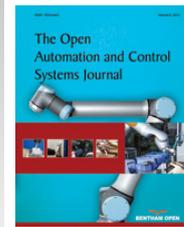




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RESEARCH ARTICLE

Quantum Linear Systems Theory

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Abstract: This paper surveys some recent results on the theory of quantum linear systems and presents them within a unified framework. Quantum linear systems are a class of systems whose dynamics, which are described by the laws of quantum mechanics, take the specific form of a set of linear quantum stochastic differential equations (QSDEs). Such systems commonly arise in the area of quantum optics and related disciplines. Systems whose dynamics can be described or approximated by linear QSDEs include interconnections of optical cavities, beam-splitters, phase-shifters, optical parametric amplifiers, optical squeezers, and cavity quantum electrodynamic systems. With advances in quantum technology, the feedback control of such quantum systems is generating new challenges in the field of control theory. Potential applications of such quantum feedback control systems include quantum computing, quantum error correction, quantum communications, gravity wave detection, metrology, atom lasers, and superconducting quantum circuits.

A recently emerging approach to the feedback control of quantum linear systems involves the use of a controller which itself is a quantum linear system. This approach to quantum feedback control, referred to as coherent quantum feedback control, has the advantage that it does not destroy quantum information, is fast, and has the potential for efficient implementation. However, the design of coherent quantum feedback controllers remains a major challenge. This paper discusses recent results concerning the synthesis of H-infinity optimal controllers for linear quantum systems in the coherent control case. An important issue which arises both in the modelling of linear quantum systems and in the synthesis of linear coherent quantum controllers is the issue of physical realizability. This issue relates to the property of whether a given set of QSDEs corresponds to a physical quantum system satisfying the laws of quantum mechanics. The paper will cover recent results relating the question of physical realizability to notions occurring in linear systems theory such as lossless bounded real systems and dual J-J unitary systems.

Keywords: Coherent control, Linear systems theory, Physical realizability, Quantum control theory, Quantum H-infinity control, Quantum systems+.

1. INTRODUCTION

Developments in quantum technology and quantum information provide an important motivation for research in the area of quantum feedback control systems; *e.g.*, see [1 - 7]. In particular, in recent years, there has been considerable interest in the feedback control and modeling of linear quantum systems; *e.g.*, see [3, 5, 8 - 26]. Such linear quantum systems commonly arise in the area of quantum optics; *e.g.*, see [27 - 29]. Feedback control of quantum optical systems has applications in areas such as quantum communications, quantum teleportation, and gravity wave detection. In particular, *linear quantum optics* is one of the possible platforms being investigated for future communication systems (see [30, 31]) and quantum computers (see [32 - 34]). Feedback control of quantum systems aims to achieve closed loop properties such as stability [35, 36], robustness [11, 37], entanglement [18, 38, 39].

Quantum linear system models have been used in the physics and mathematical physics literature since the 1980's; *e.g.*, see [26, 28, 40 - 42]. An important class of linear quantum stochastic models describe the Heisenberg evolution of the (canonical) position and momentum, or annihilation and creation operators of several independent open quantum

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harmonic oscillators that are coupled to external coherent bosonic fields, such as coherent laser beams; *e.g.*, see [8 - 13, 17, 18, 22, 25 - 28, 43, 44]). These linear stochastic models describe quantum optical devices such as optical cavities [27, 29], linear quantum amplifiers [28], and finite bandwidth squeezers [28]. Following [11, 12, 22], we will refer to this class of models as *linear quantum stochastic systems*. In particular, we consider linear quantum stochastic differential equations driven by quantum Wiener processes; see [28]. Further details on quantum stochastic differential equations and quantum Wiener processes can be found in [40, 42, 45].

This paper will survey some of the available results on the feedback control of linear quantum systems and related problems. As a survey paper, it does not present new results but rather surveys existing results in a unified framework. An important class of quantum feedback control systems involves the use of measurement devices to obtain classical output signals from the quantum system and no quantum measurements is involved. These classical signals are fed into a classical controller which may be implemented *via* analog or digital electronics and then the resulting control signal act on the quantum system *via* an actuator. However, some recent papers on the feedback control of linear quantum systems have considered the case in which the feedback controller itself is also a quantum system. Such feedback control is often referred to as coherent quantum control; *e.g.*, see [5, 6, 11, 12, 14 - 17, 46 - 48]. Due to the limitations imposed by quantum mechanics on the use of quantum measurement, the use of coherent quantum feedback control may lead to improved control system performance. In addition, in many applications, coherent quantum feedback controllers may be preferable to classical feedback controllers due to considerations of speed and ease of implementation.

One motivation for considering such coherent quantum control problems is that coherent controllers have the potential to achieve improved performance since quantum measurements inherently involve the destruction of quantum information; *e.g.*, see [34]. Also, technology is emerging which will enable the implementation of complex coherent quantum controllers (*e.g.*, see [49]) and the coherent H^∞ controllers proposed in [11] have already been implemented experimentally as described in [17]. Furthermore, coherent controllers implemented using quantum optics have the potential to operate at much higher speeds than classical controllers implemented in analog or digital electronics.

In general, quantum linear stochastic systems represented by linear Quantum Stochastic Differential Equations (QSDEs) with arbitrary constant coefficients need not correspond to physically meaningful systems. In contrast, because classical linear stochastic systems can be implemented at least approximately, using analog or digital electronics, we regard them as always being realizable. Physical quantum systems must satisfy some additional constraints that restrict the allowable values for the system matrices defining the QSDEs. In particular, the laws of quantum mechanics dictate that closed quantum systems evolve *unitarily*, implying that (in the Heisenberg picture) certain canonical observables satisfy the so-called canonical commutation relations (CCR) at all times. Therefore, to characterize physically meaningful systems [11], has introduced a formal notion of physically realizable quantum linear stochastic systems and derives a pair of necessary and sufficient characterizations for such systems in terms of constraints on their system matrices. However, the design of coherent quantum feedback controllers remains a major challenge. In this paper, we survey some methods which can be applied to this problem.

In the paper [21], the physical realizability results of [14, 15] are extended to the most general class of complex linear QSDEs. It is shown that this class of linear quantum systems corresponds to the class of real linear quantum systems considered in [11] *via* the use of a suitable state transformation.

The remainder of this paper proceeds as follows. In Section 2, we introduce the class of linear quantum stochastic systems under consideration and consider a number of different representations of these systems. We also introduce a useful special class of linear quantum systems which was considered in [14 - 16]. In Section 3, we consider the issue of physical realizability for the class of linear quantum systems under consideration. In Section 4, we will consider the problem of coherent H^∞ quantum controller synthesis. In Section 5, we present some conclusions.

2. LINEAR QUANTUM SYSTEM MODELS

In this section, we formulate the class of linear quantum system models under consideration. These linear quantum system models take the form of quantum stochastic differential equations which are derived from the quantum harmonic oscillator.

2.1. Quantum Harmonic Oscillators

We begin by considering a collection of n independent quantum harmonic oscillators which are defined on a Hilbert

space $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C})$; e.g., see [50], [42], [25]. Elements of the Hilbert space H , $\psi(x)$ are the standard complex valued wave functions arising in quantum mechanics where x is a spatial variable. Corresponding to this collection of harmonic oscillators is a vector of *annihilation operators*

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{1}$$

Each annihilation operator a_i is an unbounded linear operator defined on a suitable domain in H by

$$(a_i \psi)(x) = \frac{1}{\sqrt{2}} x_i \psi(x) + \frac{1}{\sqrt{2}} \frac{\partial \psi(x)}{\partial x_i}$$

where $\psi \in H$ is contained in the domain of the operator a_i . The adjoint of the operator a_i is denoted a_i^* and is referred to as a creation operator. The operators a_i and a_i^* are such that the following canonical commutation relations are satisfied

$$[a_i, a_j^*] = a_i a_j^* - a_j^* a_i = \delta_{ij} \tag{2}$$

where δ_{ij} denotes the Kronecker delta multiplied by the identity operator on the Hilbert space H . We also have the commutation relations

$$[a_i, a_j] = 0, [a_i^*, a_j^*] = 0. \tag{3}$$

For a general vector of operators

$$g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix},$$

on H , we use the notation

$$g^\# = \begin{bmatrix} g_1^* \\ g_2^* \\ \vdots \\ g_n^* \end{bmatrix},$$

to denote the corresponding vector of adjoint operators. Also, g^T denotes the corresponding row vector of operators $g^T = [g_1 g_2 \dots g_n]$, and $g^\dagger = (g^\#)^T$. Using this notation, the canonical commutation relations ((2)), ((3)) can be written as

$$\begin{aligned} \begin{bmatrix} [a \\ a^\#] \\ [a^\#]^\dagger \end{bmatrix} &= \begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \\ &- \left(\begin{bmatrix} a \\ a^\# \end{bmatrix}^\# \begin{bmatrix} a \\ a^\# \end{bmatrix}^T \right)^T \\ &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \end{aligned} \tag{4}$$

A state on our system of quantum harmonic oscillators is defined by a density operator ρ which is a self-adjoint

positive-semidefinite operator on H with $\text{tr}(\rho) = 1$; e.g., see [34]. Corresponding to a state ρ and an operator g on H is the quantum expectation

$$\langle g \rangle = \text{tr}(\rho g).$$

A state on the system is said to be *Gaussian* with positive-semidefinite *covariance matrix* $Q \in \mathbb{C}^{2n \times 2n}$ and *mean vector* $\alpha \in \mathbb{C}^n$ if given any vector $u \in \mathbb{C}^n$,

$$\begin{aligned} & \left\langle \exp \left(i \begin{bmatrix} u^\dagger & u^T \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha^\# \end{bmatrix} \right) \right\rangle \\ &= \exp \left(\begin{array}{c} -\frac{1}{2} \begin{bmatrix} u^\dagger & u^T \end{bmatrix} Q \begin{bmatrix} u \\ u^\# \end{bmatrix} \\ +i \begin{bmatrix} u^\dagger & u^T \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha^\# \end{bmatrix} \end{array} \right); \end{aligned}$$

e.g., see [25], [50]. Here, $u^\#$ denotes the complex conjugate of the complex vector u , u^T denotes the transpose of the complex vector u , and u^\dagger denotes the complex conjugate transpose of the complex vector u .

Note that in the zero mean case, $\alpha = 0$, the covariance matrix Q satisfies

$$Q = \left\langle \begin{bmatrix} a \\ a^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \right\rangle.$$

In the special case in which the covariance matrix Q is of the form

$$Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and the mean $\alpha = 0$, the system is said to be in the *vacuum state*. In the sequel, it will be assumed that the state on the system of harmonic oscillators is a Gaussian vacuum state. The state on the system of harmonic oscillators plays a similar role to the probability distribution of the initial conditions of a classical stochastic system.

The quantum harmonic oscillators described above are assumed to be coupled to m external independent quantum fields modelled by bosonic annihilation field operators $A_1(t), A_2(t), \dots, A_m(t)$ which are defined on separate Fock spaces F_i defined over $L^2(\mathbb{R})$ for each field operator [40], [42], [45], [39]. For each annihilation field operator $A_j(t)$, there is a corresponding creation field operator $A_j^*(t)$, which is defined on the same Fock space and is the operator adjoint of $A_j(t)$. The field operators are adapted quantum stochastic processes with forward differentials

$$d\mathcal{A}_j(t) = \mathcal{A}_j(t + dt) - \mathcal{A}_j(t)$$

and

$$d\mathcal{A}_j^*(t) = \mathcal{A}_j^*(t + dt) - \mathcal{A}_j^*(t)$$

that have the quantum Itô products [40], [42], [45], [39]:

$$\begin{aligned} d\mathcal{A}_j(t)d\mathcal{A}_k(t)^* &= \delta_{jk}dt; \\ d\mathcal{A}_j^*(t)d\mathcal{A}_k(t) &= 0; \\ d\mathcal{A}_j(t)d\mathcal{A}_k(t) &= 0; \\ d\mathcal{A}_j^*(t)d\mathcal{A}_k^*(t) &= 0. \end{aligned}$$

The field annihilation operators are also collected into a vector of operators defined as follows:

$$\mathcal{A}(t) = \begin{bmatrix} \mathcal{A}_1(t) \\ \mathcal{A}_2(t) \\ \vdots \\ \mathcal{A}_m(t) \end{bmatrix}.$$

For each i , the corresponding system state on the Fock space F_i is assumed to be a Gaussian vacuum state which means that given any complex valued function $u_i(t) \in L^2(\mathbb{R}, \mathbb{C})$, then

$$\begin{aligned} & \langle \exp(i \int_0^\infty u_i(t)^* d\mathcal{A}_i(t) + i \int_0^\infty u_i(t) d\mathcal{A}_i(t)^*) \rangle \\ &= \exp\left(-\frac{1}{2} \int_0^\infty |u(t)|^2 dt\right); \end{aligned}$$

e.g., see [40], [42], [45], [25].

In order to describe the joint evolution of the quantum harmonic oscillators and quantum fields, we first specify the *Hamiltonian operator* for the quantum system which is a Hermitian operator on H of the form

$$H = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix}$$

and $M_1 = M_1^\dagger, M_2 = M_2^T$. Here, M^\dagger denotes the complex conjugate transpose of the complex matrix M , M^T denotes the transpose of the complex matrix M , and $M^\#$ denotes the complex conjugate of the complex matrix M . Also, we specify the *coupling operator* for the quantum system to be an operator of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. Also, we write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

In addition, we define a *scattering matrix* which is a unitary matrix $S \in \mathbb{C}^{n \times n}$. These quantities then define the joint evolution of the quantum harmonic oscillators and the quantum fields according to a unitary adapted process $U(t)$ (which is an operator valued function of time) satisfying the Hudson-Parthasarathy QSDE [40], [42], [45], [23]:

$$\begin{aligned} dU(t) &= (tr [(S - I)^T d\Lambda(t)] + d\mathcal{A}(t)^\dagger L - L^\dagger S d\mathcal{A}(t) \\ &- (iH + \frac{1}{2} L^\dagger L) dt) U(t); U(0) = I, \end{aligned}$$

where $\Lambda(t) = [\Lambda_{jk}(t)]_{j,k=1,\dots,m}$. Here, the processes $\Lambda_{jk}(t)$ for $j,k = 1,\dots,m$ are adapted quantum stochastic processes referred to as *gauge processes*, and the forward differentials $d\Lambda_{jk}(t) = \Lambda_{jk}(t+dt) - \Lambda_{jk}(t)$ $j,k = 1,\dots,m$ have the quantum Itô products:

$$\begin{aligned} d\Lambda_{jk}(t)d\Lambda_{j'k'}(t) &= \delta_{kj'}d\Lambda_{jk'}(t); \\ d\mathcal{A}_j(t)d\Lambda_{kl}(t) &= \delta_{jk}d\mathcal{A}_l(t); \\ d\Lambda_{jk}d\mathcal{A}_l(t)^* &= \delta_{kl}d\mathcal{A}_j^*(t). \end{aligned}$$

Then, using the Heisenberg picture of quantum mechanics, the harmonic oscillator operators $a_i(t)$ evolve with time unitarily according to

$$a_i(t) = U(t)^* a_i U(t)$$

for $i = 1, 2, \dots, n$. Also, the linear quantum system output fields are given by

$$\mathcal{A}_i^{out}(t) = U(t)^* \mathcal{A}_i(t) U(t)$$

for $i = 1, 2, \dots, m$.

We now use the fact that for any adapted processes $X(t)$ and $Y(t)$ satisfying a quantum Itô stochastic differential equation, we have the *quantum Itô rule*

$$dX(t)Y(t) = X(t)dY(t) + dX(t)Y(t) + dX(t)dY(t);$$

e.g., see [42]. Using the quantum Itô rule and the quantum Itô products given above, as well as exploiting the canonical commutation relations between the operators in a , the following QSDEs describing the linear quantum system can be obtained (*e.g.*, see [25]):

$$\begin{aligned} da(t) &= dU(t)^* a U(t) \\ &= [F_1 \quad F_2] \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt \\ &+ [G_1 \quad G_2] \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ a(0) &= a; \\ d\mathcal{A}^{out}(t) &= dU(t)^* \mathcal{A}(t) U(t) \\ &= [H_1 \quad H_2] \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt \\ &+ [K_1 \quad K_2] \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} F_1 &= -iM_1 - \frac{1}{2}(N_1^\dagger N_1 - N_2^T N_2^\#); \\ F_2 &= -iM_2 - \frac{1}{2}(N_1^\dagger N_2 - N_2^T N_1^\#); \\ G_1 &= -N_1^\dagger S; \\ G_2 &= N_2^T S^\#; \\ H_1 &= N_1; \\ H_2 &= N_2; \\ K_1 &= S; \\ K_2 &= 0. \end{aligned} \tag{6}$$

From this, we can write

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + G \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^\# \end{bmatrix} &= H \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + K \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} F &= \begin{bmatrix} F_1 & F_2 \\ F_2^\# & F_1^\# \end{bmatrix}; G = \begin{bmatrix} G_1 & G_2 \\ G_2^\# & G_1^\# \end{bmatrix}; \\ H &= \begin{bmatrix} H_1 & H_2 \\ H_2^\# & H_1^\# \end{bmatrix}; K = \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix}. \end{aligned} \tag{8}$$

Also, the equations (6) can be re-written as

$$\begin{aligned} F &= -iJM - \frac{1}{2}JN^\dagger JN; \\ G &= -JN^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}; \\ H &= N; \\ K &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}; \end{aligned} \tag{9}$$

where

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Note that matrices of the form (8) occur commonly in the theory of linear quantum systems. It is straightforward to establish the following lemma which characterizes matrices of this form.

Lemma 1 A matrix $R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$ satisfies

$$\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_2^\# & R_1^\# \end{bmatrix}$$

if and only if

$$R\Sigma = \Sigma R^\#$$

where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

We now consider the case when the initial condition in the QSDE (5) is no longer the vector of annihilation operators (1) but rather a vector of linear combinations of annihilation operators and creation operators defined by

$$\tilde{a} = T_1 a + T_2 a^\#$$

where

$$T = \begin{bmatrix} T_1 & T_2 \\ T_2^\# & T_1^\# \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$$

is non-singular. Then, it follows from (4) that

$$\begin{aligned} & \begin{bmatrix} [\tilde{a}] & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix} \\ &= \begin{bmatrix} \tilde{a} & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix} - \left(\begin{bmatrix} \tilde{a} & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix}^T \right)^T \\ &= T \begin{bmatrix} \tilde{a} & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix} T^\dagger \\ &\quad - \left(T^\# \left(\begin{bmatrix} a & [a]^\dagger \\ [a^\#] & [a^\#]^\dagger \end{bmatrix}^T \right)^T T^T \right)^T \\ &= T \begin{pmatrix} \begin{bmatrix} a & [a]^\dagger \\ [a^\#] & [a^\#]^\dagger \end{bmatrix} \\ - \left(\begin{bmatrix} a & [a]^\dagger \\ [a^\#] & [a^\#]^\dagger \end{bmatrix}^T \right)^T \end{pmatrix} T^\dagger \\ &= \Theta \end{aligned}$$

where

$$\begin{aligned} \Theta &= T J T^\dagger \\ &= \begin{bmatrix} T_1 T_1^\dagger - T_2 T_2^\dagger & T_1 T_2^T - T_2 T_1^T \\ T_2^\# T_1^\dagger - T_1^\# T_2^\dagger & T_2^\# T_2^T - T_1^\# T_1^T \end{bmatrix}. \end{aligned} \quad (10)$$

The relationship

$$\begin{bmatrix} [\tilde{a}] & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix} = \Theta \quad (11)$$

is referred to as a *generalized commutation relation* [14 - 16]. Also, the covariance matrix corresponding to $\begin{bmatrix} \tilde{a} \\ \tilde{a}^\# \end{bmatrix}$ is given by

$$\begin{aligned} \tilde{Q} &= \left\langle \begin{bmatrix} \tilde{a} & [\tilde{a}]^\dagger \\ [\tilde{a}^\#] & [\tilde{a}^\#]^\dagger \end{bmatrix} \right\rangle \\ &= T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^\dagger \\ &= \begin{bmatrix} T_1 T_1^\dagger & T_1 T_2^T \\ T_2^\# T_1^\dagger & T_2^\# T_2^T \end{bmatrix}. \end{aligned}$$

In terms of the variables $\tilde{a}(t) = U(t)^* \tilde{a} U(t)$, the QSDEs, (7) can be rewritten as

$$\begin{aligned} \begin{bmatrix} d\tilde{a}(t) \\ d\tilde{a}(t)^\# \end{bmatrix} &= \tilde{F} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \end{bmatrix} dt + \tilde{G} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^\# \end{bmatrix} &= \tilde{H} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \end{bmatrix} dt + \tilde{K} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \tilde{F} &= \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \\ \tilde{F}_2^\# & \tilde{F}_1^\# \end{bmatrix} = T F T^{-1}; \\ \tilde{G} &= \begin{bmatrix} \tilde{G}_1 & \tilde{G}_2 \\ \tilde{G}_2^\# & \tilde{G}_1^\# \end{bmatrix} = T G; \\ \tilde{H} &= \begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 \\ \tilde{H}_2^\# & \tilde{H}_1^\# \end{bmatrix} = H T^{-1}; \\ \tilde{K} &= \begin{bmatrix} \tilde{K}_1 & \tilde{K}_2 \\ \tilde{K}_2^\# & \tilde{K}_1^\# \end{bmatrix} = K. \end{aligned} \tag{13}$$

Now, we can re-write the operators H and L defining the above collection of quantum harmonic oscillators in terms of the variables \tilde{a} as

$$H = \frac{1}{2} [\tilde{a}^\dagger \quad \tilde{a}^T] \tilde{M} \begin{bmatrix} \tilde{a} \\ \tilde{a}^\# \end{bmatrix}, \quad \tilde{L} = \tilde{N} \begin{bmatrix} \tilde{a} \\ \tilde{a}^\# \end{bmatrix}$$

where

$$\tilde{M} = (T^\dagger)^{-1} M T^{-1}, \quad \tilde{N} = N T^{-1}. \tag{14}$$

Here,

$$\tilde{M} = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 \\ \tilde{M}_2^\# & \tilde{M}_1^\# \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_2^\# & \tilde{N}_1^\# \end{bmatrix}. \tag{15}$$

Furthermore, equations (9), (13) and (14) can be combined to obtain

$$\begin{aligned} \tilde{F} &= -i\Theta \tilde{M} - \frac{1}{2} \Theta \tilde{N}^\dagger J \tilde{N}; \\ \tilde{G} &= -\Theta \tilde{N}^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}; \\ \tilde{H} &= \tilde{N}; \\ \tilde{K} &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}. \end{aligned} \tag{16}$$

Note that since S is unitary, it follows that

$$\begin{aligned} \tilde{K} J \tilde{K}^\dagger &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} S^\dagger & 0 \\ 0 & S^T \end{bmatrix} \\ &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix} \begin{bmatrix} S^\dagger & 0 \\ 0 & -S^T \end{bmatrix} \\ &= \begin{bmatrix} S S^\dagger & 0 \\ 0 & -S^\# S^T \end{bmatrix} = J. \end{aligned} \tag{17}$$

Also,

$$\begin{aligned}\tilde{K}\tilde{K}^\dagger &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix} \begin{bmatrix} S^\dagger & 0 \\ 0 & S^T \end{bmatrix} \\ &= \begin{bmatrix} SS^\dagger & 0 \\ 0 & S^\#S^T \end{bmatrix} = I.\end{aligned}\tag{18}$$

Indeed these two properties characterize all matrices non-singular \tilde{K} satisfying $\tilde{K}\Sigma = \Sigma\tilde{K}^\#$ which are of the form given in (16). Let the non-singular matrix K be such that $K\Sigma = \Sigma K^\#$, $KJK^\dagger = J$, and $KK^\dagger = I$. It follows from Lemma 1 that we can write

$$K = \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix}.$$

Also, $KK^\dagger = I$ implies

$$\begin{aligned}KK^\dagger &= \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix} \begin{bmatrix} K_1^\dagger & K_2^T \\ K_2^\dagger & K_1^T \end{bmatrix} \\ &= \begin{bmatrix} K_1K_1^\dagger + K_2K_2^\dagger & K_1K_2^T + K_2K_1^T \\ K_2^\#K_1^\dagger + K_1^\#K_2^\dagger & K_2^\#K_2^T + K_1^\#K_1^T \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\end{aligned}$$

and $KJK^\dagger = J$ implies

$$\begin{aligned}KJK^\dagger &= \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} K_1^\dagger & K_2^T \\ K_2^\dagger & K_1^T \end{bmatrix} \\ &= \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix} \begin{bmatrix} K_1^\dagger & K_2^T \\ -K_2^\dagger & -K_1^T \end{bmatrix} \\ &= \begin{bmatrix} K_1K_1^\dagger - K_2K_2^\dagger & K_1K_2^T - K_2K_1^T \\ K_2^\#K_1^\dagger - K_1^\#K_2^\dagger & K_2^\#K_2^T - K_1^\#K_1^T \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.\end{aligned}$$

The (1,1) block of these two equations imply $K_1K_1^\dagger + K_2K_2^\dagger = I$ and $K_1K_1^\dagger - K_2K_2^\dagger = I$. Hence $K_2K_2^\dagger = 0$. Therefore, $K_2 = 0$ and $K_1K_1^\dagger = I$. From this, it follows that the matrix K must be of the form given in (16).

The QSDEs (12), (13), (16) define the general class of linear quantum systems considered in this paper. Such quantum systems can be used to model a large range of devices and networks of devices arising in the area of quantum optics including optical cavities, squeezers, optical parametric amplifiers, cavity Quantum Electrodynamics (cavity QED) systems, beam splitters, and phase shifters; *e.g.*, see [3, 5, 6, 11, 17, 19, 22, 24, 26 - 29, 48].

2.2. Annihilation Operator Linear Quantum Systems

An important special case of the linear quantum systems (12), (13), (16) corresponds to the case in which the Hamiltonian operator H and coupling operator L depend only of the vector of annihilation operators a and not on the vector of creation operators $a^\#$. This class of linear quantum systems is considered in [14 - 17, 19, 20, 51] and can be used to model “passive” quantum optical devices such as optical cavities, beam splitters, phase shifters and interferometers.

This class of linear quantum systems corresponds to the case in which $\tilde{M}_2 = 0$, $\tilde{N}_2 = 0$, and $T_2 = 0$. In this case, the

linear quantum system can be modelled by the QSDEs

$$\begin{aligned} d\tilde{a}(t) &= \tilde{F}\tilde{a}(t)dt + \tilde{G}d\mathcal{A}(t) \\ d\mathcal{A}^{out}(t) &= \tilde{H}\tilde{a}(t)dt + \tilde{K}d\mathcal{A}(t) \end{aligned} \tag{19}$$

where

$$\begin{aligned} \tilde{F} &= -i\Theta_1\tilde{M}_1 - \frac{1}{2}\Theta_1\tilde{N}_1^\dagger\tilde{N}_1; \\ \tilde{G} &= -\Theta_1\tilde{N}_1^\dagger S; \\ \tilde{H} &= \tilde{N}_1; \\ \tilde{K} &= S; \\ \Theta_1 &= T_1T_1^\dagger > 0. \end{aligned} \tag{20}$$

2.3. Position and Momentum Operator Linear Quantum Systems

Note that the matrices in the general QSDEs (12), (13) are in general complex. However, it is possible to apply a particular change of variables to the system (5) so that all of the matrices in the resulting transformed QSDEs are real. This change of variables is defined as follows:

$$\begin{aligned} \begin{bmatrix} q \\ p \end{bmatrix} &= \Phi \begin{bmatrix} a \\ a^\# \end{bmatrix}; \\ \begin{bmatrix} Q(t) \\ \mathcal{P}(t) \end{bmatrix} &= \Phi \begin{bmatrix} \mathcal{A}(t) \\ \mathcal{A}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} Q^{out}(t) \\ \mathcal{P}^{out}(t) \end{bmatrix} &= \Phi \begin{bmatrix} \mathcal{A}^{out}(t) \\ \mathcal{A}^{out}(t)^\# \end{bmatrix} \end{aligned} \tag{21}$$

where the matrices Φ have the form

$$\Phi = \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \tag{22}$$

and have the appropriate dimensions. Here q is a vector of the self-adjoint position operators for the system of harmonic oscillators and p is a vector of momentum operators; *e.g.*, see [11, 12, 21, 39]. Also, $Q(t)$ and $P(t)$ are the vectors of position and momentum operators for the quantum noise fields acting on the system of harmonic oscillators. Furthermore, $Q^{out}(t)$ and $P^{out}(t)$ are the vectors of position and momentum operators for the output quantum noise fields.

It follows from (22) that

$$\Phi\Phi^\dagger = 2I \tag{23}$$

and hence

$$\Phi^{-1}\Phi^{-\dagger} = \frac{1}{2}I. \tag{24}$$

Rather than applying the transformations (21) to the quantum linear system (7) which satisfies the canonical commutation relations (4), corresponding transformations can be applied to the quantum linear system (12) which satisfies the generalized commutation relations (11). These transformations are as follows:

$$\begin{aligned}
\begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix} &= \Phi \begin{bmatrix} \tilde{a} \\ \tilde{a}^\# \end{bmatrix}; \\
\begin{bmatrix} Q(t) \\ \mathcal{P}(t) \end{bmatrix} &= \Phi \begin{bmatrix} \mathcal{A}(t) \\ \mathcal{A}(t)^\# \end{bmatrix}; \\
\begin{bmatrix} Q^{out}(t) \\ \mathcal{P}^{out}(t) \end{bmatrix} &= \Phi \begin{bmatrix} \mathcal{A}^{out}(t) \\ \mathcal{A}^{out}(t)^\# \end{bmatrix}.
\end{aligned} \tag{25}$$

When these transformations are applied to the quantum linear system (12), this leads to the following real quantum linear system:

$$\begin{aligned}
\begin{bmatrix} d\tilde{q}(t) \\ d\tilde{p}(t) \end{bmatrix} &= A \begin{bmatrix} \tilde{p}(t) \\ \tilde{q}(t) \end{bmatrix} dt + B \begin{bmatrix} dQ(t) \\ d\mathcal{P}(t) \end{bmatrix}; \\
\begin{bmatrix} dQ^{out}(t) \\ d\mathcal{P}^{out}(t) \end{bmatrix} &= C \begin{bmatrix} \tilde{q}(t) \\ \tilde{p}(t) \end{bmatrix} dt + D \begin{bmatrix} d\mathcal{P}(t) \\ dQ(t) \end{bmatrix},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
A &= \Phi \tilde{F} \Phi^{-1} \\
&= \frac{1}{2} \begin{bmatrix} \tilde{F}_1 + \tilde{F}_1^\# + \tilde{F}_2 + \tilde{F}_2^\# \\ -i(\tilde{F}_1 - \tilde{F}_1^\#) - i(\tilde{F}_2 - \tilde{F}_2^\#) \\ i(\tilde{F}_1 - \tilde{F}_1^\#) - i(\tilde{F}_2 - \tilde{F}_2^\#) \\ \tilde{F}_1 + \tilde{F}_1^\# - \tilde{F}_2 - \tilde{F}_2^\# \end{bmatrix}; \\
B &= \Phi \tilde{G} \Phi^{-1} \\
&= \frac{1}{2} \begin{bmatrix} \tilde{G}_1 + \tilde{G}_1^\# + \tilde{G}_2 + \tilde{G}_2^\# \\ -i(\tilde{G}_1 - \tilde{G}_1^\#) - i(\tilde{G}_2 - \tilde{G}_2^\#) \\ i(\tilde{G}_1 - \tilde{G}_1^\#) - i(\tilde{G}_2 - \tilde{G}_2^\#) \\ \tilde{G}_1 + \tilde{G}_1^\# - \tilde{G}_2 - \tilde{G}_2^\# \end{bmatrix}; \\
C &= \Phi \tilde{H} \Phi^{-1} \\
&= \frac{1}{2} \begin{bmatrix} \tilde{H}_1 + \tilde{H}_1^\# + \tilde{H}_2 + \tilde{H}_2^\# \\ -i(\tilde{H}_1 - \tilde{H}_1^\#) - i(\tilde{H}_2 - \tilde{H}_2^\#) \\ i(\tilde{H}_1 - \tilde{H}_1^\#) - i(\tilde{H}_2 - \tilde{H}_2^\#) \\ \tilde{H}_1 + \tilde{H}_1^\# - \tilde{H}_2 - \tilde{H}_2^\# \end{bmatrix}; \\
D &= \Phi \tilde{K} \Phi^{-1} \\
&= \frac{1}{2} \begin{bmatrix} \tilde{K}_1 + \tilde{K}_1^\# + \tilde{K}_2 + \tilde{K}_2^\# \\ -i(\tilde{K}_1 - \tilde{K}_1^\#) - i(\tilde{K}_2 - \tilde{K}_2^\#) \\ i(\tilde{K}_1 - \tilde{K}_1^\#) - i(\tilde{K}_2 - \tilde{K}_2^\#) \\ \tilde{K}_1 + \tilde{K}_1^\# - \tilde{K}_2 - \tilde{K}_2^\# \end{bmatrix}.
\end{aligned} \tag{27}$$

These matrices are all real.

Also, it follows from (10) that

$$\begin{bmatrix} [\tilde{q}] \\ [\tilde{p}] \end{bmatrix}, [\tilde{q}]^\dagger, [\tilde{p}]^\dagger = \Xi \tag{28}$$

where

$$\Xi = \Phi\Theta\Phi^\dagger = \Phi TJT^\dagger\Phi^\dagger, \tag{29}$$

which is a Hermitian matrix.

Now, we can re-write the operators H and L defining the above collection of quantum harmonic oscillators in terms of the variables \tilde{q} and \tilde{p} as

$$H = \frac{1}{2}[\tilde{q}^T \quad \tilde{p}^T]R \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix}, \begin{bmatrix} \tilde{L} \\ \tilde{L}^\# \end{bmatrix} = V \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix}$$

where

$$R = (\Phi^\dagger)^{-1}\tilde{M}\Phi^{-1}, V = \tilde{N}\Phi^{-1}. \tag{30}$$

Here

$$R = \frac{1}{4} \begin{bmatrix} \tilde{M}_1 + \tilde{M}_1^\# + \tilde{M}_2 + \tilde{M}_2^\# \\ -i(\tilde{M}_1 - \tilde{M}_1^\#) - i(\tilde{M}_2 - \tilde{M}_2^\#) \\ i(\tilde{M}_1 - \tilde{M}_1^\#) - i(\tilde{M}_2 - \tilde{M}_2^\#) \\ \tilde{M}_1 + \tilde{M}_1^\# - \tilde{M}_2 - \tilde{M}_2^\# \end{bmatrix}; \tag{31}$$

$$V = \begin{bmatrix} \tilde{N}_1 + \tilde{N}_2 & i(\tilde{N}_1 - \tilde{N}_2) \\ \tilde{N}_1^\# + \tilde{N}_2^\# & i(\tilde{N}_2^\# - \tilde{N}_1^\#) \end{bmatrix}$$

where the matrix R is real but the matrix V may be complex.

However, using (30) and (16), we can write

$$\begin{bmatrix} \tilde{L} + \tilde{L}^\# \\ \frac{\tilde{L} - \tilde{L}^\#}{i} \end{bmatrix} = \Phi \begin{bmatrix} \tilde{L} \\ \tilde{L}^\# \end{bmatrix} = \Phi V \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix} = W \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix}$$

where

$$W = \Phi V = \Phi\tilde{N}\Phi^{-1} = \frac{1}{2} \begin{bmatrix} \tilde{N}_1 + \tilde{N}_1^\# + \tilde{N}_2 + \tilde{N}_2^\# \\ -i(\tilde{N}_1 - \tilde{N}_1^\#) - i(\tilde{N}_2 - \tilde{N}_2^\#) \\ i(\tilde{N}_1 - \tilde{N}_1^\#) - i(\tilde{N}_2 - \tilde{N}_2^\#) \\ \tilde{N}_1 + \tilde{N}_1^\# - \tilde{N}_2 - \tilde{N}_2^\# \end{bmatrix}$$

is real as in (27). That is, we can write

$$\begin{bmatrix} \Re(\tilde{L}) \\ \Im(\tilde{L}) \end{bmatrix} = \frac{1}{2}W \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix}.$$

Note that the matrix $\Phi T\Phi^{-1}$ is real and

$$\Phi J\Phi^\dagger = 2i \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = 2i\tilde{J} \tag{32}$$

where

$$\tilde{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Hence, the matrix

$$\Xi = \Phi T \Phi^{-1} \Phi J \Phi^\dagger (\Phi^\dagger)^{-1} T^\dagger \Phi^\dagger$$

must be purely imaginary. Hence, we can define the real skew symmetric matrix

$$\tilde{\Theta} = -\frac{i}{2} \Xi = -\frac{i}{2} \Phi \Theta \Phi^\dagger.$$

Using this notation, (28) can be written as

$$\begin{bmatrix} [\tilde{q}] \\ [\tilde{p}] \end{bmatrix}, \begin{bmatrix} [\tilde{q}]^\dagger \\ [\tilde{p}]^\dagger \end{bmatrix} = 2i\tilde{\Theta}.$$

In addition, we note that

$$(\Phi^\dagger)^{-1} J \Phi^{-1} = \frac{i}{2} \tilde{J}. \quad (33)$$

Furthermore, equations (16), (27), and (31) can be combined to obtain

$$\begin{aligned} A &= -i\Xi R - \frac{1}{2} \Xi V^\dagger J V \\ &= 2\tilde{\Theta} R - \frac{1}{2} \Xi W^T \Phi^{-\dagger} J \Phi^{-1} W \\ &= 2\tilde{\Theta} R - \frac{i}{4} \Xi W^T \tilde{J} W \\ &= 2\tilde{\Theta} R + \frac{1}{2} \tilde{\Theta} W^T \tilde{J} W; \\ B &= -\Xi V^\dagger J \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix} \Phi^{-1} \\ &= -\Xi W^T \Phi^{-\dagger} J \Phi^{-1} \Phi \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix} \Phi^{-1} \\ &= -\frac{i}{2} \Xi W^T \tilde{J} D \\ &= \tilde{\Theta} W^T \tilde{J} D; \\ C &= W; \\ D &= \frac{1}{2} \begin{bmatrix} S + S^\# & i(S - S^\#) \\ i(S - S^\#) & S + S^\# \end{bmatrix}. \end{aligned} \quad (34)$$

Now from (27), we have $D = \Phi \tilde{K} \Phi^{-1}$ and $D^T = \Phi^\dagger \tilde{K}^\dagger \Phi^\dagger$ and hence,

$$\begin{aligned} D D^T &= \Phi \tilde{K} \Phi^{-1} \Phi^{-\dagger} \tilde{K}^\dagger \Phi^\dagger \\ &= \frac{1}{2} \Phi \tilde{K} \tilde{K}^\dagger \Phi^\dagger \\ &= \frac{1}{2} \Phi \Phi^\dagger \\ &= I \end{aligned} \quad (35)$$

using (24), (18) and (23); *i.e.*, D is an orthogonal matrix.

Also, we have

$$\begin{aligned}
 D\tilde{J}D^T &= -\frac{i}{2}\Phi\tilde{K}\Phi^{-1}\Phi J\Phi^\dagger\Phi^{-\dagger}\tilde{K}\Phi^\dagger \\
 &= -\frac{i}{2}\Phi\tilde{K}J\tilde{K}^\dagger\Phi^\dagger \\
 &= -\frac{i}{2}\Phi J\Phi^\dagger \\
 &= \tilde{J}
 \end{aligned} \tag{36}$$

using (32) and (17); *i.e.*, D is a symplectic matrix.

Conversely suppose a matrix $D = \Phi\tilde{K}\Phi^{-1}$ satisfies $DD^T = I$ and $D\tilde{J}D^T = \tilde{J}$. It follows in a similar fashion to above that $\tilde{K}\tilde{K}^\dagger = I$ and $\tilde{K}J\tilde{K}^\dagger = J$. Hence, as in Section 2.1 the matrix \tilde{K} must be of the form in (16). Thus, the matrix D must be of the form in (34).

3. PHYSICAL REALIZABILITY

Not all QSDEs of the form (12), (13) correspond to physical quantum systems. This motivates a notion of physical realizability which has been considered in the papers [11, 12, 14 - 16, 19 - 21, 38, 51]. This notion is of particular importance in the problem of coherent quantum feedback control in which the controller itself is a quantum system. In this case, if a controller is synthesized using a method such as quantum H^∞ control [11, 14, 16] or quantum linear quadratic Gaussian (LQG) control [12, 38], it important that the controller can be implemented as a physical quantum system [19, 22]. We first consider the issue of physical realizability in the case of general linear quantum systems and then we consider the issue of physical realizability for the case of annihilator operator linear quantum system of the form considered in Subsection 2.2.

3.1. Physical Realizability for General Linear Quantum Systems

The formal definition of physically realizable QSDEs requires that they can be realized as a system of quantum harmonic oscillators.

Definition 1 QSDEs of the form (12), (13) are physically realizable if there exist complex matrices $\Theta = \Theta^\dagger$, $\tilde{M} = \tilde{M}^\dagger$, \tilde{N} , S such that $S^\dagger S = I$, Θ is of the form in (10), \tilde{M} is of the form in (15), and (16) is satisfied.

A version of the following theorem was presented in [21]; see also [11, 12] for related results.

Theorem 1 The QSDEs (12), (13) are physically realizable if and only if there exists a complex matrix $\Theta = \Theta^\dagger$ of the form in (10) such that

$$\begin{aligned}
 \tilde{F}\Theta + \Theta\tilde{F}^\dagger + \tilde{G}J\tilde{G}^\dagger &= 0; \\
 \tilde{G} &= -\Theta\tilde{H}^\dagger J\tilde{K}; \\
 \tilde{K}J\tilde{K}^\dagger &= J; \\
 \tilde{K}\tilde{K}^\dagger &= I.
 \end{aligned} \tag{37}$$

Proof. If there exist matrices $\Theta = \Theta^\dagger$, $\tilde{M} = \tilde{M}^\dagger$, \tilde{N} , S such that $S^\dagger S = I$, \tilde{M} is of the form in (15), Θ is of the form in (10), and (16) is satisfied, then it follows by straightforward substitution that the first equation in (27) will be satisfied and

$$\begin{aligned}
 \tilde{G} &= -\Theta\tilde{H}^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}; \\
 \tilde{K} &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}.
 \end{aligned} \tag{38}$$

Then, the remaining equations in (37) follow using (17) and (18).

Conversely, suppose there exists a complex matrix $\Theta = \Theta^\dagger$ of the form in (10) such that (37) is satisfied. Also, as shown at the end of Section II.A, the conditions $\tilde{K}J\tilde{K}^\dagger = J$ and $\tilde{K}\tilde{K}^\dagger = I$ imply that there exists a complex matrix S such

that $S^\dagger S = I$ and \tilde{K} is of the form $\tilde{K} = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}$.

Also, let

$$\begin{aligned}\tilde{M} &= \frac{i}{2}(\Theta^{-1}\tilde{F} - \tilde{F}^\dagger\Theta^{-1}); \\ \tilde{N} &= \tilde{H}.\end{aligned}$$

It is straightforward to verify that this matrix \tilde{M} is Hermitian. Also, it follows from (37) that

$$\tilde{G} = -\Theta\tilde{N}^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}$$

as required. Furthermore, using $S^\dagger S = I$, it now follows that

$$\tilde{G}J\tilde{G}^\dagger = \Theta\tilde{N}^\dagger J\tilde{N}\Theta.$$

Hence, (37) implies

$$\tilde{F}\Theta + \Theta\tilde{F}^\dagger + \Theta\tilde{N}^\dagger J\tilde{N}\Theta = 0$$

and hence

$$\tilde{F}^\dagger\Theta^{-1} = -\Theta^{-1}\tilde{F} - \tilde{N}^\dagger J\tilde{N}$$

From this, it follows that

$$\tilde{M} = \frac{i}{2}(2\Theta^{-1}\tilde{F} + \tilde{N}^\dagger J\tilde{N})$$

and hence,

$$\tilde{F} = -i\Theta\tilde{M} - \frac{1}{2}\Theta\tilde{N}^\dagger J\tilde{N}$$

as required. Hence, (16) is satisfied.

We now use Lemma 1 to show that \tilde{M} is of the form in ((13)). Indeed, we have $T\Sigma = \Sigma T^\#, T^\#\Sigma = \Sigma T, T^{-1}\Sigma = \Sigma(T^\#)^{-1}, (T^\#)^{-1}\Sigma = \Sigma T^{-1}, \tilde{F}\Sigma = \Sigma\tilde{F}^\#, \tilde{F}^\#\Sigma = \Sigma\tilde{F}$, and $\Sigma J = -J\Sigma$, and $\Sigma J = -J\Sigma$. Hence,

$$\begin{aligned}\Sigma\tilde{M}^\# &= -\frac{i}{2} \begin{pmatrix} \Sigma(T^T)^{-1}J(T^\#)^{-1}\tilde{F}^\# \\ -\Sigma\tilde{F}^T(T^T)^{-1}J(T^\#)^{-1} \end{pmatrix} \\ &= \frac{i}{2} \begin{pmatrix} (T^\dagger)^{-1}JT^{-1}\tilde{F} \\ -\tilde{F}^\dagger(T^\dagger)^{-1}JT^{-1} \end{pmatrix} \Sigma \\ &= \tilde{M}\Sigma.\end{aligned}$$

Therefore, it follows from Lemma 1 that \tilde{M} is of the form in (15) and hence, the QSDEs (12), (13) are physically realizable.

Remark 1 In the canonical case when $T = I$ and $\Theta = J$, the physical realizability equations (37) become

$$\begin{aligned} \tilde{F}J + J\tilde{F}^\dagger + \tilde{G}J\tilde{G}^\dagger &= 0; \\ \tilde{G} &= -J\tilde{H}^\dagger J\tilde{K}; \\ \tilde{K}J\tilde{K}^\dagger &= J; \\ \tilde{K}\tilde{K}^\dagger &= I. \end{aligned} \tag{39}$$

Following the approach of [21], we now relate the physical realizability of the QSDEs (12), (13) to the dual (J, J) -unitary property of the corresponding transfer function matrix

$$\Gamma(s) = \begin{bmatrix} \Gamma_{11}(s) & \Gamma_{12}(s) \\ \Gamma_{21}(s) & \Gamma_{22}(s) \end{bmatrix} = \tilde{H}(sI - \tilde{F})^{-1}\tilde{G} + \tilde{K}. \tag{40}$$

Definition 2 (See [21, 52].) A transfer function matrix $\Gamma(s)$ of the form (40) is dual (J, J) -unitary if

$$\Gamma(s)J\Gamma^\dagger(s) = J$$

for all $s \in \mathbb{C}_+$.

Here, $\Gamma^\dagger(s) = \Gamma(-s^*)^\dagger$ and \mathbb{C}_+ denotes the set $\{s \in \mathbb{C} : \Re[s] \geq 0\}$.

Theorem 2 The transfer function matrix (40) corresponding to the QSDEs (12), (13) is dual (J, J) -unitary if and only if

$$\tilde{K}J\tilde{K}^\dagger = J, \tag{41}$$

and there exists a Hermitian matrix Θ such that

$$\begin{aligned} \tilde{F}\Theta + \Theta\tilde{F}^\dagger + \tilde{G}J\tilde{G}^\dagger &= 0; \\ \tilde{K}J\tilde{G}^\dagger + \tilde{H}\Theta &= 0. \end{aligned} \tag{42}$$

Theorem 3 (See also [21]) If the QSDEs (12), (13) are physically realizable, then the corresponding transfer function matrix (40) is dual (J, J) -unitary.

Conversely, suppose the QSDEs (12), (13) satisfy the following conditions:

1. The transfer function matrix (40) corresponding to the QSDEs (12), (13) is dual (J, J) -unitary;
- 2.

$$\tilde{K}\tilde{K}^\dagger = I; \tag{43}$$

3. The Hermitian matrix Θ satisfying (42) is of the form in (10).

Then, the QSDEs (12), (13) are physically realizable.

Proof. If the QSDEs (12), (13) are physically realizable, then it follows from Theorem 1 that there exist complex matrices $\Theta = \Theta^\dagger$ and S such that $S^\dagger S = I$ and equations (37) are satisfied. However, $\tilde{K}J\tilde{K}^\dagger = J$ and $\tilde{K}\tilde{K}^\dagger = I$ imply $\tilde{K}J = J\tilde{K}$ and hence, it follows from (37) that

$$\tilde{K}J\tilde{G}^\dagger + \tilde{H}\Theta = 0.$$

That is, the conditions (42) are satisfied and hence it follows from Theorem 2 that the transfer function matrix (40) corresponding to the QSDEs (12), (13) is dual (J, J) -unitary.

Conversely, if the QSDEs (12), (13) satisfy conditions (i) - (iii) of the theorem, then it follows from Theorem 2 that there exists a Hermitian matrix Θ of the form in (10) such that equations (42) are satisfied. Hence,

$$\tilde{K}J\tilde{G}^\dagger + \tilde{H}\Theta = 0. \quad (44)$$

Furthermore, we have $\tilde{K}J\tilde{K}^\dagger = J$ and $\tilde{K}\tilde{K}^\dagger = I$ and therefore $\tilde{K}J = J\tilde{K}$. Hence, (44) implies

$$\tilde{G}^\dagger + \tilde{K}^\dagger J \tilde{H} \Theta = 0.$$

From this it follows that equations (37) are satisfied. Thus, it follows from Theorem 1 that the QSDEs (12), (13) are physically realizable.

3.2. Physical Realizability for Annihilator Operator Linear Quantum Systems

For annihilator operator linear quantum systems described by QSDEs of the form (19) the corresponding formal definition of physical realizability is as follows.

Definition 3 (See [14, 15, 51].) The QSDEs of the form (19) are said to be physically realizable if there exist matrices $\Theta_1 = \Theta_1^\dagger > 0$, $\tilde{M}_1 = \tilde{M}_1^\dagger$, \tilde{N} , and S such that $S^\dagger S = I$ and (20) is satisfied.

The following theorem from [14, 15, 51] gives a characterization of physical realizability in this case.

Theorem 4 The QSDEs (19) are physically realizable if and only if there exists a complex matrix $\Theta_1 = \Theta_1^\dagger > 0$ such that

$$\begin{aligned} \tilde{F}\Theta_1 + \Theta_1\tilde{F}^\dagger + \tilde{G}\tilde{G}^\dagger &= 0; \\ \tilde{G} &= -\Theta_1\tilde{H}^\dagger\tilde{K}; \\ \tilde{K}^\dagger\tilde{K} &= I. \end{aligned} \quad (45)$$

In the case of QSDEs of the form (19), the issue of physical realizability is determined by the lossless bounded real property of the corresponding transfer function matrix

$$\Gamma(s) = \tilde{H}(sI - \tilde{F})^{-1}\tilde{G} + \tilde{K}. \quad (46)$$

Definition 4 (See also [53].) The transfer function matrix (46) corresponding to the QSDEs (19) is said to be lossless bounded real if the following conditions hold:

1. \tilde{F} is a Hurwitz matrix; *i.e.*, all of its eigenvalues have strictly negative real parts;
- 2.

$$\Gamma(i\omega)^\dagger\Gamma(i\omega) = I$$

for all $\omega \in \mathbb{R}$.

Definition 5 (See also [14, 15, 51].) The QSDEs (19) are said to define a minimal realization of the transfer function matrix (46) if the following conditions hold:

1. *Controllability*

$$\text{rank} [\tilde{G} \quad \tilde{F}\tilde{G} \quad \tilde{F}^2\tilde{G} \quad \dots \quad \tilde{F}^{n-1}\tilde{G}] = n;$$

2. *Observability*

$$\text{rank} \begin{bmatrix} \tilde{H} \\ \tilde{H}\tilde{F} \\ \tilde{H}\tilde{F}^2 \\ \vdots \\ \tilde{H}\tilde{F}^{n-1} \end{bmatrix} = n.$$

The following theorem, which is a complex version of the standard lossless bounded real lemma, gives a state space characterization of the lossless bounded real property.

Theorem 5 (Complex Lossless Bounded Real Lemma; *e.g.*, see [14, 15, 53]). Suppose the QSDEs (19) define a minimal realization of the transfer function matrix (46). Then the transfer function (46) is lossless bounded real if and only if there exists a Hermitian matrix $X > 0$ such that

$$\begin{aligned} X\tilde{F} + \tilde{F}^\dagger X + \tilde{H}^\dagger \tilde{H} &= 0; \\ \tilde{H}^\dagger \tilde{K} &= -X\tilde{G}; \\ \tilde{K}^\dagger \tilde{K} &= I. \end{aligned} \tag{47}$$

Combining Theorems 4 and 5 leads to the following result which provides a complete characterization of the physical realizability property for minimal QSDEs of the form (19).

Theorem 6 (See [14, 15, 51].) Suppose the QSDEs (19) define a minimal realization of the transfer function matrix (46). Then, the QSDEs (19) are physically realizable if and only if the transfer function matrix (46) is lossless bounded real.

The following theorem from [14, 16], is useful in synthesizing coherent quantum controllers using state space methods.

Theorem 7 (See [14, 16].) Suppose the matrices F, G_1, H_1 define a minimal realization of the transfer function matrix

$$\Gamma_1(s) = H_1(sI - F)^{-1}G_1.$$

Then, there exists matrices G_2 and H_2 such that the following QSDEs of the form (19)

$$\begin{aligned} d\tilde{a}(t) &= F\tilde{a}(t)dt \\ &+ \begin{bmatrix} G_2 & G_1 \end{bmatrix} \begin{bmatrix} d\mathcal{A}_1(t) \\ d\mathcal{A}_2(t) \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}_1^{out}(t) \\ d\mathcal{A}_2^{out}(t) \end{bmatrix} &= \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tilde{a}(t)dt \\ &+ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} d\mathcal{A}_1(t) \\ d\mathcal{A}_2(t) \end{bmatrix} \end{aligned} \tag{48}$$

are physically realizable if and only if F is Hurwitz and

$$\|H_1(sI - F)^{-1}G_1\|_\infty \leq 1. \tag{49}$$

3.3. Physical Realizability for Position and Momentum Operator Linear Quantum Systems

It is often convenient to consider the physical realizability of real quantum linear systems of the form (26). This can be achieved by applying the transformations (25) and the equations (34), (32), (33), (36), (35) to obtain the following corollary of Theorem 1.

Corollary 1 (See also [11, 12].) The QSDEs (26) are physically realizable if and only if there exists a real matrix $\Theta = \Theta^T$ such that

$$\begin{aligned} A\tilde{\Theta} + \tilde{\Theta}A^T + B\tilde{J}B^T &= 0; \\ B &= \tilde{\Theta}C^T\tilde{J}D; \\ D\tilde{J}D^T &= \tilde{J}; \\ DD^T &= I. \end{aligned} \tag{50}$$

Remark 2 For real QSDEs of the form (26) with corresponding transfer function

$$Y(s) = C(sI - A)^{-1}B + D$$

It is straightforward using equations (25) to verify that this transfer function is related to the transfer function (40) of the corresponding complex QSDEs (12) according to the relation

$$Y(s) = \Phi \Gamma(s) \Phi^{-1}. \quad (51)$$

Now if the real QSDEs (26) are physically realizable, it follows that the corresponding complex QSDEs (12), (13) are physically realizable. Hence, using Theorem 3, it follows that the corresponding transfer function matrix (40) is dual (J, J) -unitary; *i.e.*,

$$\Gamma(s) J \Gamma^*(s) = J$$

for all $s \in \mathbb{C}_+$. Therefore, it follows from (51) and (32) that

$$Y(s) \tilde{J} Y^*(s) = \tilde{J}$$

for all $s \in \mathbb{C}_+$. Also, as in the discussion at the end of Section II.C, the conditions (41), (43) are equivalent to the conditions

$$\begin{aligned} D \tilde{J} D^T &= \tilde{J}; \\ D D^T &= I. \end{aligned}$$

4. COHERENT QUANTUM H^∞ CONTROL

In this section, we formulate a coherent quantum control problem in which a linear quantum system is controlled by a feedback controller which is itself a linear quantum system. The fact that the controller is to be a quantum system means that any controller synthesis method needs to produce controllers which are physically realizable. The problem we consider is the quantum H^∞ control problem in which it is desired to design a coherent controller such that the resulting closed loop quantum system is stable and attenuates specified disturbances acting on the system; see [11, 14, 16]. In the standard quantum H^∞ control problem such as considered in [11, 14, 16], the quantum noises are averaged out and only the external disturbance is considered. Other approaches to coherent quantum controller design include the coherent LQG approach. However the H^∞ approach has the advantage that it is more computationally tractable than the coherent LQG approach. Also, as in the case of classical H^∞ control, this approach enables the design of control systems which are robust to uncertainties in the plant model.

4.1. Coherent H^∞ Control of General Quantum Linear Systems

In this subsection, we formulate the coherent quantum H^∞ control problem for a general class of quantum systems of the form (12), (13).

We consider quantum *plants* described by linear complex quantum stochastic models of the following form defined in an analogous way to the QSDEs (12), (13):

$$\begin{aligned} \begin{bmatrix} d\tilde{a}(t) \\ d\tilde{a}(t)^\# \end{bmatrix} &= F \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix}; \\ dz(t) &= H_1 \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \end{bmatrix} dt + K_{12} du(t); \\ dy(t) &= H_2 \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} K_{20} & K_{21} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix} \end{aligned} \quad (52)$$

where all of the matrices in these QSDEs have a form as in (13). Here, the input

$$dw(t) = \begin{bmatrix} \beta_w(t)dt + d\mathcal{A}(t) \\ \beta_w^\#(t)dt + d\mathcal{A}(t)^\# \end{bmatrix}$$

represents a disturbance signal where $\beta_w(t)$ is an adapted process; see [11, 14, 42]. The signal $u(t)$ is a control input of the form

$$du(t) = \begin{bmatrix} \beta_u(t)dt + dB(t) \\ \beta_u^\#(t)dt + dB(t)^\# \end{bmatrix}$$

where $\beta_u(t)$ is an adapted process. The quantity

$$dv(t) = \begin{bmatrix} d\mathcal{C}(t) \\ d\mathcal{C}(t)^\# \end{bmatrix}$$

represents any additional quantum noise in the plant. The quantities $\begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}$, $\begin{bmatrix} dB(t) \\ dB(t)^\# \end{bmatrix}$ and $\begin{bmatrix} d\mathcal{C}(t) \\ d\mathcal{C}(t)^\# \end{bmatrix}$ are quantum noises of the form described in Section 2.

In the coherent quantum H^∞ control problem, we consider controllers which are described by QSDEs of the form (12), (13) as follows:

$$\begin{aligned} \begin{bmatrix} d\hat{a}(t) \\ d\hat{a}(t)^\# \end{bmatrix} &= F_c \begin{bmatrix} \hat{a}(t) \\ \hat{a}(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix} \\ \begin{bmatrix} du(t) \\ du_0(t) \\ du_1(t) \end{bmatrix} &= \begin{bmatrix} H_c \\ H_{c_0} \\ H_{c_1} \end{bmatrix} \begin{bmatrix} \hat{a}(t) \\ \hat{a}(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} K_c & 0 & 0 \\ 0 & K_{c_0} & 0 \\ 0 & 0 & K_{c_1} \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix} \end{aligned} \tag{53}$$

where all of the matrices in these QSDEs have a form as in (13). Here the quantities

$$dw_{c_0} = \begin{bmatrix} d\mathcal{A}_c(t) \\ d\mathcal{A}_c(t)^\# \end{bmatrix}, dw_{c_1} = \begin{bmatrix} d\mathcal{B}_c(t) \\ d\mathcal{B}_c(t)^\# \end{bmatrix}$$

are controller quantum noises of the form described in Section 2. Also, the outputs du and du_1 are unused outputs of the controller which have been included so that the controller can satisfy the definition of physical realizability given in Definition 1.

Corresponding to the plant (52) and (53), we form the closed loop quantum system by identifying the output of the plant dy with the input to the controller dy , and identifying the output of the controller du with the input to the plant du . This leads to the following closed-loop QSDEs:

$$\begin{aligned}
d\eta(t) &= \begin{bmatrix} F & G_2 H_c \\ G_c H_2 & F_c \end{bmatrix} \eta(t) dt \\
&+ \begin{bmatrix} G_0 & G_2 & 0 \\ G_c K_{20} & G_{c_0} & G_{c_1} \end{bmatrix} \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix} \\
&+ \begin{bmatrix} G_1 \\ G_c K_{21} \end{bmatrix} dw(t); \\
dz(t) &= \begin{bmatrix} H_1 & K_{12} H_c \end{bmatrix} \eta(t) dt \\
&+ \begin{bmatrix} 0 & K_{12} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw_{c_0}(t) \\ dw_{c_1}(t) \end{bmatrix}
\end{aligned} \tag{54}$$

where

$$\eta(t) = \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^\# \\ \hat{a}(t) \\ \hat{a}(t)^\# \end{bmatrix}.$$

For a given quantum plant of the form (52), the coherent quantum H^∞ control problem involves finding a physically realizable quantum controller (53) such that the resulting closed loop system (54) is such that the following conditions are satisfied: [(i)]

1. The matrix

$$F_{cl} = \begin{bmatrix} F & G_2 H_c \\ G_c H_2 & F_c \end{bmatrix} \tag{55}$$

is Hurwitz;

2. The closed loop transfer function

$$\Gamma_{cl}(s) = H_{cl}(sI - F_{cl})^{-1} G_{cl}$$

satisfies

$$\| \Gamma_{cl}(s) \|_\infty < 1 \tag{56}$$

where

$$H_{cl} = \begin{bmatrix} H_1 & K_{12} H_c \end{bmatrix}, G_{cl} = \begin{bmatrix} G_1 \\ G_c K_{21} \end{bmatrix}.$$

Remark 3 In the paper [11], a version of the coherent quantum H^∞ control problem is solved for linear quantum systems described by real QSDEs which are similar to those in (26). In this case, the problem is solved using a standard two Riccati equation approach such as given in [54, 55]. A result is given in [11] which shows that any H^∞ controller which is synthesized using the two Riccati equation approach can be made physically realizable by adding suitable additional quantum noises. Alternative solutions to the problem could be obtained using linear matrix inequality (LMI) approaches to the H^∞ control problem. However, the Riccati equation approach has the advantage that it is numerically less demanding than the LMI approach.

4.2. Coherent H^∞ Control of Annihilator Operator Quantum Linear Systems

In this subsection, we consider the special case of coherent quantum H^∞ control for annihilation operator quantum linear systems of the form considered in Subsection 2.2 and present the Riccati equation solution to this problem obtained in [14, 15]. The quantum H^∞ control problem being considered is the same as considered in Subsection 4.1 but we restrict attention to annihilation operator plants of the form (19) as follows:

$$\begin{aligned} d\tilde{a}(t) &= F\tilde{a}(t)dt + \begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix}; \\ dz(t) &= H_1\tilde{a}(t)dt + K_{12}du(t); \\ dy(t) &= H_2\tilde{a}(t)dt + \begin{bmatrix} K_{20} & K_{21} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix}. \end{aligned} \tag{57}$$

Also, we restrict attention to annihilation operator controllers of the form (19) as follows:

$$\begin{aligned} d\hat{a}(t) &= F_c\hat{a}(t)dt \\ &+ \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix}; \\ \begin{bmatrix} du(t) \\ du_0(t) \\ du_1(t) \end{bmatrix} &= \begin{bmatrix} H_c \\ H_{c_0} \\ H_{c_1} \end{bmatrix} \hat{a}(t)dt \\ &+ \begin{bmatrix} K_c & 0 & 0 \\ 0 & K_{c_0} & 0 \\ 0 & 0 & K_{c_1} \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix}. \end{aligned} \tag{58}$$

The quantum plant (57) is assumed to satisfy the following assumptions:

1. $K_{12}^\dagger K_{12} = E_1 > 0$;
2. $K_{21} K_{21}^\dagger = E_2 > 0$;
3. The matrix $\begin{bmatrix} F - i\omega I_n & G_2 \\ H_1 & K_{12} \end{bmatrix}$ is full rank for all $\omega \geq 0$;
4. The matrix $\begin{bmatrix} F - i\omega I_n & G_1 \\ H_2 & K_{21} \end{bmatrix}$ is full rank for all $\omega \geq 0$.

The results will be stated in terms of the following pair of complex algebraic Riccati equations:

$$\begin{aligned} (F - G_2 E_1^{-1} K_{12}^\dagger H_1)^\dagger X + X(F - G_2 E_1^{-1} K_{12}^\dagger H_1) \\ + X(G_1 G_1^\dagger - G_2 E_1^{-1} G_2^\dagger)X \\ + H_1^\dagger (I - K_{12} E_1^{-1} K_{12}^\dagger) H_1 = 0; \end{aligned} \tag{59}$$

$$\begin{aligned} (F - G_1 K_{21}^\dagger E_2^{-1} H_2)Y + Y(F - G_1 K_{21}^\dagger E_2^{-1} H_2)^\dagger \\ + Y(H_1^\dagger H_1 - H_2^\dagger E_2^{-1} H_2)Y \\ + G_1 (I - K_{21}^\dagger E_2^{-1} K_{21}) G_1^\dagger = 0. \end{aligned} \tag{60}$$

The solutions to these Riccati equations will be required to satisfy the following conditions.

1. The matrix $F - G_2 E_1^{-1} K_{12}^\dagger H_1 + (G_1 G_1^\dagger - G_2 E_1^{-1} G_2^\dagger) X$ is Hurwitz; i.e., X is a stabilizing solution to (59).
2. The matrix $F - G_1 K_{21}^\dagger E_2^{-1} H_2 + Y(H_1^\dagger H_1 - H_2^\dagger E_2^{-1} H_2)$ is Hurwitz; i.e., Y is a stabilizing solution to (60).
3. The matrices X and Y satisfy

$$\rho(XY) < 1 \quad (61)$$

where $\rho(\cdot)$ denotes the spectral radius.

If the above Riccati equations have suitable solutions, a quantum controller of the form (58) is constructed as follows:

$$\begin{aligned} F_c &= F + G_2 H_c - G_c H_2 + (G_1 - G_c K_{21}) G_1^\dagger X; \\ G_c &= (I - YX)^{-1} (Y H_2^\dagger + G_1 K_{21}^\dagger) E_2^{-1}; \\ H_c &= -E_1^{-1} (g^2 G_2^\dagger X + K_{12}^\dagger H_1). \end{aligned} \quad (62)$$

The following Theorem is presented in [14, 15].

Theorem 8 Necessity: Consider a quantum plant (57) satisfying the above assumptions. If there exists a quantum controller of the form (58) such that the resulting closed-loop system satisfies the conditions (55), (56), then the Riccati equations (59) and (60) will have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying (61).

Sufficiency: Suppose the Riccati equations (59) and (60) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying (61). If the controller (58) is such that the matrices F_c , G_c , H_c are as defined in (62), then the resulting closed-loop system will satisfy the conditions (55), (56).

Note that this theorem does not guarantee that a controller defined by (58), (62) will be physically realizable. However, if the matrices defined in (62) are such that

$$\| H_c (sI - F_c)^{-1} G_c \| < 1,$$

then it follows from Theorem 7 that a corresponding physically realizable controller of the form (58) can be constructed.

CONCLUSION

In this paper, we have surveyed some recent results in the area of quantum linear systems theory and the related area of coherent quantum H^∞ control. However, a number of other recent results on aspects of quantum linear systems theory have not been covered in this paper. These include results on coherent quantum LQG control (see [11, 38]), and model reduction for quantum linear systems (see [51]). These results have not been covered since the emphasis in this paper is on the H^∞ approach to coherent quantum control. Also, it is not possible to provide a direct comparison between H^∞ control methods and LQG control methods since their performance indices are measuring different quantities.

Furthermore, in order to apply synthesis results on coherent quantum feedback controller synthesis, it is necessary to realize a synthesized feedback controller transfer function using physical optical components such as optical cavities, beam-splitters, optical amplifiers, and phase shifters. In a recent paper [22], this issue was addressed for a general class of coherent linear quantum controllers. An alternative approach to this problem is addressed in [19] for the class of annihilation operator linear quantum systems considered in Subsection 2.2 and [14, 15, 16]. For this class of quantum systems, an algorithm is given to realize a physically realizable controller transfer function in terms of a cascade connection of optical cavities and phase shifters.

An important application of both classical and coherent feedback control of quantum systems is in enhancing the property of entanglement for linear quantum systems. Entanglement is an intrinsically quantum mechanical notion which has many applications in the area of quantum computing and quantum communications.

To conclude, we have surveyed some of the important advances in the area of linear quantum control theory. However, many important problems in this area remain open and the area provides a great scope for future research.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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