t_0) = x_0$, we conclude

$$x_0 = \sum_{k=1}^{m=2n} c_k p_k. \tag{21}$$

Let

$$P := [p_1,\cdots,p_m] \tag{22}$$

and

$$c = [c_1,\cdots,c_m]^T. \tag{23}$$

Then,

$$Pc = x_0. \tag{24}$$

Since matrix P is regular, the solution c of matrix equation (24) is uniquely determined. For the solution of (24), we need *not* $(H4)$. Any solution method can be applied, for example, Gaussian elimination. However, under the additional condition $(H4)$, according to paper [5], there is a biorthogonal system of eigenvectors $\{p_1,\cdots,p_m\},\{u_1^*,\cdots,u_m^*\}$ so that $c_k, k = 1,\cdots,m = 2n$ can be calculated by $c_k = (x_0, u_k^*), k = 1,\cdots,m$, which is numerically very effective. Without hypothesis $(H4)$, one can use the biorthogonalization method of the paper [6] to preserve the formulae $c_k = (x_0, u_k^*), k = 1,\cdots,m$.

3.2. General Square Matrix A

a. Hypotheses on A

Again, first, we formulate some hypotheses on matrix A.

$(H1')$ $A \in IR^{m \times m}$,

$(H2')$ $T^{-1}AT = J = diag(J_i(\lambda_i))_{i=1,\cdots,r}$ where $J_i(\lambda_i) \in C^{m_i \times m_i}$ are the canonical Jordan forms,

$(H3')$ $\lambda_i = \lambda_i(A) \ne 0, i = 1,\cdots,r$,

$(H4')$ $\lambda_i \ne \lambda_j, i \ne j, i,j = 1,\cdots,r$,

(HS') $m = 2n$, $r = 2\rho$, and the principal vectors $p_1^{(1)},\cdots,p_{m_1}^{(1)};\cdots;p_1^{(\rho)},\cdots,p_{m_\rho}^{(\rho)};\overline{p}_1^{(1)},\cdots,\overline{p}_{m_1}^{(1)};\cdots;\overline{p}_1^{(\rho)},\cdots,\overline{p}_{m_\rho}^{(\rho)}$

form a basis of $\mathbb{C}^m$.

We mention that for the special hypothesis (HS') similar remarks hold as for (HS) in the case of diagonalizable matrices A.

Let $(H1')$, $(H2')$, and (HS') be fulfilled and $Ap_k^{(l)} = \lambda_l p_k^{(l)} + p_{k-1}^{(l)}, k = 1,\cdots,m_l, l = 1,\cdots,r$, where the indices are chosen such that $\lambda_{\rho+l} = \overline{\lambda}_l, l = 1,\cdots,\rho$ and $p_k^{(\rho+l)} = \overline{p}_k^{(l)}, k = 1,\cdots,m_l, l = 1,\cdots,\rho$. The vectors $p_k^{(l)}$ are the principal vectors of stage k corresponding to the eigenvalue λ_l of A.

b. Representation of the Basis $x_k^{(l,r)}(t), x_k^{(l,i)}(t)$, $k = 1,\cdots,m_l, l = 1,\cdots,p$

Under the hypotheses $(H1')$, $(H2')$, and (HS'), from [4] we obtain the following *real basis functions* for the ODE $\dot{x} = Ax$:

$$\begin{aligned} x_k^{(l,r)}(t) = e^{\lambda_l^{(r)}(t-t_0)} \Big\{ &\cos\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l,r)}(t-t_0) + p_k^{(l,r)}\Big] \\ &- \sin\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l,i)}(t-t_0) + p_k^{(l,i)}\Big]\Big\}, \\ x_k^{(l,i)}(t) = e^{\lambda_l^{(r)}(t-t_0)} \Big\{ &\sin\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l,r)}(t-t_0) + p_k^{(l,r)}\Big] \\ &+ \cos\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l,i)}(t-t_0) + p_k^{(l,i)}\Big]\Big\}, \end{aligned} \tag{25}$$

$k = 1,\cdots,m_l, l = 1,\cdots,\rho$, where

$p_k^{(l)} = p_k^{(l,r)} + i\,p_k^{(l,i)}$

is the decomposition of $p_k^{(l)}$ into its real and imaginary part.

c. New Derivation of the Two-Sided Bounds on $x(t)$

First, we state the following lemma.

Lemma 4: Let the hypotheses $(H1')$, $(H2')$, and (HS') be fulfilled and $p_k^{(l,r)} = Re\,p_k^{(l)}, p_k^{(l,i)} = Im\,p_k^{(l)}$, $k = 1,\cdots,m_l, l = 1,\cdots,\rho$. Then, the vectors $p_k^{(l,r)}\, p_k^{(l,i)}$, $k = 1,\cdots,m_l, l = 1,\cdots,\rho$ are linearly independent.

Proof: The proof is similar to that of Lemma 1 and therefore omitted.

Now, let $u_k^{(l)*}, k = 1,\cdots,m_l$ be the principal vectors of stage k of A^* corresponding to the eigenvalue $\overline{\lambda}_l, l = 1,\cdots,r = 2\rho$. Under $(H1')$, $(H2')$, and (HS'), the solution $x(t)$ of (1) has the form

$$x(t) = \sum_{l=1}^{r=2\rho}\sum_{k=1}^{m_l} c_{1k}^{(l)} x_k^{(l)}(t) = \sum_{l=1}^{\rho}\sum_{k=1}^{m_l}[c_{1k}^{(l)} x_k^{(l)}(t) + c_{2k}^{(l)} \overline{x}_k^{(l)}(t)].$$

with uniquely determined coefficients $c_{1k}^{(l)}, k = 1,\cdots,m_l, l = 1,\cdots,r = 2\rho$. Using the relations

$$c_{1k}^{(l)} = (x_0, u_k^{(l)*}),\ k=1,\cdots,m_l,\ l=1,\cdots,\rho$$
$$c_{2k}^{(l)} = c_{1k}^{(\rho+l)} = \overline{c}_{1k}^{(l)},\ l=1,\cdots,\rho \qquad (26)$$

(see [4], Section 3.2 for the last relation), then the *spectral abscissa of* A *with respect to the initial vector* $x_0 \in IR^n$ is

$$\begin{aligned} v_0 := v_{x_0}[A] &:= \max_{l=1,\cdots,r=2\rho}\{\lambda_l^{(r)}(A)|x_0 \perp M_{\bar\lambda_l(A^*)} := [u_1^{(l)*},\cdots,u_{m_l}^{(l)*}]\} \\ &= \max_{l=1,\cdots,r=2\rho}\{\lambda_l^{(r)}(A)|c_{1k}^{(l)} \neq 0 \text{ for at least one } k\in\{1,\cdots,m_l\}\} \\ &= \max_{l=1,\cdots,\rho}\{\lambda_l^{(r)}(A)|c_{1k}^{(l)} \neq 0 \text{ for at least one } k\in\{1,\cdots,m_l\}\} \\ &= \max_{l=1,\cdots,\rho}\{\lambda_l^{(r)}(A)|x_0 \perp M_{\bar\lambda_l(A^*)} = [u_1^{(l)*},\cdots,u_{m_l}^{(l)*}]\} \end{aligned} \qquad (26)$$

Index Sets

For the sequel, we need the following index sets:

$$J_{v_0} := \{l_0 \in IN | 1 \le l_0 \le \rho \text{ and } \lambda_{l_0}^{(r)}(A) = v_0\} \qquad (27)$$

and

$$\begin{aligned} J_{v_0}^- &:= \{1,\cdots,\rho\}\setminus J_{v_0} \\ &= \{l_0^- \in IN | 1 \le l_0^- \le \rho \text{ and } \lambda_{l_0^-}^{(r)}(A) < v_0\}. \end{aligned} \qquad (28)$$

Starting Point: Appropriate Representation of *x(t)*

We have

$$x(t) = \sum_{l=1}^{\rho}\sum_{k=1}^{m_l}[c_k^{(l,r)}x_k^{(l,r)}(t) + c_k^{(l,i)}x_k^{(l,i)}(t)]$$

with

$$c_k^{(l,r)} = 2Rec_{1k}^{(l)}, \quad c_k^{(l,i)} = -2Imc_{1k}^{(l)}, \quad k=1,\cdots,m_l, l=1,\cdots,\rho$$

(cf. [4]). Thus, due (25),

$$x(t) = \sum_{l=1}^{\rho} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t) \qquad (29)$$

with

$$\begin{aligned} f_k^{(l)}(t) := \ & c_k^{(l,r)}\Big\{\cos\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{k-1}^{(l,r)}(t-t_0)+p_k^{(l,r)}\Big] \\ & -\sin\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{k-1}^{(l,i)}(t-t_0)+p_k^{(l,i)}\Big]\Big\} \\ + \ & c_k^{(l,i)}\Big\{\sin\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{k-1}^{(l,r)}(t-t_0)+p_k^{(l,r)}\Big] \\ & +\cos\lambda_l^{(i)}(t-t_0)\Big[p_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{k-1}^{(l,i)}(t-t_0)+p_k^{(l,i)}\Big]\Big\} \end{aligned} \qquad (31)$$

$k=1,\cdots,m_l, l=1,\cdots,\rho$.

Estimate from above

From (30), (31), for every $\varepsilon > 0$, one has immediately

$$\|x(t)\| \le X_1(\varepsilon)e^{(v_0+\varepsilon)(t-t_0)},\ t \ge t_0. \qquad (32)$$

Estimate from below

From (30), one obtains

$$\begin{aligned} \|x(t)\| &= \|\sum_{l=1}^{\rho} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| \\ &\ge \|\sum_{l\in J_{\nu_0}} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| - \|\sum_{l\in J_{\nu_0}^-} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| \\ &= \|\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| e^{\nu_0(t-t_0)} - \|\sum_{l\in J_{\nu_0}^-} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\|, \quad t \ge t_0. \end{aligned} \qquad (33)$$

Now,

$$\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} f_k^{(l)}(t) \neq 0, \quad t \ge t_0, \qquad (34)$$

since $p_k^{(l,r)}, p_k^{(l,i)}, k=1,\cdots,m_l, l=1,\cdots,\rho$ are linearly independent according to the above Lemma 4. Further, the functions $f_k^{(l)}(t), t \ge t_0, k=1,\cdots,m_l, l=1,\cdots,\rho$ are periodic with period $2\pi/\lambda_k^{(i)}, k=1,\cdots,m_l, l=1,\cdots,\rho$ if $\lambda_l^{(i)} \neq 0$, $k=1,\cdots,\rho$. This entails

$$\|\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| \ge \inf_{t\ge t_0}\|\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| =: X_{v_0} > 0, \quad t \ge t_0, \qquad (35)$$

which remains valid also if $\lambda_l^{(i)} = 0$ for some or all $l \in J_{v_0}$. With this,

$$\begin{aligned} \|\sum_{l\in J_{\nu_0}^-} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| &= \|\sum_{l\in J_{\nu_0}^-} e^{(\lambda_k^{(r)}-\nu_0)(t-t_0)}\sum_{k=1}^{m_l} f_k^{(l)}(t)\| e^{\nu_0(t-t_0)} \\ &\le \frac{X_{\nu_0}}{2}e^{\nu_0(t-t_0)}, \quad t \ge t_1 \ge t_0 \end{aligned} \qquad (36)$$

for sufficiently large $t_1 \ge t_0$ since $\lambda_l^{(r)} - v_0 < 0, l \in J_{v_0^-}$.

Two-sided bound on *x(t)* by $e^{v_0(t-t_0)}$

Summarizing, from (32), (33), (35), (36), we obtain

Theorem 5: (Two-sided bound on $x(t)$ by $e^{v_0(t-t_0)}$)

Let the hypotheses $(H1'), (H2')$, and (HS^1) be fulfilled. Then, there exists a constant $X_0 > 0$ and for every $\varepsilon > 0$ a constant $X_1(\varepsilon) > 0$ such that

$$X_0 e^{v_0(t-t_0)} \le \|x(t)\| \le X_1(\varepsilon)e^{(v_0+\varepsilon)(t-t_0)}, \quad t \ge t_1 \ge t_0 \qquad (37)$$

for sufficiently large t_1, where the upper bound holds for $t_1 = t_0$. If $x(t) \neq 0, t \ge t_0$, then $t_1 = t_0$. Further, if the index of every eigenvalue $\lambda = \lambda(A)$ with $Re\lambda = \nu_0$ is equal to unity, then $\varepsilon = 0$ can be chosen.

Proof: (37) has been proven; the rest is left to the reader.

d. Two-sided Bound on $x(t)$ by $\|\psi(t)\|$

Let

$$A^* u_k^{(l)^*} = \overline{\lambda}_l u_k^{(l)^*} + u_{k-1}^{(l)^*}, \quad k = 1,\cdots,m_l,\ l = 1,\cdots,r = 2\rho$$

and

$$p_{x_0,k-1}^{(l)}(t-t_0) := (x_0, p_1^{(l)}\frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + p_{k-1}^{(l)}(t-t_0) + p_k^{(l)}), \qquad (38)$$

$k = 1,\cdots,m_l,\ l = 1,\cdots,r = 2\rho$ as well as

$$\psi_k^{(l)}(t) := p_{x_0,k-1}^{(l)}(t-t_0)e^{\lambda_l^{(r)}(t-t_0)}, \qquad (39)$$

$k = 1,\cdots,m_l,\ l = 1,\cdots,r = 2\rho$; further, let

$$\psi^{(l)}(t) := [\psi_1^{(l)}(t),\cdots,\psi_{m_l}^{(l)}(t)]^T, \qquad (40)$$

$l = 1,\cdots,r = 2\rho$ and

$$\psi(t) := [\psi_1(t)^T,\cdots,\psi_r(t)^T]^T. \qquad (41)$$

Herewith, we get

Theorem 6: (Two-sided bound on $x(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1'),(H2')$, and (HS^1) be fulfilled. Then, there exist constants $\xi_0 > 0$ and $\xi_1 > 0$ such that

$$\xi_0 \|\psi(t)\| \le \|x(t)\| \le \xi_1 \|\psi(t)\|, \quad t \ge t_0. \qquad (42)$$

Proof: This follows from [2], Theorem 13 and the equivalence of norms in finite-dimensional spaces.

e. Determination of the Constants $c_k^{(l)}, k = 1, \cdots, m_l, l = 1,\cdots,r$ without Hypothesis $(H4')$,

i.e., without $\lambda_i \ne \lambda_j, i \ne j, i,j = 1,\cdots,r$

We start with the representation

$$x(t) = \sum_{l=1}^{r}\sum_{k=1}^{m_l} c_k^{(l)} x_k^{(l)}(t), \quad t \ge t_0.$$

Substituting $t = t_0$ and using $x(t_0) = x_0$, we conclude

$$x_0 = \sum_{l=1}^{r}\sum_{k=1}^{m_l} c_k^{(l)} p_k^{(l)}. \qquad (43)$$

Let

$$P := [p_1^{(1)},\cdots,p_{m_1}^{(1)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}] \qquad (44)$$

and

$$c^{(l)} = [c_1^{(l)},\cdots,c_{m_l}^{(l)}]^T, \qquad (45)$$

$l = 1,\cdots,r$ as well as

$$c = [c^{(1)T},\cdots,c^{(r)T}]^T. \qquad (46)$$

Then,

$$Pc = x_0. \qquad (47)$$

Since matrix P is regular, the solution c of matrix equation (47) is uniquely determined. For the solution of (47), we need *not* $(H4^1)$. Any solution method can be applied, for example, Gaussian elimination. However, under the additional condition $(H4^1)$, according to paper [5], there is a biorthogonal system of principal vectors $\{p_1^{(1)},\cdots,p_{m_1}^{(1)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}\},\{v_1^{(1)^*},\cdots,v_{m_1}^{(1)^*};\cdots;v_1^{(r)^*},\cdots,v_{m_r}^{(r)^*}\}$ with $v_k^{(l)^*} = u_{m_l-k+1}^{(l)^*}, k = 1,\cdots,m_l, l = 1,\cdots,r$. Without hypothesis $(H4^1)$, one can use the biorthogonalization method of the paper [6] to construct a biorthogonal system.

4. THE STATE-SPACE DESCRIPTION OF $M\ddot{y} + B\dot{y} + Ky = 0, y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0$

Let $M,B,K \in IR^{n\times n}$ and $y_0,\dot{y}_0 \in IR^n$. Further, let M be regular. The matrices M, B, and K are the mass, damping, and stiffness matrices, as the case may be; y_0 is the initial displacement and $\dot{y}_0$ the initial velocity. We study the initial value problem

$$M\ddot{y} + B\dot{y} + Ky = 0, y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0, \qquad (48)$$

where $y(t)$ is the sought displacement and $z(t) = \dot{y}(t)$ the associated velocity.

State-Space Description

Let

$$x := \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x_0 := \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix}, \qquad (49)$$

and

$$A = \left[\begin{array}{c|c} O & E \\ \hline -M^{-1}K & -M^{-1}B \end{array}\right]; \qquad (50)$$

x is called *state vector* and A *system matrix*. Herewith, (48) is equivalent to

$$\dot{x} = Ax, x(t_0) = x_0. \qquad (51)$$

In the sequel, we need only the special form of $x(t)$.

5. TWO-SIDED BOUNDS ON *y(t)*

In this section, we derive bounds on $y(t)$ corresponding to those on $x(t)$ in Section 3.

5.1. Diagonalizable Matrix A

a. Hypotheses on A

We suppose $(H1),(H2)$, and (HS); $(H3)$ is not needed, and $(H4)$ is needed only if the coefficients c_k are to be calculated by $c_k = (x_0,u_k^*), k = 1,\cdots,n$.

b. Representation of the Basis $y_k^{(r)}(t), y_k^{(i)}(t), k = 1,\cdots,n$

As in [4], let

$$x_k^{(r)}(t) := \begin{bmatrix} y_k^{(r)}(t) \\ \dot{y}_k^{(r)}(t) \end{bmatrix}, \quad x_k^{(i)}(t) := \begin{bmatrix} y_k^{(i)}(t) \\ \dot{y}_k^{(i)}(t) \end{bmatrix}, \quad p_k := \begin{bmatrix} q_k \\ r_k \end{bmatrix},$$

$k = 1, \cdots, m = 2n$. Then, from (2),

$$\begin{aligned} y_k^{(r)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\cos \lambda_k^{(i)}(t-t_0) q_k^{(r)} - \sin \lambda_k^{(i)}(t-t_0) q_k^{(i)}\right], \\ y_k^{(i)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\sin \lambda_k^{(i)}(t-t_0) q_k^{(r)} + \cos \lambda_k^{(i)}(t-t_0) q_k^{(i)}\right], \end{aligned} \tag{52}$$

$k = 1, \cdots, n$.

c. Derivation of the Two-Sided Bounds on y(t) by $e^{\nu_0 (t-t_0)}$

Starting point: Appropriate representation of *y(t)*

From (7), (8), we conclude

$$y(t) = \sum_{k=1}^{n} e^{\lambda_k^{(r)}(t-t_0)} g_k(t) \tag{53}$$

with

$$\begin{aligned} g_k(t) &:= c_k^{(r)}[\cos \lambda_k^{(i)}(t-t_0) q_k^{(r)} - \sin \lambda_k^{(i)}(t-t_0) q_k^{(i)}] \\ &+ c_k^{(i)}[\sin \lambda_k^{(i)}(t-t_0) q_k^{(r)} + \cos \lambda_k^{(i)}(t-t_0) q_k^{(i)}], \end{aligned} \tag{54}$$

$k = 1, \cdots, n$.

Estimate from above

From (49)and (16), one has immediately

$$\| y(t) \|_2 \le \| x(t) \|_2 \le X_{1,2}\, e^{\nu_0 (t-t_0)}, t \ge t_0;$$

due to the equivalence of norms in finite-dimensional spaces, this entails

$$\| y(t) \| \le Y_1 e^{\nu_0 (t-t_0)}, t \ge t_0, \tag{55}$$

for a constant $Y_1 > 0$.

Estimate from below

Here, we have to investigate two cases.

Case 1:

$$\sum_{k \in J_{\nu_0}} g_k(t) \neq 0, \quad t \ge t_0. \tag{56}$$

We mention that the corresponding inequality (11) for $x(t)$ *could be proven* by use of Lemma 1. Now, from (56), it follows

$$\| \sum_{k \in J_{\nu_0}} g_k(t) \| \ge \inf_{t \ge t_0} \| \sum_{k \in J_{\nu_0}} g_k(t) \| =: Y_{\nu_0} > 0, \quad t \ge t_0. \tag{57}$$

Similarly as for $x(t)$, here

$$\| y(t) \| \ge \frac{Y_{\nu_0}}{2} e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{58}$$

for sufficiently large t_1.

Case 2:

$$\sum_{k \in J_{\nu_0}} g_k(t) = 0, \; \textit{for at least one } t \ge t_0. \tag{59}$$

In this case, (57) and therefore (58) remain valid only for $Y_{\nu_0} = 0$.

Two-sided bound on *y(t)* by $e^{\nu_0 (t-t_0)}$

Summarizing, we obtain

Theorem 7: (Two-sided bound on $y(t)$ by $e^{\nu_0 (t-t_0)}$)

Let the hypotheses $(H1), (H2)$, and (HS) be fulfilled and additionally condition (56). Then, there exist constants $Y_0 > 0$ and $Y_1 > 0$ such that

$$Y_0 e^{\nu_0 (t-t_0)} \le \| y(t) \| \le Y_1 e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{60}$$

for sufficiently large t_1, where the upper bound holds for $t_1 = t_0$. If $y(t) \neq 0, t \ge t_0$, then $t_1 = t_0$. If, instead of (56), condition (59) is fulfilled, then the lower bound is only valid with $Y_0 = 0$.

According to Lemma 1, a sufficient algebraic condition for (11) is the linear independence of $p_k^{(r)}, p_k^{(r)}, k \in J_{\nu_0}$. Similarly, we have

Sufficient algebraic condition for $\sum_{k \in J_{\nu_0}} g_k(t) \neq 0, \quad t \ge t_0$:

$q_k^{(r)}, q_k^{(i)}, k \in J_{\nu_0}$ are linearly independent.

Sufficient algebraic condition for

$\sum_{k \in J_{\nu_0}} g_k(t) = 0,$ *for at least one* $t \ge t_0$:

$J_{\nu_0} = \{k_0\}$ and $q_{k_0}^{(r)}, q_{k_0}^{(i)}$ are linearly dependent, e.g.,

$$q_{k_0}^{(i)} = \sigma_{k_0} q_{k_0}^{(r)}. \tag{61}$$

Because then,

$$\begin{aligned} g_{k_0}(t) &:= c_{k_0}^{(r)}[\cos \lambda_{k_0}^{(i)}(t-t_0) q_{k_0}^{(r)} - \sin \lambda_{k_0}^{(i)}(t-t_0) q_{k_0}^{(i)}] \\ &+ c_{k_0}^{(i)}[\sin \lambda_{k_0}^{(i)}(t-t_0) q_{k_0}^{(r)} + \cos \lambda_{k_0}^{(i)}(t-t_0) q_{k_0}^{(i)}] \\ &= \left[A_{k_0} \cos \lambda_{k_0}^{(i)}(t-t_0) + B_{k_0} \sin \lambda_{k_0}^{(i)}(t-t_0)\right] q_{k_0}^{(r)}, t \ge t_0, \end{aligned} \tag{62}$$

with

$$\begin{aligned} A_{k_0} &:= c_{k_0}^{(r)} + \sigma_{k_0} c_{k_0}^{(i)}, \\ B_{k_0} &:= c_{k_0}^{(r)} - \sigma_{k_0} c_{k_0}^{(i)}. \end{aligned} \tag{63}$$

Since the factor in the bracket of $g_{k_0}(t)$ takes on the value zero, we have proven that (61) is sufficient for (59).

Theorem 8: (Two-sided bound on $y(t)$ by $\| \psi(t) \|$)

Let the hypotheses $(H1), (H2)$, and (HS) be fulfilled and additionally (56). Then, there exist constants $\eta_0 > 0$ and $\eta_1 > 0$ such that for sufficiently large $t_1 \ge t_0$,

$$\eta_0 \| \psi(t) \| \le \| y(t) \| \le \eta_1 \| \psi(t) \|, \quad t \ge t_1 \ge t_0, \tag{64}$$

with $\psi(t)$ defined by (18), where $t_1 = t_0$ if $y(t) \neq 0, t \geq t_0$.

Proof: From (16) and (60), it follows

$$\frac{Y_0}{X_1} \| x(t) \| \leq Y_0 \, e^{\nu_0 (t-t_0)} \leq \| y(t) \| \leq Y_1 \, e^{\nu_0 (t-t_0)} \leq \frac{Y_1}{X_0} \| x(t) \|, \quad t \geq t_1 \geq t_0. \tag{65}$$

From (19), we infer

$$\frac{Y_0}{X_1} \xi_0 \|\psi(t)\| \leq \frac{Y_0}{X_1} \|x(t)\| \leq Y_0 \, e^{\nu_0 (t-t_0)} \leq \|y(t)\| \leq Y_1 \, e^{\nu_0 (t-t_0)} \leq \frac{Y_1}{X_0} \|x(t)\| \leq \frac{Y_1}{X_0} \xi_1 \|\psi(t)\|,$$

$t \geq t_1 \geq t_0$; set $\eta_0 := \frac{Y_0}{X_1} \xi_0$ and $\eta_1 := \frac{Y_1}{X_0} \xi_1$. Then, $\eta_0 > 0$ and $\eta_1 > 0$, and (64) is proven.

5.2. General Square Matrix A

a. Hypotheses on A

We suppose $(H1'), (H2')$, and (HS'); $(H3')$ is not needed, and $(H4')$ is needed only if the coefficients $c_{1k}^{(l)}$ are to be computed by $c_{1k}^{(l)} = (x_0, u_{m_l-k+1}^{(l)\,*}), k = 1, \cdots, m_l, l = 1, \cdots, \rho$.

b. Representation of the Basis $y_k^{(l,r)}(t), y_k^{(l,i)}(t), k = 1, \cdots, m_l, l = 1, \cdots, p$

As in [4], let

$$y_k^{(l,r)}(t) := \begin{bmatrix} y_k^{(l,r)}(t) \\ \dot{y}_k^{(l,r)}(t) \end{bmatrix}, \quad y_k^{(l,i)}(t) := \begin{bmatrix} y_k^{(l,i)}(t) \\ \dot{y}_k^{(l,i)}(t) \end{bmatrix}, \quad p_k^{(l)} := \begin{bmatrix} q_k^{(l)} \\ r_k^{(l)} \end{bmatrix},$$

$k = 1, \cdots, m_l, l = 1, \cdots, \rho$. Then, from (25),

$$\begin{aligned} y_k^{(l,r)}(t) = e^{\lambda_l^{(r)}(t-t_0)} \Big\{ & \cos \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,r)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,r)}(t-t_0) + q_k^{(l,r)} \right] \\ & - \sin \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,i)}(t-t_0) + q_k^{(l,i)} \right] \Big\}, \\ y_k^{(l,i)}(t) = e^{\lambda_l^{(r)}(t-t_0)} \Big\{ & \sin \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,r)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,r)}(t-t_0) + q_k^{(l,r)} \right] \\ & + \cos \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,i)}(t-t_0) + q_k^{(l,i)} \right] \Big\}, \end{aligned} \tag{66}$$

$k = 1, \cdots, m_l, l = 1, \cdots, \rho$.

c. Two-sided Bound on $y(t)$ by $e^{\nu_0(t-t_0)}$

Starting Point: Appropriate Representation of *y(t)*

We have

$$y(t) = \sum_{l=1}^{\rho} e^{\lambda_l^{(r)}(t-t_0)} \sum_{k=1}^{m_l} g_k^{(l)}(t) \tag{67}$$

with

$$\begin{aligned} g_k^{(l)}(t) \;:=\; & c_k^{(l,r)} \Big\{ \cos \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,r)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,r)}(t-t_0) + q_k^{(l,r)} \right] \\ & - \sin \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,i)}(t-t_0) + q_k^{(l,i)} \right] \Big\} \\ + \; & c_k^{(l,i)} \Big\{ \sin \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,r)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,r)}(t-t_0) + q_k^{(l,r)} \right] \\ & + \cos \lambda_l^{(i)}(t-t_0) \left[q_1^{(l,i)} \frac{(t-t_0)^{k-1}}{(k-1)!} + \cdots + q_{k-1}^{(l,i)}(t-t_0) + q_k^{(l,i)} \right] \Big\} \end{aligned} \tag{68}$$

$k = 1, \cdots, m_l, l = 1, \cdots, \rho$.

Estimate from above

From (49) and (37), for every $\varepsilon > 0$, there exists a constant $X_{1,2}(\varepsilon) > 0$ such that

$$\|y(t)\|_2 \leq \|x(t)\|_2 \leq X_{1,2}(\varepsilon) e^{(\nu_0+\varepsilon)(t-t_0)}, \; t \geq t_0;$$

due to the equivalence of norms in finite-dimensional spaces, this entails

$$\|y(t)\| \leq Y_1(\varepsilon) e^{(\nu_0+\varepsilon)(t-t_0)}, \; t \geq t_0, \tag{69}$$

for a constant $Y_1(\varepsilon) > 0$.

Estimate from below

Here, we have to investigate two cases.

Case 1:

$$\sum_{l \in J_{\nu_0}} \sum_{k=1}^{m_l} g_k^{(l)}(t) \neq 0, \quad t \geq t_0. \tag{70}$$

We mention that the corresponding inequality (34) for $x(t)$ *could be proven* by Lemma 4. Now, from (70), it follows

$$\| \sum_{l \in J_{\nu_0}} \sum_{k=1}^{m_l} g_k^{(l)}(t) \| \geq \inf_{t \geq t_0} \| \sum_{l \in J_{\nu_0}} \sum_{k=1}^{m_l} g_k^{(l)}(t) \| =: Y_{\nu_0} > 0, \quad t \geq t_0. \tag{71}$$

Similarly as for $x(t)$, here

$$\| y(t) \| \geq \frac{Y_{\nu_0}}{2} e^{\nu_0 (t-t_0)}, \quad t \geq t_1 \geq t_0 \tag{72}$$

for sufficiently large t_1.

Case 2:

$$\sum_{l \in J_{\nu_0}} \sum_{k=1}^{m_l} g_k^{(l)}(t) = 0, \; \textit{for at least one } t \geq t_0. \tag{73}$$

In this case, (71) and therefore (72) remain valid only for $Y_{\nu_0} = 0$.

Two-sided bound on $y(t)$ by $e^{\nu_0(t-t_0)}$

Summarizing, we obtain

Theorem 9: (Two-sided bound on $y(t)$ by $e^{\nu_0(t-t_0)}$)

Let the hypotheses $(H1^{'}),(H2^{'})$, and (HS') be fulfilled and additionally (70). Then, there exists a constant $Y_0>0$ and for every $\varepsilon>0$ a constant $Y_1(\varepsilon)>0$ such that

$$Y_0 e^{\nu_0(t-t_0)} \le \| y(t) \| \le Y_1(\varepsilon) e^{(\nu_0+\varepsilon)(t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{74}$$

for sufficiently large t_1, where $t_1=t_0$ if $y(t)\neq 0, t\ge t_0$. If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=\nu_0$ is equal to unity, then $\varepsilon=0$ can be chosen. If, instead of (70), condition (73) is fulfilled, then the lower bound is only valid with $Y_0=0$.

Sufficient Condition for $Y_{\nu_0}>0$

We want to give a sufficient condition for $Y_{\nu_0}>0$. As a preparation, we first introduce some abbreviations and prove a lemma.

Abbreviations:

$$\begin{array}{llll}
\underline{j=1}: & \tilde{q}_1^{(l,r)}(t) & := & q_1^{(l,r)}, \\
 & \tilde{q}_1^{(l,i)}(t) & := & q_1^{(l,i)}, \\
\underline{j=2}: & \tilde{q}_2^{(l,r)}(t) & := & q_1^{(l,r)}(t-t_0)+q_2^{(l,r)}, \\
 & \tilde{q}_2^{(l,i)}(t) & := & q_1^{(l,i)}(t-t_0)+q_2^{(l,i)}, \\
 & & \vdots & \\
\underline{j=m_l}: & \tilde{q}_{m_l}^{(l,r)}(t) & := & q_1^{(l,r)}\dfrac{(t-t_0)^{m_l-1}}{(m_l-1)!}+\cdots+q_{m_l-1}^{(l,r)}(t-t_0)+q_{m_l}^{(l,r)}, \\
 & \tilde{q}_{m_l}^{(l,i)}(t) & := & q_1^{(l,i)}\dfrac{(t-t_0)^{m_l-1}}{(m_l-1)!}+\cdots+q_{m_l-1}^{(l,i)}(t-t_0)+q_{m_l}^{(l,i)},
\end{array}$$

$l\in J_{\nu_0}$, for every fixed $t\ge t_0$. With these abbreviations, we have

Lemma 10: Let the hypotheses $(H1^{'}),(H2^{'})$, and (HS') be fulfilled. Then, for every fixed $t\ge t_0$, the vectors $\tilde{q}_k^{(l,r)}(t),\tilde{q}_k^{(l,i)}(t),k=1,\cdots,m_l,l\in J_{\nu_0}$ are linearly independent if and only if the vectors $q_k^{(l,r)},q_k^{(l,i)},k=1,\cdots,m_l,l\in J_{\nu_0}$ are linearly independent.

Proof: $\Rightarrow$: Let $t=t_0$. Then, $\tilde{q}_k^{(l,r)}(t_0)=q_k^{(l,r)},\tilde{q}_k^{(l,i)}(t_0)=q_k^{(l,i)},k=1,\cdots,m_l,l\in J_{\nu_0}$ so that the assertion follows.

$\Leftarrow$: Let $t\ge t_0$ be fixed and let

$$\sum_{l\in J_{\nu_0}}\left\{\left[\tilde{\alpha}_1^{(l,r)}\tilde{q}_1^{(l,r)}(t)+\tilde{\alpha}_2^{(l,r)}\tilde{q}_2^{(l,r)}(t)+\cdots+\tilde{\alpha}_{m_l}^{(l,r)}\tilde{q}_{m_l}^{(l,r)}(t)\right]+\left[\tilde{\alpha}_1^{(l,i)}\tilde{q}_1^{(l,i)}(t)+\tilde{\alpha}_2^{(l,i)}\tilde{q}_2^{(l,i)}(t)+\cdots+\tilde{\alpha}_{m_l}^{(l,i)}\tilde{q}_{m_l}^{(l,i)}(t)\right]\right\} = 0.$$

This means

$$\begin{aligned}\sum_{l\in J_{\nu_0}}\Bigg\{&\Bigg[\tilde{\alpha}_1^{(l,r)}q_1^{(l,r)}+\tilde{\alpha}_2^{(l,r)}\left(q_1^{(l,r)}(t-t_0)+q_2^{(l,r)}\right)+\cdots \\
&+\tilde{\alpha}_{m_l}^{(l,r)}\left(q_1^{(l,r)}\frac{(t-t_0)^{m_l-1}}{(m_l-1)!}+\cdots+q_{m_l-1}^{(l,r)}(t-t_0)+q_{m_l}^{(l,r)}\right)\Bigg] \\
&+\Bigg[\tilde{\alpha}_1^{(l,i)}q_1^{(l,i)}+\tilde{\alpha}_2^{(l,i)}\left(q_1^{(l,i)}(t-t_0)+q_2^{(l,i)}\right)+\cdots \\
&+\tilde{\alpha}_{m_l}^{(l,i)}\left(q_1^{(l,i)}\frac{(t-t_0)^{m_l-1}}{(m_l-1)!}+\cdots+q_{m_l-1}^{(l,i)}(t-t_0)+q_{m_l}^{(l,i)}\right)\Bigg]\Bigg\} = 0,\end{aligned}$$

from which we infer

$$\begin{array}{rcl}
\tilde{\alpha}_1^{(l,r)} + (t-t_0)\tilde{\alpha}_2^{(l,r)} + \dfrac{(t-t_0)^2}{2}\tilde{\alpha}_3^{(l,r)} + \cdots + \dfrac{(t-t_0)^{m_l-1}}{(m_l-1)!}\tilde{\alpha}_{m_l}^{(l,r)} & = & 0, \\
\tilde{\alpha}_2^{(l,r)} + (t-t_0)\tilde{\alpha}_3^{(l,r)} + \cdots + \dfrac{(t-t_0)^{m_l-2}}{(m_l-2)!}\tilde{\alpha}_{m_l}^{(l,r)} & = & 0, \\
\cdots & & \\
\tilde{\alpha}_{m_l-1}^{(l,r)} + (t-t_0)\tilde{\alpha}_{m_l}^{(l,r)} & = & 0, \\
\tilde{\alpha}_{m_l}^{(l,r)} & = & 0.
\end{array}$$

This delivers

$$\tilde{\alpha}_1^{(l,r)}=\tilde{\alpha}_2^{(l,r)}=\cdots=\tilde{\alpha}_{m_l}^{(l,r)}, l\in J_{\nu_0};$$

correspondingly,

$$\tilde{\alpha}_1^{(l,i)}=\tilde{\alpha}_2^{(l,i)}=\cdots=\tilde{\alpha}_{m_l}^{(l,i)}, l\in J_{\nu_0},$$

so that the assertion follows.

Remark: The linear independence of the vectors for every $t\ge t_0$ is much a stronger statement than the linear independence of the associated functions.

Corollary 11: (Sufficient condition for $Y_{\nu_0}>0$ or $Y_0>0$)

Let $q_k^{(l,r)},q_k^{(l,i)},k=1,\cdots,m_l,l\in J_{\nu_0}$ be linearly independent. Then, $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}g_k^{(l)}(t)\neq 0, t\ge t_0$, so that $Y_{\nu_0}>0$ and therefore $Y_0=Y_{\nu_0}/2>0$, where $Y_{\nu_0}=\inf_{t\ge t_0}\|\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}g_k^{(l)}(t)\|$.

Proof: Assume that $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}g_k^{(l)}(\tilde{t})=0$ for at least one $\tilde{t}\ge t_0$. From the linear independence of $q_k^{(l,r)},q_k^{(l,i)},k=1,\cdots,m_l,l\in J_{\nu_0}$ and Lemma 10, one infers

$$\begin{array}{rcl}
c_k^{(l,r)}\cos\lambda_k^{(l)}(\tilde{t}-t_0)+c_k^{(l,i)}\sin\lambda_k^{(l)}(\tilde{t}-t_0) & = & 0, \\
-c_k^{(l,r)}\sin\lambda_k^{(l)}(\tilde{t}-t_0)+c_k^{(l,i)}\cos\lambda_k^{(l)}(\tilde{t}-t_0) & = & 0,
\end{array}$$

$k=1,\cdots,m_l, l\in J_{\nu_0}$. Thereby, we conclude $c_k^{(l,r)}=c_k^{(l,i)}=0, l\in J_{\nu_0}$. This is a contradiction to $c_{1k}^{(l)}=(c_k^{(l,r)}-c_k^{(l,i)})/2\neq 0$ for at least one $k=1,\cdots,m_l, l\in J_{\nu_0}$.

By a proof similar to that in Lemma 1, we have Sufficient algebraic condition for $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l} g_k^{(l)}(t)\neq 0,\quad t\geq t_0$: $q_k^{(l,r)}, q_k^{(l,i)}, k=1,\cdots,m_l, l\in J_{\nu_0}$ are linearly independent.

Likewise we have Sufficient algebraic condition for $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l} g_k^{(l)}(t)=0, \textit{ for at least one } t\geq t_0$: $J_{\nu_0}=\{l_0\}$ and $m_{l_0}=1$ as well as $q_1^{(l_0,r)}, q_1^{(l_0,i)}$ are linearly dependent, say, $q_1^{(l_0,i)}=\sigma_{l_0} q_1^{(l_0,r)}$.

d. Two-Sided Bound on $y(t)$ *by* $\|\psi(t)\|$

Let (70) be fulfilled so that (74) holds. Then, we have

Theorem 12: (Two-sided bound on $y(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1'), (H2')$, and (HS') be fulfilled and additionally (70). Then, for every $\varepsilon>0$ there exist constants $\eta_0(\varepsilon)>0$ and $\eta_1(\varepsilon)>0$ such that

$$\eta_0(\varepsilon)\|\psi(t)\|e^{-\varepsilon(t-t_0)}\leq\|y(t)\|\leq\eta_1(\varepsilon)\|\psi(t)\|e^{\varepsilon(t-t_0)},\quad t\geq t_1\geq t_0 \tag{75}$$

for sufficiently large t_1, where $t_1=t_0$ if $y(t)\neq 0, t\geq t_0$. Here, $\psi(t)$ is defined by (41). If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=\nu_0$ is equal to unity, then $\varepsilon=0$ can be chosen.

Proof: From (37),(74), it follows

$$\frac{Y_0}{X_1(\varepsilon)}\|x(t)\|e^{-\varepsilon(t-t_0)}\leq Y_0 e^{\nu_0(t-t_0)}\leq\|y(t)\|\leq Y_1(\varepsilon)e^{(\nu_0+\varepsilon)(t-t_0)}\leq\frac{Y_1(\varepsilon)}{X_0}\|x(t)\|e^{\varepsilon(t-t_0)}, \tag{76}$$

$t\geq t_1\geq t_0$. From (42), we infer

$$\frac{Y_0}{X_1(\varepsilon)}\xi_0\|\psi(t)\|e^{-\varepsilon(t-t_0)}\leq\|y(t)\|\leq\frac{Y_1(\varepsilon)}{X_0}\xi_1\|\psi(t)\|e^{\varepsilon(t-t_0)},\quad t\geq t_1\geq t_0;$$

set $\eta_0(\varepsilon):=\frac{Y_0}{X_1(\varepsilon)}\xi_0$ and $\eta_1(\varepsilon):=\frac{Y_1(\varepsilon)}{X_0}\xi_1$. Then, $\eta_0(\varepsilon)>0$ and $\eta_1(\varepsilon)>0$, and (75) is proven.

6. TWO-SIDED BOUNDS ON $z(t)=\dot y(t)$

In this section, we state bounds on $z(t)=\dot y(t)$ corresponding to those on $y(t)$ in Section 5. However, no proofs are given.

6.1. Diagonalizable Matrix A

a. Hypothesis on A

Again, we suppose $(H1), (H2)$, and (HS); further, $(H3)$ guarantees that the vectors $r_k=\lambda_k q_k, k=1,\cdots,m=2n$ are linearly dependent, if the vectors $q_k, k=1,\cdots,m$ are so, but later we need the stronger condition (80) below; $(H4)$ is needed only if the coefficients c_k are to be calculated by $c_k=(x_0,u_k^*), k=1,\cdots,m=2n$.

b. Representation of $\dot y_k^{(r)}(t), \dot y_k^{(i)}(t), k=1,\cdots,n$

From (2),

$$\begin{aligned}\dot y_k^{(r)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\cos\lambda_k^{(i)}(t-t_0)r_k^{(r)}-\sin\lambda_k^{(i)}(t-t_0)r_k^{(i)}\right],\\ \dot y_k^{(i)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\sin\lambda_k^{(i)}(t-t_0)r_k^{(r)}+\cos\lambda_k^{(i)}(t-t_0)r_k^{(i)}\right],\end{aligned} \tag{77}$$

$k=1,\cdots,n$.

Remark: Here, it is wise to take the form (77) or [4], (57), and *not* the form [4], (58) since with the form (77), it is easy to carry over the results of Section 5 from $x(t)$ to $\dot y(t)$.

c. Two-Sided Bound on $z(t)=\dot y(t)$ *by* $e^{\nu_0(t-t_0)}$

Starting Point: Appropriate Representation of $z(t)=\dot y(t)$

One has

$$z(t)=\dot y(t)=\sum_{k=1}^{n} e^{\lambda_k^{(r)}(t-t_0)}h_k(t) \tag{78}$$

with

$$\begin{aligned}h_k(t) &:= c_k^{(r)}[\cos\lambda_k^{(i)}(t-t_0)r_k^{(r)}-\sin\lambda_k^{(i)}(t-t_0)r_k^{(i)}]\\ &+ c_k^{(i)}[\sin\lambda_k^{(i)}(t-t_0)r_k^{(r)}+\cos\lambda_k^{(i)}(t-t_0)r_k^{(i)}],\end{aligned} \tag{79}$$

$k=1,\cdots,n$.

Case 1:

$$\sum_{k\in J_{\nu_0}} h_k(t)\neq 0,\quad t\geq t_0. \tag{80}$$

Case 2:

$$\sum_{k\in J_{\nu_0}} h_k(t)=0, \textit{ for at least one } t\geq t_0. \tag{81}$$

Here,

Theorem 13: (Two-sided bound on $z(t)=\dot y(t)$ by $e^{\nu_0(t-t_0)}$)

Let the hypotheses $(H1), (H2)$, and (HS) be fulfilled and additionally (80). Then, there exist constants $Z_0>0$ and $Z_1>0$ such that

$$Z_0 e^{\nu_0(t-t_0)}\leq\|z(t)\|\leq Z_1 e^{\nu_0(t-t_0)},\quad t\geq t_1\geq t_0 \tag{82}$$

for sufficiently large t_1, where the upper bound holds for $t_1=t_0$. If $z(t)\neq 0, t\geq t_0$, then $t_1=t_0$. If, instead of (80), condition (81) is fulfilled, then the lower bound is only valid with $Z_0=0$.

Sufficient algebraic condition for $\sum_{k\in J_{v_0}} h_k(t)\neq 0, \quad t\geq t_0$: $r_k^{(r)}, r_k^{(i)}, k\in J_{v_0}$ are linearly independent.

Sufficient algebraic condition for $\sum_{k\in J_{v_0}} h_k(t)=0$, *for at least one* $t\geq t_0$: $J_{v_0}=\{k_0\}$ and $r_{k_0}^{(r)}, r_{k_0}^{(i)}$ are linearly dependent, e.g.,

$$r_{k_o}^{(i)}=\mu_{k_o} r_{k_o}^{(r)}. \tag{83}$$

d. Two-Sided Bound on $z(t) = \dot{y}(t)$ by $\|\psi(t)\|$

One obtains

Theorem 14: (Two-sided bound on $z(t)=\dot{y}(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1)$, $(H2)$, and (HS) be fulfilled and additionally (80). Then, there exist constants $\zeta_0>0$ and $\zeta_1>0$ such that for sufficiently large $t_1\geq t_0$,

$$\zeta_0\|\psi(t)\|\leq\|z(t)\|\leq\zeta_1\|\psi(t)\|, \quad t\geq t_1\geq t_0, \tag{84}$$

with $\psi(t)$ defined by (18), where $t_1=t_0$ if $z(t)\neq 0, t\geq t_0$.

6.2. General Square Matrix A

a. Hypotheses on A

We suppose $(H1'), (H2')$, and (HS'); further, $(H3')$ guarantees that the vectors $r_k^{(l)}, k=1,\cdots,m_l, l\in J_{v_0}$ are linearly dependent, if the vectors $q_k^{(l)}, k=1,\cdots,m_l, l\in J_{v_0}$ are so, but later we need the stronger condition (88) below; $(H4')$ is needed only if the coefficients $c_{1k}^{(l)}$ are to be computed by $c_{1k}^{(l)}=(x_0, u_{m_l-k+1}^{(l)*}), k=1,\cdots,m_l, l=1,\cdots,\rho$.

b. Representation of the Basis $\dot{y}_k^{(l,r)}(t), \dot{y}_k^{(l,i)}(t), k=1,...,m_l, l=1,...,p$

One gets

$$\begin{aligned}
\dot{y}_k^{(l,r)}(t) &= e^{\lambda_l^{(r)}(t-t_0)}\Big\{\cos\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,r)}(t-t_0)+r_k^{(l,r)}\Big] \\
&\quad -\sin\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,i)}(t-t_0)+r_k^{(l,i)}\Big]\Big\}, \\
\dot{y}_k^{(l,i)}(t) &= e^{\lambda_l^{(r)}(t-t_0)}\Big\{\sin\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,r)}(t-t_0)+r_k^{(l,r)}\Big] \\
&\quad +\cos\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,i)}(t-t_0)+r_k^{(l,i)}\Big]\Big\},
\end{aligned} \tag{85}$$

$k=1,\cdots,m_l, l=1,\cdots,\rho$.

c. Two-Sided Bound on $z(t) = \dot{y}(t)$

Starting point: Appropriate Representation of $z(t) = \dot{y}(t)$

We have

$$z(t)=\dot{y}(t)=\sum_{l=1}^{\rho} e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l} h_k^{(l)}(t) \tag{86}$$

with

$$\begin{aligned}
h_k^{(l)}(t) &:= c_k^{(l,r)}\Big\{\cos\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,r)}(t-t_0)+r_k^{(l,r)}\Big] \\
&\quad -\sin\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,i)}(t-t_0)+r_k^{(l,i)}\Big]\Big\} \\
&\quad + c_k^{(l,i)}\Big\{\sin\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,r)}(t-t_0)+r_k^{(l,r)}\Big] \\
&\quad +\cos\lambda_l^{(i)}(t-t_0)\Big[r_1^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+r_{k-1}^{(l,i)}(t-t_0)+r_k^{(l,i)}\Big]\Big\}
\end{aligned} \tag{87}$$

$k=1,\cdots,m_l, l=1,\cdots,\rho$.

Case 1:

$$\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} h_k^{(l)}(t)\neq 0, \quad t\geq t_0. \tag{88}$$

Case 2:

$$\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} h_k^{(l)}(t)=0, \text{ for at least one } t\geq t_0. \tag{89}$$

We have

Theorem 15: (Two-sided bound on $z(t)=\dot{y}(t)$ by $e^{v_0(t-t_0)}$)

Let the hypotheses $(H1')$, $(H2')$, and (HS') be fulfilled and additionally (88). Then, there exists a constant $Z_0>0$ and for every $\varepsilon>0$ a constant $Z_1(\varepsilon)>0$ such that

$$Z_0 e^{v_0(t-t_0)}\leq\|z(t)\|\leq Z_1(\varepsilon)e^{(v_0+\varepsilon)(t-t_0)}, \quad t\geq t_1\geq t_0 \tag{90}$$

for sufficiently large t_1, where $t_1=t_0$ if $z(t)\neq 0, t\geq t_0$. If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=v_0$ is equal to unity, then $\varepsilon=0$ can be chosen. If, instead of (88), condition (89) is fulfilled, then the lower bound is only valid with $Z_0=0$.

Sufficient algebraic condition for $\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} h_k^{(l)}(t)\neq 0, \quad t\geq t_0$: $r_k^{(l,r)}, r_k^{(l,i)}, k=1,\cdots,m_l, l\in J_{v_0}$ are linearly independent.

Sufficient algebraic condition for $\sum_{l\in J_{v_0}}\sum_{k=1}^{m_l} h_k^{(l)}(t)=0$, *for at least one* $t\geq t_0$: $J_{v_0}=\{l_0\}$ and $m_{l_0}=1$ as well as $r_1^{(l_0,r)}, r_1^{(l_0,i)}$ are linearly dependent, say,

$$r_1^{(l_0,i)}=\mu_{l_0} r_1^{(l_0,r)}. \tag{91}$$

d. Two-Sided Bound on $z(t)=\dot{y}(t)$ by $\|\psi(t)\|$

We have

Theorem 16: (Two-sided bound on $z(t)=\dot{y}(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1'),(H2')$, and (HS') be fulfilled and additionally (88). Then, for every $\varepsilon>0$ there exist constants $\zeta_0(\varepsilon)>0$ and $\zeta_1(\varepsilon)>0$ such that

$$\zeta_0(\varepsilon)\|\psi(t)\|e^{-\varepsilon(t-t_0)}\le\|z(t)\|\le\zeta_1(\varepsilon)\|\psi(t)\|e^{\varepsilon(t-t_0)},\quad t\ge t_1\ge t_0 \tag{92}$$

for sufficiently large t_1, where $t_1=t_0$ if $y(t)\ne 0, t\ge t_0$. Here, $\psi(t)$ is defined by (41). If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=\nu_0$ is equal to unity, then $\varepsilon=0$ can be chosen.

7. TWO-SIDED BOUNDS ON $x_S(t)$ WITH $S\subset\{1,\ldots,m=2n\}$

In this subsection, we derive two-sided bounds on $x_S(t)$ similar to those on $x(t)$, where $S\subset\{1,\cdots,m=2n\}$ is any subset.

7.1. Diagonalizable Matrix A

Important special cases of S are as follows:

$$\begin{array}{lll} S=\{1,\cdots,m=2n\} & \Rightarrow\ x_S(t)=x(t), & p_{S,k}=p_k,\ k=1,\cdots,m=2n\\ S=\{1,\cdots,n\} & \Rightarrow\ x_S(t)=y(t), & p_{S,k}=q_k,\ k=1,\cdots,n\\ S=\{n+1,\cdots,2n\} & \Rightarrow\ x_S(t)=\dot{y}(t), & p_{S,k}=r_k,\ k=1,\cdots,n\\ S=\{j_0 | 1\le j_0\le n\} & \Rightarrow\ x_S(t)=y_{j_0}(t), & p_{S,k}=(q_k)_{j_0},\ k=1,\cdots,n\\ S=\{n+j_0 | 1\le j_0\le n\} & \Rightarrow\ x_S(t)=\dot{y}_{j_0}(t), & p_{S,k}=(r_k)_{j_0},\ k=1,\cdots,n \end{array} \tag{93}$$

where $(q_k)_{j_0}$ means the j_0 th component of vector q_k and so on.

We suppose that $(H1),(H2)$, and (HS); instead of $(H3)$, condition (97) below will be used here.

b. Representation of the Basis $x_{S,k}^{(r)}(t), x_{S,k}^{(i)}(t), k=1,\cdots,n$

From (2),

$$\begin{aligned} x_{S,k}^{(r)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\cos\lambda_k^{(i)}(t-t_0)p_{S,k}^{(r)}-\sin\lambda_k^{(i)}(t-t_0)p_{S,k}^{(i)}\right],\\ x_{S,k}^{(i)}(t) &= e^{\lambda_k^{(r)}(t-t_0)}\left[\sin\lambda_k^{(i)}(t-t_0)p_{S,k}^{(r)}+\cos\lambda_k^{(i)}(t-t_0)p_{S,k}^{(i)}\right], \end{aligned} \tag{94}$$

with

$$\begin{aligned} p_{S,k}^{(r)} &= Re\{p_{S,k}\},\\ p_{S,k}^{(i)} &= Im\{p_{S,k}\}, \end{aligned}$$

$k=1,\cdots,n$,

c. Two-Sided Bound on $x_S(t)$ by $e^{\nu_0(t-t_0)}$

Starting point: Appropriate Representation of $x_S(t)$

We have

$$x_S(t)=\sum_{k=1}^{n}e^{\lambda_k^{(r)}(t-t_0)}f_{S,k}(t) \tag{95}$$

with

$$\begin{aligned} f_{S,k}(t) &:= c_k^{(r)}[\cos\lambda_k^{(i)}(t-t_0)p_{S,k}^{(r)}-\sin\lambda_k^{(i)}(t-t_0)p_{S,k}^{(i)}]\\ &+ c_k^{(i)}[\sin\lambda_k^{(i)}(t-t_0)p_{S,k}^{(r)}+\cos\lambda_k^{(i)}(t-t_0)p_{S,k}^{(i)}], \end{aligned} \tag{96}$$

$k=1,\cdots,n$.

<u>Case 1:</u>

$$\sum_{k\in J_{\nu_0}} f_{S,k}(t)\ne 0,\quad t\ge t_0. \tag{97}$$

<u>Case 2:</u>

$$\sum_{k\in J_{\nu_0}} f_{S,k}(t)=0,\ \textit{for at least one}\ t\ge t_0. \tag{98}$$

Here,

Theorem 17: (Two-sided bound on $x_S(t)$ by $e^{\nu_0(t-t_0)}$)

Let the hypotheses $(H1),(H2)$, and (HS) be fulfilled and additionally (97). Then, there exist constants $X_{S,0}>0$ and $X_{S,1}>0$ such that

$$X_{S,0}\,e^{\nu_0(t-t_0)}\le\|x_S(t)\|\le X_{S,1}\,e^{\nu_0(t-t_0)},\quad t\ge t_1\ge t_0 \tag{99}$$

for sufficiently large t_1, where the upper bound holds for $t_1=t_0$. If $x_S(t)\ne 0, t\ge t_0$, then $t_1=t_0$. If, instead of (97), condition (98) is fulfilled, then the lower bound is only valid with $X_{S,0}=0$.

Sufficient algebraic condition for $\sum_{k\in J_{\nu_0}} f_{S,k}(t)\ne 0,\ t\ge t_0$:

$p_{S,k}^{(r)}, p_{S,k}^{(i)}, k\in J_{\nu_0}$ are linearly independent.

Sufficient algebraic condition for $\sum_{k\in J_{\nu_0}} f_{S,k}(t)=0,\ \textit{for at least one}\ t\ge t_0$:

$J_{\nu_0}=\{k_0\}$ and $p_{S,k_0}^{(r)}, p_{S,k_0}^{(i)}$ are linearly dependent, e.g.,

$$p_{S,k_0}^{(i)}=\sigma_{S,k_0}\,p_{S,k_0}^{(r)}. \tag{100}$$

d. Two-Sided Bound on $x_S(t)$ by $\|\psi(t)\|$

One obtains

Theorem 18: (Two-sided bound on $x_S(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1),(H2)$, and (HS) be fulfilled and additionally (97). Then, there exist constants $\xi_{S,0}>0$ and $\xi_{S,1}>0$ such that for sufficiently large $t_1\ge t_0$,

$$\xi_{S,0}\|\psi(t)\|\le\|x_S(t)\|\le\xi_{S,1}\|\psi(t)\|,\quad t\ge t_1\ge t_0, \tag{101}$$

with $\psi(t)$ defined by (18), where $t_1=t_0$ if $x_S(t)\ne 0, t\ge t_0$.

7.2. General Square Matrix A

Similarly as in 7.1, we have the following important special cases of S:

$$\begin{array}{lll} S=\{1,\cdots,m=2n\} & \Rightarrow & x_S(t)=x(t),\quad p_{S,k}^{(l)}=p_k^{(l)},\ k=1,\cdots,m_l,\ l=1,\cdots,m=2\rho \\ S=\{1,\cdots,n\} & \Rightarrow & x_S(t)=y(t),\quad p_{S,k}^{(l)}=q_k^{(l)},\ k=1,\cdots,m_l,\ l=1,\cdots,m=2\rho \\ S=\{n+1,\cdots,2n\} & \Rightarrow & x_S(t)=\dot{y}(t),\quad p_{S,k}^{(l)}=r_k^{(l)},\ k=1,\cdots,m_l,\ l=1,\cdots,m=2\rho \end{array} \quad (102)$$

and so on.

a. Hypotheses on A

We suppose $(H1^{'}),(H2^{'})$, and $(HS^{'})$; instead of $(H3^{'})$, condition (106) below is used here.

b. Representation of the Basis $x_{S,k}^{(l,r)}(t), x_{S,k}^{(l,i)}(t), k=1,\cdots,m_l$, $l=1,\cdots,\rho$

From (25),

$$\begin{aligned} x_{S,k}^{(l,r)}(t) &= e^{\lambda_l^{(r)}(t-t_0)}\Big\{\cos\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,r)}(t-t_0)+p_{S,1}^{(l,r)}\Big] \\ &\quad -\sin\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,i)}(t-t_0)+p_{S,1}^{(l,i)}\Big]\Big\}, \\ x_{S,k}^{(l,i)}(t) &= e^{\lambda_l^{(r)}(t-t_0)}\Big\{\sin\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,r)}(t-t_0)+p_{S,k}^{(l,r)}\Big] \\ &\quad +\cos\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,i)}(t-t_0)+p_{S,1}^{(l,i)}\Big]\Big\}, \end{aligned} \quad (103)$$

$k=1,\cdots,m_l, l=1,\cdots,\rho$.

c. Two-Sided Bound on $x_S(t)$ by $e^{\nu_0(t-t_0)}$

Starting point: Appropriate Representation of $x_S(t)$

We have

$$x_S(t)=\sum_{l=1}^{\rho}e^{\lambda_l^{(r)}(t-t_0)}\sum_{k=1}^{m_l}f_{S,k}^{(l)}(t) \quad (104)$$

with

$$\begin{aligned} f_{S,k}^{(l)}(t) \;:=\; & c_k^{(l,r)}\Big\{\cos\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,r)}(t-t_0)+p_{S,k}^{(l,r)}\Big] \\ & -\sin\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,i)}(t-t_0)+p_{S,k}^{(l,i)}\Big]\Big\} \\ + \; & c_k^{(l,i)}\Big\{\sin\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,r)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,r)}(t-t_0)+p_{S,k}^{(l,r)}\Big] \\ & +\cos\lambda_l^{(i)}(t-t_0)\Big[p_{S,1}^{(l,i)}\frac{(t-t_0)^{k-1}}{(k-1)!}+\cdots+p_{S,k-1}^{(l,i)}(t-t_0)+p_{S,k}^{(l,i)}\Big]\Big\} \end{aligned} \quad (105)$$

$k=1,\cdots,m_l, l=1,\cdots,\rho$.

Case 1:

$$\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}f_{S,k}^{(l)}(t)\neq 0,\quad t\geq t_0. \quad (106)$$

Case 2:

$$\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}f_{S,k}^{(l)}(t)=0,\ \textit{for at least one } t\geq t_0. \quad (107)$$

We have

Theorem 19: (Two-sided bound on $z(t)=\dot{y}(t)$ by $e^{\nu_0(t-t_0)}$)

Let the hypotheses $(H1^{'})$, $(H2^{'})$, (HS') and be fulfilled and additionally (106). Then, there exists a constant $X_{S,0}>0$ and for every $\varepsilon>0$ a constant $X_{S,1}(\varepsilon)>0$ such that

$$X_{S,0}\,e^{\nu_0(t-t_0)}\leq\|x_S(t)\|\leq X_{S,1}(\varepsilon)e^{(\nu_0+\varepsilon)(t-t_0)},\quad t\geq t_1\geq t_0 \quad (108)$$

for sufficiently large t_1, where $t_1=t_0$ if $x_S(t)\neq 0, t\geq t_0$. If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=\nu_0$ is equal to unity, i.e., $\iota(\lambda)=1$, then $\varepsilon=0$ can be chosen. If, instead of (106), condition (107) is fulfilled, then the lower bound is only valid with $X_{S,0}=0$.

Sufficient algebraic condition for $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}f_{S,k}^{(l)}(t)\neq 0, t\geq t_0$:

$p_{S,k}^{(l,r)}, p_{S,k}^{(l,i)}, k=1,\cdots,m_l, l\in J_{\nu_0}$ are linearly independent.

Sufficient algebraic condition for $\sum_{l\in J_{\nu_0}}\sum_{k=1}^{m_l}f_{S,k}^{(l)}(t)\neq 0,\textit{for at least one} t\geq t_0$ $J_{\nu_0}=\{l_0\}$ and $m_{l_0}=1$ as well as $p_{S,1}^{(l_0,r)}, p_{S,1}^{(l_0,i)}$ are linearly dependent, say,

$$p_{S,1}^{(l_0,i)}=\sigma_{S,l_0}\,p_{S,1}^{(l_0,r)}. \quad (109)$$

d. Two-Sided Bound on $x_S(t)$ by $\|\psi(t)\|$

We have

Theorem 20: (Two-sided bound on $x_S(t)$ by $\|\psi(t)\|$)

Let the hypotheses $(H1^{'})$, $(H2^{'})$, and (HS') be fulfilled and additionally (106). Then, for every $\varepsilon>0$ there exist constants $\xi_{S,0}(\varepsilon)>0$ and $\xi_{S,1}(\varepsilon)>0$ such that

$$\xi_{S,0}(\varepsilon)\|\psi(t)\|e^{-\varepsilon(t-t_0)}\leq\|x_S(t)\|\leq\xi_{S,1}(\varepsilon)\|\psi(t)\|e^{\varepsilon(t-t_0)},\quad t\geq t_1\geq t_0 \quad (110)$$

for sufficiently large t_1, where $t_1=t_0$ if $x_S(t)\neq 0, t\geq t_0$. Here, $\psi(t)$ is defined by (41). If the index of every eigenvalue $\lambda=\lambda(A)$ with $Re\lambda=\nu_0$ is equal to unity, then $\varepsilon=0$ can be chosen.

8. APPLICATIONS

8.1. The Vibration Problem

Consider the multi-mass vibration model in Fig. (**1**).

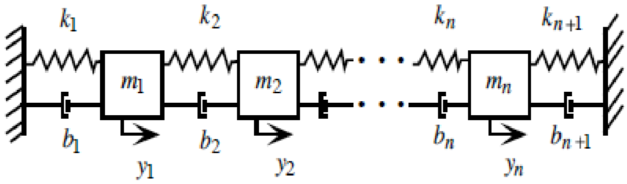

Fig. (1). Multi-mass vibration model.

The associated initial-value problem is given by

$$M\ddot{y}+B\dot{y}+Ky=0,\quad y(0)=y_0,\ \dot{y}(0)=\dot{y}_0$$

where $y=[y_1,\cdots,y_n]^T$ and

$$M=\begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_n \end{bmatrix},$$

$$B=\begin{bmatrix} b_1+b_2 & -b_2 & & & \\ -b_2 & b_2+b_3 & -b_3 & & \\ & -b_3 & b_3+b_4 & -b_4 & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & -b_n & b_n+b_{n+1} \end{bmatrix},$$

$$K=\begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & -k_3 & k_3+k_4 & -k_4 & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{bmatrix},$$

or, in the *state-space description* $\dot{x}(t)=Ax(t),\quad x(0)=x_0$,

where the *state vector* x is given by $x=[y^T,z^T]^T,\ z=\dot{y}$, and where the *system matrix* A has the form

$$A=\left[\begin{array}{c|c} O & E \\ \hline -M^{-1}K & -M^{-1}B \end{array}\right].$$

As in [2], we specify the values as

$$\begin{aligned} m_j &= 1,\ j=1,\cdots,n \\ k_j &= 1,\ j=1,\cdots,n+1 \end{aligned}$$

and

$$b_j = \begin{cases} 1/2, & j\ even \\ 1/4, & j\ odd. \end{cases}$$

With the above numerical values, we have $M=E$,

$$B=\begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & & & \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & & \\ & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & \\ & & \ddots & \ddots & \ddots \\ & & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ & & & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

(if n is even), and

$$K=\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Further, let

$$y_0=[-1,1,-1,1,-1]^T,\quad \dot{y}_0=[0,0,0,0,0]^T$$

so that $x_0=[y_0^T,z_0^T]^T$ with $z_0=\dot{y}_0$. We choose **n=5** in this paper so that $m=2n=10$. Thus, $M,B,K\in IR^{5\times 5}$ and $A\in IR^{10\times 10}$. Finally,

$t_0=0.$

Here, we obtain $\lambda_i\neq\lambda_j, i\neq j, i,j=1,\cdots,m=10$; therefore, A is diagonalizable, and $\varepsilon=0$ can be set. Further, for $u_0=T^{-1}x_0$, we have $u_{0,j}\neq 0, j=1,\cdots,m=10$ so that $v_0=v_{x_0}[A]=v[A]=max_{j=1,\cdots,10}Re\lambda_j(A)\doteq -0.050239$ (see [1]). We remark that $BM^{-1}K\neq KM^{-1}B$.

Enumerating the eigenvalues such that $\lambda_j^{(i)}>0$ and $\lambda_j^{(r)}$ are increasing for $j=1,\cdots,5$ as well as $\lambda_{j+5}=\overline{\lambda}_j$, $j=1,\cdots,5$, we obtain

$$\begin{aligned} \lambda_1 &= -0.69976063878053+1.79598147815975i, \\ \lambda_2 &= -0.56266837404074+1.61635870164386i, \\ \lambda_3 &= -0.37500000000000+1.36358901432946i, \\ \lambda_4 &= -0.18733162595926+0.99452168646559i, \\ \lambda_5 &= -0.05023936121946+0.51637145071101i, \end{aligned}$$

and thus $J_{v_0}=\{5\}$. Further,

$$p_5 == \begin{bmatrix} q_5 \\ \hline r_5 \end{bmatrix} = \begin{bmatrix} 0.25786399565391+0.01653656897030i \\ 0.44351627218028+0.00006053419538i \\ 0.51248089685635 \\ 0.44351627218028+0.00006053419538i \\ 0.25461690120244-0.01653656897030i \\ \hline -0.02149393453213+0.13232281886012i \\ -0.02231323233506+0.22901609968036i \\ -0.02574671289524+0.26463050417139i \\ -0.02231323233506+0.22901609968036i \\ -0.00425277836311+0.13230768531127i \end{bmatrix}.$$

Here, $q_5^{(r)}, q_5^{(i)}$ resp. $r_5^{(r)}, r_5^{(i)}$ are linearly independent. Thus, the constant Y_0 in Theorem 7 resp. the constant Z_0 in Theorem 13 is positive.

8.2. Two-Sided Bounds on *y(t)*

In Section 5.1, we have derived the bounds

$$Y_{0,2}\, e^{\nu_0(t-t_0)} \le \| y(t) \|_2 \le Y_{1,2}\, e^{\nu_0(t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{111}$$

and

$$\eta_{0,2} \, \| \psi(t) \| \le \| y(t) \|_2 \le \eta_{1,2} \, \| \psi(t) \|, \quad t \ge t_1 \ge t_0, \tag{112}$$

with positive constants $Y_{0,2}, Y_{1,2}$ and $\eta_{0,2}, \eta_{1,2}$. The best constants $Y_{1,2}$ and $\eta_{1,2}$ are obtained by the differential calculus of norms. For example, the optimal constant $\eta_{1,2}$ is computed from the conditions

$$\| y(t_{c,2}) \|_2 \overset{!}{=} \eta_{1,2} \, \| \psi(t_{c,2}) \|_2,$$

$$D_+ \| y(t_{c,2}) \|_2 \overset{!}{=} \eta_{1,2} D_+ \| \psi(t_{c,2}) \|_2,$$

which leads to

$$\frac{D_+ \| y(t_{c,2}) \|_2}{\| y(t_{c,2}) \|_2} = \frac{D_+ \| \psi(t_{c,2}) \|_2}{\| \psi(t_{c,2}) \|_2}$$

or

$$D_+ \| y(t_{c,2}) \| \, \| \psi(t_{c,2}) \|_2 - D_+ \| \psi(t_{c,2}) \|_2 \, \| y(t_{c,2}) \| = 0.$$

When the point of contact $t_{c,2}$ between $y = \| y(t) \|_2$ and $y = \| \psi(t) \|_2$ has been computed from this nonlinear algebraic equation in $t_{c,2}$, then $\eta_{1,2}$ is obtained from

$$\eta_{1,2} = \frac{\| y(t_{c,2}) \|}{\| \psi(t_{c,2}) \|_2}.$$

The lower bound $y = Y_{0,2}\, e^{\nu_0(t-t_0)}$ resp. $y = \eta_{0,2} \| \psi(t) \|_2$ meets the curve $y = \| y(t) \|_2$ at the point $t_{c,2}$, where it has a kink like $|t|^{1/2}$ at $t=0$ (in contrast to the curve $y = \| x(t) \|_2$) as can be clearly seen from the plot of $y = \| y(t) \|_2$ for $25 \le t \le 50$ (not presented here). Therefore, $t_{c,2}$ cannot be determined by the differential calculus of norms; instead, it must be computed from

$$t_{c,2} = \min_{j=1,2,\cdots} \| y(t_{j,2}) \|_2,$$

where $t_{j,2}, j = 1, 2, \cdots$ are the local minima of $y = \| y(t) \|_2$. In the sequel, we use the additional index l for *lower bound* and the additional index u for the *upper bound*. In this way, for (111), we get

$$t_{c,l,2} \doteq 27.591842, \quad Y_{0,2} \doteq 0.00307318,$$

as well as

$$t_{c,u,2} \doteq 0.014088, \quad Y_{1,2} \doteq 2.236857.$$

In Fig. (**2**), the curve $y = \| y(t) \|_2$ and the upper bound $y = Y_{1,2}\, e^{\nu_0(t-t_0)}$ are drawn.

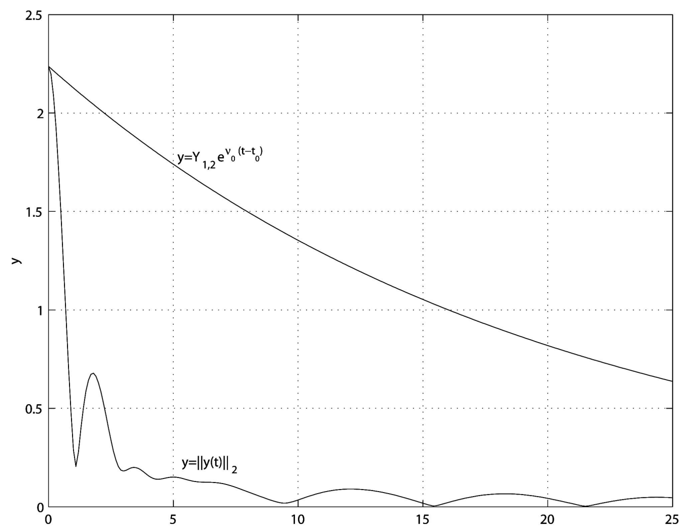

Fig. (2). $y = \| y(t) \|_2$ and *optimal* upper bound $y = Y_{1,2}\, e^{\nu_0(t-t_0)}$

For (112), we obtain

$$t_{c,l,2} \doteq 27.591842, \quad \eta_{0,2} \doteq 0.028839,$$

as well as

$$t_{c,u,2} \doteq 48.885432, \quad \eta_{1,2} \doteq 1.557559.$$

In Fig. (**3**), the curve $y = \| y(t) \|_2$ and the upper bound $y = \eta_{1,2} \| \psi(t) \|_2$ are plotted, and in Fig. (**4**), the curve $y = \| y(t) \|_2$ along with the two-sided bounds $y = \eta_{0,2} \| \psi(t) \|_2$ and $y = \eta_{1,2} \| \psi(t) \|_2$. The upper bound $y = \eta_{1,2} \| \psi(t) \|_2$ depends on x_0 and adapts faster to the curve $y = \| y(t) \|_2$ than the upper bound $y = Y_{1,2}\, e^{\nu_0(t-t_0)}$; but, in the initial

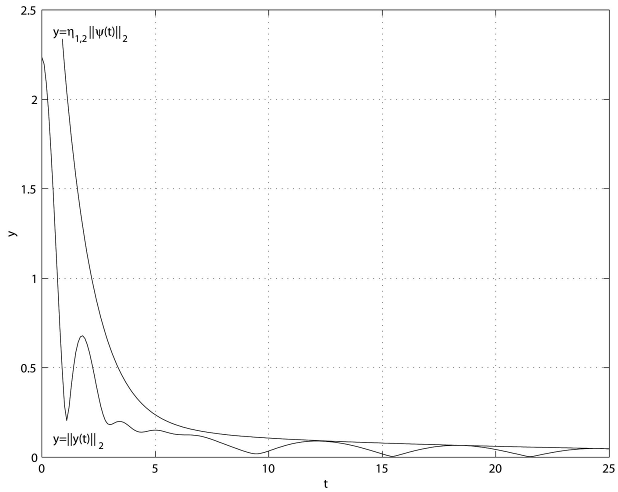

Fig. (3). $y = \| y(t) \|_2$ and *optimal* upper bound $y = \eta_{1,2} \| \psi(t) \|_2$

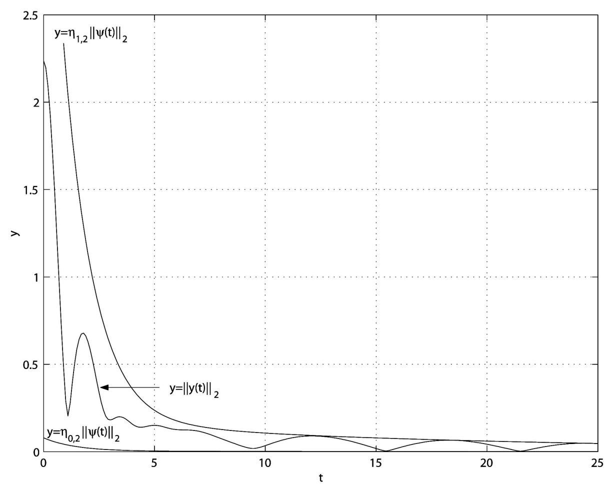

Fig. (4). $y = \| y(t) \|_2$ and *optimal* two-sided bounds $y = \eta_{0,2} \| \psi(t) \|_2$ and $y = \eta_{1,2} \| \psi(t) \|_2$

domain, $y = \eta_{1,2} \| \psi(t) \|_2$ is worse than $y = Y_{1,2}\, e^{\nu_0 (t - t_0)}$. This can be remedied, however, by the method described in [7].

8.3. Two-Sided Bounds on *z(t)* = *ẏ(t)*

In Section 6.1, we have derived the bounds

$$Z_{0,2}\, e^{\nu_0 (t-t_0)} \le \| z(t) \|_2 \le Z_{1,2}\, e^{\nu_0 (t-t_0)}, \quad t \ge t_1 \ge t_0 \tag{113}$$

and

$$\zeta_{0,2} \,\| \psi(t) \| \le \| z(t) \|_2 \le \zeta_{1,2} \,\| \psi(t) \|, \quad t \ge t_1 \ge t_0, \tag{114}$$

with positive constants $Z_{0,2}, Z_{1,2}$ and $\zeta_{0,2}, \zeta_{1,2}$. The best constants are computed like $Y_{1,2}, Y_{1,2}$ and $\eta_{1,2}, \eta_{1,2}$ in Section 8.2. For (113), we get

$$t_{c,l,2} \doteq 42.613832,$$
$$Z_{0,2} \doteq 0.002200,$$

as well as

$$t_{c,u,2} \doteq 0.688631,$$
$$Z_{1,2} \doteq 2.758394.$$

In Fig. (**5**), the curve $y = \| z(t) \|_2$ and the upper bound $y = Z_{1,2}\, e^{\nu_0 (t-t_0)}$ are drawn.

For (114), we obtain

$$t_{c,l,2} \doteq 24.357362,$$
$$\zeta_{0,2} \doteq 0.016083,$$

as well as

$$t_{c,u,2} \doteq 0.879345,$$
$$\zeta_{1,2} \doteq 1.632277.$$

In Fig. (**6**), the curve $y = \| z(t) \|_2$ and the upper bound $y = \zeta_{1,2} \| \psi(t) \|_2$ are plotted, and in Fig. (**7**), the curve $y = \| z(t) \|_2$ along with the two-sided bounds $y = \zeta_{0,2} \| \psi(t) \|_2$ and $y = \zeta_{1,2} \| \psi(t) \|_2$. Similar remarks to those at the end of Section 8.2 hold.

8.4. Computational Aspects

In this subsection, we say something about the used computer equipment and the computation time.

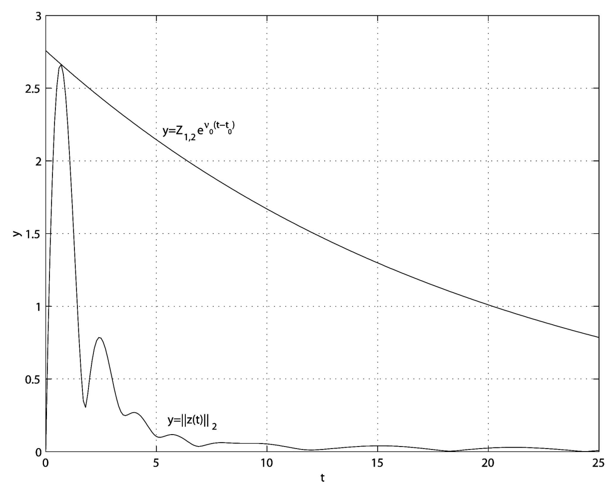

Fig. (5). $y = \| z(t) \|_2$ and *optimal* upper bound $y = Z_{1,2}\, e^{\nu_0 (t-t_0)}$

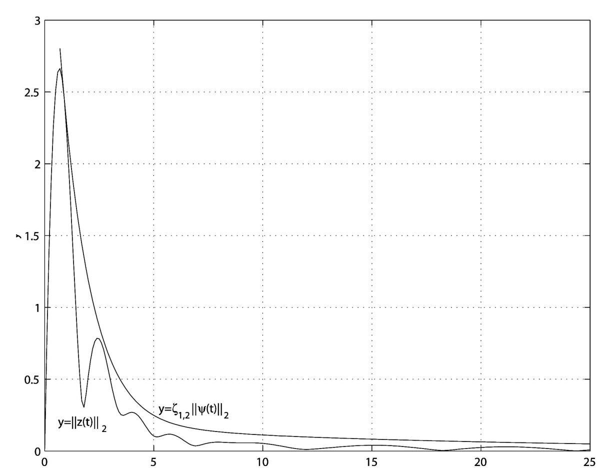

Fig. (6). $y = \| z(t) \|_2$ and *optimal* upper bound $y = \zeta_{1,2} \| \psi(t) \|_2$

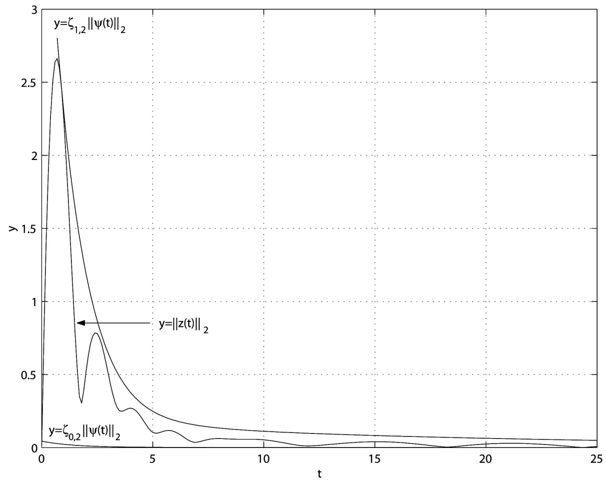

Fig. (7). $y = \| z(t) \|_2$ and *optimal* two-sided bounds $y = \zeta_{0,2} \| \psi(t) \|_2$ and $y = \zeta_{1,2} \| \psi(t) \|_2$

(i) As to the *computer equipment*, the following hardware was available: a Pentium II CPU at 300 MHz, an 8 GB mass storage facility, two SDRAM 64 MB high-speed memories. As software package for the computations, we used 368-Matlab, Version 4.2.c, for the generation of the figures, Version 6.0, in order to be able to caption them.

(ii) The *computation time* t of an operation was determined by the command sequence *t1=clock; operation; t=etime(clock,t1)*; it is put out in seconds rounded to two decimal places, by MATLAB. For example, to compute the points of contact and to generate the table of values $t, y(t), y_u(t), y_l(t), t = 0(0.1)25$ for Figs. (**4** and **7**), we obtained $t_4 = 2.26s$ and $t_7 = 1.93s$.

9. CONCLUSION

In this paper, for the vibration problem $M\ddot{y} + B\dot{y} + Ky = 0, y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0$, two-sided bounds on the displacement vector $y(t)$ and the velocity vector $\dot{y}(t)$ are derived. These bounds have the same shapes as the two-sided bounds on the solution $x(t)$ of the corresponding state-space problem $\dot{x} = Ax, x(t_0) = x_0$. Even two-sided bounds on any quantity $x_S(t)$ where $S \subset \{1, \cdots, m = 2n\}$ are obtained. The differential calculus of norms is used to compute the optimal constants in the upper bounds, whereas the best lower bounds must be determined via the local minima since the curves have kinks there. Along with the papers [4-6], one is now able to handle the solution of the above vibration problem in nearly the same way as for one-mass models. Therefore, the papers [4-6] and this paper make a major contribution to *Computational Engineering*, especially to *Computational Mechanics*. It could turn out to be of great value also in *Computational Electrics*. In retrospect, a mathematician might be tempted to derive the two-sided bounds on $x_S(t)$ first and to obtain the two-sided bounds on $x(t)$, $y(t)$, and $z(t) = \dot{y}(t)$ as special cases. We have not done this here since the results are mainly of interest to engineers and since we think the presented way will be more convenient and simpler to understand for them. The method described here is also applicable to models with system matrix $A \in IR^{m \times m}$ where m is an odd natural number such as $A \in IR^{3 \times 3}$ from [3], Fig. 3.1, when the appropriate adaptations are made for the solution bases.

10. COMMENTS ON THE REFERENCES

The References [1, 2, 4-16] contain the most important contributions to the author's current research area. In [3, 17-22] the reader finds dynamical problems of interest with respect to the present paper. In the References [23-26] there is some material on Linear Algebra useful in the context of the paper. The references [27] and [28] are on functional analytical methods used in the author's work. Finally, [29] is a useful reference book on numerical solution methods of ordinary differential equations consulted by the author in his work.

11. OUTLOOK ON FUTURE WORK

The question naturally arises as to whether the method presented for the IVP $\dot{x} = Ax, x(t_0) = x_0$ in this paper can be carried over to more general differential equations. In order to assess the chances to be able to do this, we want to look back to what was possible in the past work. What can be said is the following: In [12], it was possible to treat problems with periodic system matrix, i.e. the IVP $\dot{x} = A(t)x, x(t_0) = x_0$ with $A(t) = A(t + t_p)$ in a similar way as for $\dot{x} = Ax, x(t_0) = x_0$, more precisely, it was possible to derive an upper bound on $x(t)$ of the same form as for the case of a constant matrix. Further, in [7], the same held for the quasilinear IVP $\dot{x} = Ax + h(t, x), x(t_0) = x_0$. Therefore, one can be optimistic to carry over the results of the present paper to the case of an IVP with periodic system matrix and to the case of a quasilinear IVP. These issues will be the subject of the author's pertinent future scientific work.

ACKNOWLEDGEMENT

The author would like to thank one of the referees for some useful comments.

REFERENCES

[1] Kohaupt L. Computation of optimal two-sided bounds for the asymptotic behavior of free linear dynamical systems with application of the differential calculus of norms. J Comput Math Optimi 2006; 2: 127-73.

[2] Kohaupt L. Solution of the matrix eigenvalue problem $VA + A^*V = \mu V$ with applications to the study of free linear systems. J Comp Appl Math 2008; 213: 142-64.

[3] Müller PC, Schiehlen WO. Linear Vibrations. Dordrecht Boston Lancaster: Martinus Nijhoff Publishers 1985.

[4] Kohaupt L. Solution of the vibration problem $M\ddot{y} + B\dot{y} + Ky = 0$, $y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0$ without the hypothesis $BM^{-1}K = KM^{-1}B$ or $B = \alpha M + \beta K$. Appl Math Sci 2008; 2: 1989-2024.

[5] Kohaupt L. Construction of a biorthogonal system of principal vectors for the matrices A and A^* with applications to $\dot{x} = Ax, x(t_0) = x_0$. J Comput Math Optimi 2007; 3: 163-92.

[6] Kohaupt L. A biorthogonalization method with applications to the explicit representation of the solution $x(t)$ of $\dot{x} = Ax, x(t_0) = x_0$. Appl Math Sci 2008; 2: 961-74.

[7] Kohaupt L. New upper bounds for free linear and nonlinear vibration systems with applications to the differential calculus of norms. Applied Mathematical Modeling 2004; 4; 367-88.
[8] Kohaupt L. Second Logarithmic Derivative of a Complex Matrix in the Chebyshev Norm. SIAM J Matrix Anal Appl 1999; 21: 382-9.
[9] Kohaupt L. Differential calculus for some p-norms of the fundamental matrix with applications. J Comp Appl Math 2001; 135: 1-21.
[10] Kohaupt L. Differential calculus for p-norms of complex-valued vector functions with applications. J Comp Appl Math 2002; 145: 425-57.
[11] Kohaupt L. Extension and further development of the differential calculus for matrix norms with applications. J Comp Appl Math 2003; 156: 433-56.
[12] Kohaupt L. Differential calculus for the matrix norms $|\cdot|_1$ and $|\cdot|_\infty$ with applications to asymptotic bounds for periodic linear systems. Int J Comp Math 2004; 81: 81-101.
[13] Kohaupt L. Illustration of the logarithmic derivatives by examples suitable for classroom teaching. Rocky Mountain J Math 2005; 35: 1595-629.
[14] Kohaupt L. New upper bounds for excited vibration systems with applications of the differential calculus of norms. Int J Comp Math 2007; 84: 1035-57.
[15] Kohaupt L. Two-sided bounds on the difference $R(t) = \Phi(t) - \Psi(t)$ between the continuous and discrete evolution as well as on $R(t)x_0 = \Phi(t)x_0 - \Psi(t)x_0$ with application of the differential calculus of norms. In: Alvarez MP, Ed. Leading-Edge Applied Mathematical Modeling Research 2008; pp. 319-340. Nova Science Publishers, Inc. ISBN 978-1-60021-977-1.
[16] Kohaupt L. Contributions to optimal bounds on the solution of ordinary differential euations with vibration behavior. Habilitation Thesis, TU Freiberg. Aachen: Shaker Verlag 2009.
[17] Falk S. Eigenwerte gedämpfter Schwingungssysteme und Eingrenzung ihrer Eigenwerte. Archive Appl Mech (Ingenieur-Archiv) 1960; 29: 436-44.
[18] Falk S. Klassifikation gedämpfter Schwingungssysteme bei Gültigkeit der Bequemlichkeitshypothese. Archive Appl Mech (Ingenieur-Archiv) 1978; 47: 57-66.
[19] Knowles JK. On damped linear dynamical systems. Archive Appl Mech (Ingenieur-Archiv) 1985; 55: 321-32.
[20] Müller FH, Baseler J. Zur Eigenformmethode nicht proportional gedämpfter Schwingungssysteme. Archive Appl Mech (Ingenieur-Archiv) 1985; 55: 348-57.
[21] Thomson WT. Theory of Vibration with Applications. Englewood Cliffs, New Jersey: Prentice-Hall 1971.
[22] Waller H, Krings W. Matrizenmethoden in der Maschinen- und Bauwerksdynamik. Mannheim Wien Zürich: Bibliographisches Institut 1975.
[23] Gantmacher FR. Matrizentheorie. Berlin Heidelberg New York Tokyo: Springer-Verlag 1986.
[24] Halmos PR. Finite-Dimensional Vector Spaces. New York Heidelberg Berlin: Springer-Verlag 1974.
[25] Niemeyer H, Wermuth E. Lineare Algebra. Braunschweig Wiesbaden: Vieweg 1987.
[26] Zurmühl R, Falk S. Matrizen und ihre Anwendungen, Teil 1. Berlin Heidelberg New York Tokyo: Springer-Verlag 1984.
[27] Kato T. Perturbation Theory for Linear Operators. New York: Springer 1966.
[28] Taylor AE. Introduction to Functional Analysis. New York London: John Wiley & Sons 1958.
[29] Hairer E, Nørset SP, Wanner G. Solving Ordinary Differential Equations I. Berlin Heidelberg New York London Paris Tokyo Hong Kong Barcelona Budapest: Springer-Verlag 1993.

Received: September 14, 2010 Revised: December 03, 2010 Accepted: January 03, 2011