

Volterra Type Integral Equation by the Whittaker Cardinal Expansion

K. Maleknejad^{*,a} and M. Alizadeh^b

^aDepartment of Mathematics, Iran University of Science & Technology, Narmak, Tehran 16846 13114, Iran

^bFaculty of Science, Islamic Azad University of Kerman, Kerman, Iran

Abstract: In this study, we use Whittaker cardinal expansion and collocation method to approximate the solution of Volterra integral equation of the second kind numerically which is arisen in many mathematical, industrial and medical problems. Then convergence of this method will be discussed in L_α space for every positive α . At the end of the paper, some numerical examples are presented to guarantee the accuracy of this method.

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INTRODUCTION

Volterra type integral equation has been arisen in many mathematical and engineering field, so that solving this kind of problems are more efficient and useful in many research branches. Analytical solution of this kind of equation is not accessible in general form of equation and we can only get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy [1-5]. In this study we want to discuss on convergence of projection method with Sinc function then present a numerical solution to this type of equation.

Consider Volterra integral equation of the second kind

$$f(s) = \int_a^s k(s,t)f(t)dt + g(s) \quad -\infty < c \leq s \leq d < \infty \quad (1)$$

where $k(s,t)$ and $g(s)$ are known functions, but $f(t)$ is an unknown function. In the operator form of this equation, we have

$$(I - \mathcal{K})f = g \quad (2)$$

where I is the identity operator and \mathcal{K} is an integral operator defined by

$$\mathcal{K} : X \rightarrow X$$

$$\mathcal{K}f = \int_a^s k(s,t)f(t)dt$$

with X is a normed space.

Background

Sinc function is defined on the whole real line by

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Now, we define the sequence of Sinc basis function for positive h by

$$S(k,h)(x) = \frac{\sin(\pi(x - kh) / h)}{\pi(x - kh) / h}$$

where k is an integer.

Definition 1 Let $h > 0$, and let $W(\frac{\pi}{h})$ denote the family of functions f that are analytic in \mathbb{C} (set of all complex functions), such that

$$\int |f(t)|^2 dt < \infty$$

and for all z in \mathbb{C} (set of all complex numbers)

$$|f(z)| \leq ce^{\frac{\pi|z|}{h}}$$

with c a positive constant.

Definition 2 Let $d > 0$ and D_d denote the domain

$$D_d = \{\omega \in \mathbb{C} : |Im(\omega)| < d\}$$

Now for every positive α, β we define $L_{\alpha,\beta}(D_d)$ is a space of functions f that are analytic in D_d such that for some $c > 0$ and all $z \in D_d$ we have

$$|f(z)| \leq c \frac{|e^{\alpha z}|}{(1 + |e^z|)^{\alpha+\beta}}.$$

*Address correspondence to this author at the Department of Mathematics, Iran University of Science & Technology, Narmak, Tehran 16846 13114, Iran; E-mail: maleknejad@iust.ac.ir

If $\alpha = \beta$, we have $L_{\alpha,\alpha}(D_d)$, but for simplicity we denote $L_\alpha(D_d)$ instead of $L_{\alpha,\alpha}(D_d)$.

Theorem 1 Let $h > 0$, the sequence $\{h^{-\frac{1}{2}}S(k,h)\}_{k=-\infty}^\infty$ is complete orthonormal sequence in $W(\frac{\pi}{h})$. Every f in $W(\frac{\pi}{h})$ has the cardinal series representation

$$f(x) = C(f,h)(x) = \sum_{k=-\infty}^\infty f(kh)S(k,h)(x)$$

Proof. [6] page 91.

For $h > 0$, the series $C(f,h)(x)$ is called the whittaker cardinal expansion of f whenever this series converges [6].

PROJECTION METHOD

In projection methods such as collocation method for solving Eq.(2), we assume that $X_n \subset X = L_\alpha$ and $\{\phi_i\}$ $i = 1, \dots, n$ are basis functions for X_n . By this assumption we try to solve integral equation of the second kind in a space with finite elements [7, 8].

In this paper, X is $W(\frac{\pi}{h})$ for every positive h which is defined in next section and X_n is a space constructed by Sinc basis functions. By using Whittaker cardinal expansion we can approximate every function in this space.

Now, we assume X_n is a subspace of $W(\frac{\pi}{h})$ constructed by orthonormal sequence $\{h^{-1/2}S(k,h)\}_{k=-N}^N$ [6]. Thus, we approximate an unknown function $f(s)$ by this orthonormal sequence, so by sampling theorem we have

$$f(t) \approx C_N(f,h)(t) = \sum_{k=-N}^N f(kh)S(k,h)(t) \quad \forall h > 0$$

then by substituting this in integral equation we get

$$\begin{aligned} & \sum_{k=-N}^N f(kh)S(k,h)(s) \\ &= \int_a^s k(s,t) \sum_{k=-N}^N f(kh)S(k,h)(t)dt + g(s). \end{aligned}$$

Now, we define residual equation by

$$\begin{aligned} R_N(s) &= \sum_{k=-N}^N f(kh)S(k,h)(s) \\ &- \int_a^s k(s,t) \sum_{k=-N}^N f(kh)S(k,h)(t)dt - g(s) \end{aligned}$$

for determining the unknown coefficients $f(kh)$'s we select some collocation points such that

$$R_N(s_i) = 0 \quad i = 0, 1, \dots, 2N$$

in this paper collocation points are

$$s_i = a + \frac{i(d-a)}{2N} \quad i = 0, 1, \dots, 2N$$

so that we have a system of linear equations $A_N X = b_N$ where

$$A_N = [S(k,h)(s_i) - \int_a^{s_i} k(s_i,t)S(k,h)(t)dt]_{k=-N}^N \quad (3)$$

$$b_N = [g(s_i)] \quad i = 0, 1, \dots, 2N \quad (4)$$

$$X^T = [f(kh)]_{k=-N}^N \quad (5)$$

In this system of linear equations we need to determine $\int_a^{s_i} k(s_i,t)S(k,h)(t)dt$. In this paper, we use a known numerical integration formula known as Clenshaw-Curtis method [9] to estimate this integral term and evaluate the elements of coefficient matrix A .

MAIN RESULTS

In this section, we want to investigate the convergence of this method for Volterra integral equation of the second kind by collocation method. To achieve this goal, firstly, we introduce definitions and theorems of Sinc approximation in strip [6], then use them to prove the convergence of our method.

Definition 3 Let $\lambda > 0$ and D_λ denote the domain

$$D_\lambda = \{\omega \in \mathbb{C} : |Im(\omega)| < \lambda\}$$

Now for every positive α, β we define $L_{\alpha,\beta}(D_\lambda)$ is a space of functions f that are analytic in D_λ such that for some $c > 0$ and all $z \in D_\lambda$ we have

$$|f(z)| \leq c \frac{|e^{\alpha z}|}{(1 + |e^z|)^{\alpha+\beta}}$$

If $\alpha = \beta$ we have $L_{\alpha,\alpha}(D_\lambda)$ but for simplicity we denote $L_\alpha(D_\lambda)$ instead of $L_{\alpha,\alpha}(D_\lambda)$.

Theorem 2 Let $f \in L_\alpha(D_\lambda)$ for $\alpha > 0$ and taking $h = (\frac{\pi\lambda}{\alpha N})^{1/2}$ then there exists a positive number c_1 , depending only on f, λ, α, y such that for $s = 2$ or $s = \infty$

$$\begin{aligned} & \|E_N(f,h)(\cdot + iy)\|_s \\ &= \left\| f(\cdot + iy) - \sum_{k=-N}^N f(kh)S(k,h)(\cdot + iy) \right\|_s \\ &\leq c_1 N^{(1-1/s)/2} \exp\left\{-\left(\frac{\pi\alpha}{\lambda}\right)^{1/2}(\lambda - |y|)N^{1/2}\right\} \end{aligned}$$

Proof. [6] page 137.

In this paper we choose $s = \infty$ and for the real case we can substitute $y = 0$ so that if $(\pi\alpha\lambda)^{1/2} = c_2$ then

$$\|E_N(f,h)(\cdot)\|_\infty \leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\}.$$

Theorem 3 By regard to Eq.(1) where $k(s, t)$ is continuous on square $[c, d] \times [a, d]$ and $f(t), g(t) \in L_\alpha([a, d])$ for $\alpha > 0$, if we define

$$P_N^{num}(f)(t) = \sum_{k=-N}^N f^{num}(kh)S(k, h)(t)$$

as a Sinc approximation of $f(t)$ where $f^{num}(kh)$ is the solution of linear system of equations constructed by collocation method then

$$\begin{aligned} & \|f - P_N^{num}(f)\|_\infty \\ & \leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\} \{1 + c_3 \|A_N^{-1}\| \frac{2}{\pi} (3 + \log(N))\} \end{aligned}$$

Proof. First of all we define the following projection operator

$$f(t) \approx P_N^{num}(f)(t) = \sum_{k=-N}^N f^{num}(kh)S(k, h)(t) \quad (6)$$

In this projection operator $f^{num}(kh)$ is the solution of linear system of equations $A_N X = b_N$, so that $f^{num}(kh)$ contains truncation error and round off error. Another projection that we need in this theorem is

$$P_N(f)(t) = \sum_{k=-N}^N f(kh)S(k, h)(t)$$

where $f(kh)$ contains only truncation error. Practically, we solve

$$P_N^{num}(f)(s) = \int_a^s k(s, t) P_N^{num}(f)(t) dt + g(s)$$

to find $f^{num}(kh)$. So that we have

$$[f^{num}(kh)]_{k=-N}^N = A_N^{-1} [g(s_i)] \quad i = 0, 1, \dots, 2N$$

But if we substitute $P_N(f)(t) = \sum_{k=-N}^N f(kh)S(k, h)(t)$ then we define the new right hand side for Eq.(1) that is denoted by \hat{g} and estimated as follow;

$$\begin{aligned} & P_N(f)(s) + f(s) - P_N(f)(s) \\ & = \int_a^s k(s, t) [f(t) - P_N(f)(t) + P_N(f)(t)] dt + g(s) \end{aligned}$$

$$P_N(f)(s) - \int_a^s k(s, t) P_N(f)(t) dt = g(s) - f(s) +$$

$$P_N(f)(s) + \int_a^s k(s, t) [f(t) - P_N(f)(t)] dt$$

then

$$\hat{g}(s) = g(s) - f(s) + P_N(f)(s) +$$

$$\int_a^s k(s, t) [f(t) - P_N(f)(t)] dt$$

now we have

$$[f(kh)]_{k=-N}^N = A_N^{-1} [\hat{g}(s_i)] \quad i = 0, 1, \dots, 2N$$

so that

$$\sup_{k \in S_N} |f^{num}(kh) - f(kh)| \leq \|A_N^{-1}\| \sup_{i \in S_N} |g(s_i) - \hat{g}(s_i)|$$

where S_N is all integers belong to $[-N, N]$ and $\|A_N^{-1}\|$ is the maximum norm on columns or rows, so

$$\sup_{i \in S_N} |g(s_i) - \hat{g}(s_i)| \leq \sup_{i \in S_N} |P_N(f)(s_i) - f(s_i) +$$

$$\int_a^s k(s_i, t) [f(t) - P_N(f)(t)] dt |$$

$$\leq \sup_{i \in S_N} |P_N(f)(s_i) - f(s_i)|$$

$$+ \sup_{i \in S_N} | \int_a^s k(s_i, t) [f(t) - P_N(f)(t)] dt |$$

$$\leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\}$$

$$+ \sup_{t \in [a, d]} |k(s_i, t)| \sup_{t \in [a, d]} |[f(t) - P_N(f)(t)]| d - a |$$

$$\leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\} + M |d - a| c_1 N^{1/2} \exp\{-c_2 N^{1/2}\}$$

where $M = \sup_{t \in [a, d]} |k(s_i, t)|$ and let $(1 + M |d - a|) = c_3$ so

$$\sup_{i \in S_N} |g(s_i) - \hat{g}(s_i)| \leq c_1 c_3 N^{1/2} \exp\{-c_2 N^{1/2}\}$$

and

$$\sup_{k \in S_N} |f^{num}(kh) - f(kh)| \leq \|A_N^{-1}\| c_1 c_3 N^{1/2} \exp\{-c_2 N^{1/2}\}.$$

Now we have

$$\sup_{t \in [a, d]} |P_N(f)(t) - P_N^{num}(f)(t)|$$

$$= \sup_{t \in [a, d]} | \sum_{k=-N}^N [f(kh) - f^{num}(kh)] S(k, h)(t) |$$

$$\leq \sup_{k \in S_N} |f(kh) - f^{num}(kh)| \sup_{t \in [a, d]} \sum_{k=-N}^N |S(k, h)(t)|$$

$$\leq \|A_N^{-1}\| c_1 c_3 N^{1/2} \exp\{-c_2 N^{1/2}\} \left(\frac{2}{\pi} \{3 + \log(N)\} \right)$$

By regard to the [6]

$$\sup_{t \in [a, d]} \sum_{k=-N}^N |S(k, h)(t)| \leq \frac{2}{\pi} \{3 + \log(N)\}$$

Finally, we have

$$\|f - P_N^{num}(f)\|$$

$$\leq \|f - P_N(f)\| + \|P_N(f) - P_N^{num}(f)\|$$

$$\leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\} +$$

$$\|A_N^{-1}\| c_1 c_3 N^{1/2} \exp\{-c_2 N^{1/2}\} \frac{2}{\pi} \{3 + \log(N)\}$$

$$\leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\} \{1 + c_3 \|A_N^{-1}\| \frac{2}{\pi} (3 + \log(N))\}$$

and proof of this theorem is completed.

NUMERICAL EXAMPLES

In this section, we use relations (3),(4),(5) to convert an integral equation to system of linear equations, for this result we choose $\alpha = 1$ and $\lambda = \pi$ so that $h = \frac{\pi}{\sqrt{N}}$. In this procedure we use Clenshaw-Curtis method to estimate $\int_a^{s_i} k(s_i, t)S(k, h)(t)dt$ numerically. In these examples E_N is defined by

$$E_N = \max_i |f(t_i) - P_N^{num}(f)(t_i)| \quad i = 0, 1, \dots, 2N$$

where t_i 's are the collocation points.

In the following, we present some examples then numerical results are shown in Table 1, and these results are easily Comparable against other numerical methods which were appeared in [5-9] in points of convergence rate and error bound.

Example 1. In this example we solve Eq.(1) with

$$k(s, t) = \frac{(s - t)^2}{1 + t^2}$$

$$g(s) = \sqrt{1 + s^2} - s(2 - \frac{3\sqrt{1 + s^2}}{2}) + (\frac{1}{2} - s^2)arcsinh(s)$$

where the exact solution is $f(t) = \sqrt{1 + t^2}$, and E_N is evaluated in $[0, 1]$.

Example 2. In this example we solve Eq.(1) with

$$k(s, t) = s \sin(t + s) - t \cos(t - s)$$

$$g(s) = \frac{1}{4}((5 + s^2 + 2s \cos(2s))\sin(s) - (2s^2 + s)\cos(s))$$

where the exact solution is $f(t) = \sin(t)$, and E_N is evaluated in $[0, \frac{\pi}{2}]$.

Table 1. Numerical Results for Examples

| N | E_N for Ex.1 | E_N for Ex.2 |
|---|--------------------------|--------------------------|
| 3 | 3.8315×10^{-3} | 5.6884×10^{-2} |
| 5 | 8.8190×10^{-7} | 5.1365×10^{-6} |
| 7 | 7.8724×10^{-11} | 1.7283×10^{-12} |
| 9 | 3.2124×10^{-11} | 1.2213×10^{-15} |

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