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Some Observations About Dissipative Properties for Dynamical Systems Under Change of Variables

Guillermo Fernández-Anaya* and José-Job Flores-Godoy

Departamento de Física y Matemáticas, Universidad Iberoamericana, Paseo de la Reforma #880, Lomas de Santa Fe México, D.F. 01219, México

Abstract: This note gives results about the preservation of some dissipative properties of systems under a change of variables. In the textbooks it is not mentioned explicitly the relationship between the equations associated with the dynamics of a system and the selected Lyapunov function to establish its stability property, when a change of coordinates is used. Based on the fact that Lyapunov stability is preserved under these changes of variables, it is shown that various forms of dissipativity can be preserved. In addition, we will show that the input-state stability (ISS), integral input-to-state stability (iISS) and input/output to state stability (IOSS) can be preserved under this class of transformation. Some examples are given to show these results.

Keywords: Preservation of dissipativity, preservation of input-state stability, preservation of integral input-to-state stability, preservation of input/output to state stability, change of coordinates.

1. INTRODUCTION

The notion of dissipativity in dynamical systems was introduced in the early 1970s. This concept was originally stated by Willems in the seminal papers [1, 2] and it was later extended and explored in [3]. This concept generalizes the idea of a Lyapunov function and has found applications in diverse areas such as stability theory, chaos and synchronization theory, system norm estimation, and robust control.

Dissipativity is one of the most important concepts in systems and control theory, both from the theoretical point of view as well as from the practical perspective. Dissipativity theory is based on a characterization of open systems by a dissipation inequality between the storage variation and a supply rate. The storage reflects the energy stored in the system's internal components. The supply rate governs the exchange of energy with the external world. In many mechanical and electrical engineering applications, dissipativity is related to the notion of energy. This fundamental ideas are strongly related with concepts like passivity and finite gain, see [1-4], and constitute a fundamental basis of the development of the robustness analysis [5-8], for applications of the dissipativity ideas on power systems, and for further developments on dissipativity and input-to state stability, see [9-14].

Now days there are several definitions uses for dissipativity in the literature, in this work we used the definitions given in [14, 15]. We aim our study at the conditions needed for a change of coordinates to leave invariant the property of dissipativity of a nonlinear dynamical system, such a study has not been pursued as can be seen in books like [13, 15, 16] or in specialized articles on the topic like [10, 11, 14]. Here westudy this invariance for different definitions of dissipativity and the input-state stability (ISS), integral input-to-state stability (iISS) and input/output to state stability (IOSS). Our objective is to clarify and contribute for a better understanding of the preservation of some types of stability when using diffeomorphism and also to motivate new venues of study to this topic.

2. PRESERVING LYAPUNOV STABILITY

Usually it is not stated explicitly on text books how a Lyapunov function associated to a dynamical system is transformed by a change of coordinates, i.e., a diffeomorphism (in general, any mapping such that its Jacobian matrix is invertible), of the dynamical system. It is therefore important to know what class of change of coordinate's leaves invariant a particular definition of stability. In this work we will consider the following class of change of coordinates:

Definition 1. Let us consider a smooth map $\alpha : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$, with Ω an open domain in R^n that contains the origin, i. e., $0 \in \Omega$; also $\alpha(0) = 0$, $\alpha(v(t)) \neq 0$ for $v(t) \neq 0$ in Ω; the inverse of the Jacobian matrix $\left(\frac{\partial \alpha(v(t))}{\partial v}\right)^{-1}$ exists in Ω; all of these conditions imply that α is a diffeomorphism [17, 18].

The set of all smooth maps that satisfies Definition 1 will be denoted by $\mathrm{Diff}_{n,\Omega}$.

In what follows we present examples on how a diffeomorphism preserves some types of stability. For example, we consider the standard definition of stability for

*Address correspondence to this author at the Departamento de Física y Matemáticas, Universidad Iberoamericana, Paseo de la Reforma #880, Lomas de Santa Fe México, D.F. 01219, México;
Tel: 55 59 50 40 71; E-mail: guillermo.fernandez@uia.mx

a linear time invariant system. Let us consider a linear time invariant system described by

$$\dot{x}(t) = Ax(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$. The system (1) is stable in the sense of Lyapunov if and only if for the Lyapunov function $V(x(t)) = x^\top(t)Px(t)$, its time derivative along the trajectories of (1) is such that

$$\dot{V}(x(t)) = x^\top(t)\left(PA + A^\top P\right)x(t) \leq 0,$$

for all $x(t) \in \mathbb{R}^n$ and $P = P^\top \geq 0$.

Now consider the change of variable defined by $x(t) = \alpha(v(t))$, with $\alpha(\cdot) \in \mathrm{Diff}_{n,R^n}$, such that system (1) is transformed into a nonlinear system

$$\dot{v}(t) = \left(\frac{\partial\alpha(v(t))}{\partial v}\right)^{-1} A\alpha((v(t)). \tag{2}$$

Then the stability property of system (1)is preserved for the transformed system given in (2) as long as the following properties are satisfied

$$\begin{aligned} V(\alpha(v(t))) &= \alpha^\top(v(t))P\alpha(v(t)) \geq 0, \\ \dot{V}(\alpha(v(t))) &= \alpha^\top(v(t))\left[PA + A^\top P\right]\alpha(v(t)) \leq 0. \end{aligned}$$

Similar results can be drawn for asymptotic stability with the corresponding hypothesis [16].

For the case of a stable autonomous nonlinear system of the form

$$\dot{x}(t) = f(x(t)), \tag{3}$$

where $f(\cdot)$ is a differentiable function, with Lyapunov function $V(x(t))$, i.e. $V(\cdot)$ satisfies the conditions of Lyapunov's second theorem for stability [16]. Consider a mapping $x(t) = \alpha(v(t))$ with $\alpha(\cdot) \in \mathrm{Diff}_{n,\Omega}$, then the system in (3) can be transformed into the following nonlinear system

$$\dot{v}(t) = \left(\frac{\partial\alpha(v(t))}{\partial v}\right)^{-1} f\alpha(v(t)). \tag{4}$$

Now, by the chain rule applied to the Lyapunov function $V(\alpha(v(t)))$, with

$$\nabla_v^\top V(\alpha(v)) = \nabla_x^\top V(x)\Big|_{x=\alpha(v)} \frac{\partial\alpha(v)}{\partial v}, \tag{5}$$

for its gradient, with $\nabla_v^\top V(v) : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ and $\nabla_x^\top V(x) : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ row vectors. Again the Lyapunov stability is preserved, because the following inequality is satisfies in a neighborhood of the origin

$$\begin{aligned} \dot{V}(\alpha(v(t))) &= \nabla_v^\top V(\alpha(v))\dot{v}(t), \\ &= \nabla_x^\top V(x)\Big|_{\alpha(v)} f(\alpha(v(t))) \leq 0, \end{aligned} \tag{6}$$

since the substitution of $x(t)$ by the map $\alpha(v(t))$ preserves the following inequality in a neighborhood of the origin

$$\dot{V}(x(t)) = \nabla_x^\top V(x)\, f(x(t)) \leq 0.$$

Also, the asymptotic stability can be preserved under suitable conditions and the proof is similar.

3. PRESERVATION OF DISSIPATIVE SYSTEMS

In this section we present some results, for a class of well-known dynamical systems, on preservation of dissipativity. Consider the following state-space representation affine in the input and output:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \\ y(t) &= h(x(t)) + j(x(t))u(t), \end{aligned} \tag{7}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $j : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, are smooth functions of $x(t)$ with $f(0) = 0$ and $h(0) = 0$.

Definition 2. [15] The system (7) is dissipative with respect to the supply rate $w(u, y)$ and energy storage function $V(x(t))$, if for all admissible $u(\cdot)$ and all $t_1 \geq t_0$ one has

$$\int_{t_0}^{t_1} w(u(t), y(t))\,dt \geq V(x(t_1)) - V(x(t_0)),$$

along the trajectories of the system (7), with $x(t_0) = 0$.

The following assumptions are made for this paper: (i) The state space of the system (7) is reachable from the origin; (ii) The available storage function $V(\cdot)$, when it exists, is a differentiable function of x; (iii) the supply rate considered is the following:

$$\begin{aligned} w(u, y) &= y^\top Qy + 2y^\top Su + u^\top Ru, \\ &= \begin{bmatrix} y^\top & u^\top \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \end{aligned} \tag{8}$$

with Q and R symmetric matrices.

The following fundamental theorem is known as the nonlinear Kalman-Yakubovich-Popov Lemma.

Lemma 3. [15] The nonlinear system (7) is dissipative in the sense of Definition 2 with respect to the supply rate $w(u, y)$ in (8) if and only if there exists functions $V : \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \to \mathbb{R}^q$, $W : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ (for some integer q), with $V(\cdot)$ differentiable, such that:

$$\begin{aligned} V(x) &\geq 0, \\ V(0) &= 0, \\ \nabla^\top V(x)f(x) &= h^\top(x)Qh(x) - L^\top(x)L(x), \\ \frac{1}{2}g^\top(x)\nabla V(x) &= \hat{S}^\top(x)h(x) - W^\top(x)L(x), \\ \hat{R}(x) &= W^\top(x)W(x), \end{aligned} \tag{9}$$

where

$$\begin{aligned}\hat{S}(x) &\triangleq Qj(x) + S,\\ \hat{R}(x) &= R + j^\top(x)S + S^\top j(x) + j^\top(x)Qj(x).\end{aligned} \tag{10}$$

For the transformation $x(t) = \alpha(v(t))$, induced by the map $\alpha(v(t)) \in \mathrm{Diff}_{n,\Omega}$, i. e., if $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $\alpha(v(t)) = (\alpha_1(v(t)), \alpha_2(v(t)), \ldots, \alpha_n(v(t)))$ then $x_i(t) = \alpha_i(v(t))$ for $i = 1, \ldots, n$. The system (8) is transformed in:

$$\begin{aligned}\left(\frac{\partial \alpha(v)}{\partial v}\right)\dot{v}(t) &= f(\alpha(v(t))) + g(\alpha(v(t)))u(t),\\ z(t) &= h(\alpha(v(t))) + j(\alpha(v(t)))u(t),\end{aligned} \tag{11}$$

where $\frac{\partial \alpha(v)}{\partial v}$ is the Jacobian matrix of $\alpha(v)$.

Lemma 4 is the basic result from which Proposition 5 is proved.

Lemma 4. The nonlinear system (7) is dissipative with respect to the supply rate(8), and $\alpha \in \mathrm{Diff}_{n,R^n}$. Then

$$\begin{aligned}V(\alpha(v(t))) &\geq 0,\\ V(\alpha(0)) &= 0\\ \nabla^\top V(x)\big|_{\alpha(v)} f(\alpha(v)) &= h^\top(\alpha(v))Qh(\alpha(v)) - L^\top(\alpha(v))L(\alpha(v)),\\ \frac{1}{2}g^\top(\alpha(v))\left[\nabla V(x)\big|_{\alpha(v)}\right] &= \hat{S}^\top(\alpha(v))h(\alpha(v)) - W^\top(\alpha(v))L(\alpha(v)),\\ \hat{R}(\alpha(v)) &= W^\top(\alpha(v))W(\alpha(v)),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\hat{S}(\alpha(v)) &\triangleq Qj(\alpha(v)) + S,\\ \hat{R}(\alpha(v)) &= R + j^\top(\alpha(v))S + S^\top j(\alpha(v)) + j^\top(\alpha(v))Qj(\alpha(v)),\end{aligned} \tag{13}$$

where the notation $\nabla^\top V(x)\big|_{\alpha(v)}$ means the gradient $\nabla^\top V(x)$ evaluated at $\alpha(v)$.

Proof. Making the substitution $x(t)$ by the map $\alpha(v(t)$ in the equations (9) and (10), taking into account that $\alpha \in \mathrm{Diff}_{n,R^n}$, this substitution is also an automorphism in the commutative ring of the differentiable functions $f: \mathbb{R}^n \to \mathbb{R}^n$, and recalling that an automorphism preserves product of functions, addition of functions, constant functions and identities, we obtain the equations (12) and (13).

The following proposition shows that the dissipativity property of a system presented in Definition 2 is preserved under diffeomorphism.

Proposition 5. Consider the same hypothesis of the Lemma 3, and $\alpha \in \mathrm{Diff}_{n,R^n}$. The system (7) is dissipative with respect to the supply rate $w(u,y)$ in (8) if and only if the system

$$\begin{aligned}\dot{v}(t) &= \left(\frac{\partial \alpha(v)}{\partial v}\right)^{-1}\left[f(\alpha(v(t))) + g(\alpha(v(t)))u(t)\right],\\ z(t) &= h(\alpha(v(t))) + j(\alpha(v(t)))u(t),\end{aligned} \tag{14}$$

is dissipative with respect to the supply rate $w(u,z) = z^\top Qz + 2z^\top Su + u^\top Ru$.

Proof. To simplify the notation we will drop the dependence on t in the following. Using (8)-(14), Lemma 3 and Lemma 4, we have

$$\begin{aligned}w(u,z) &= z^\top Qz + 2z^\top Su + u^\top Ru,\\ &= \big(h(\alpha(v)) + j(\alpha(v)))u\big)^\top Q\big(h(\alpha(v)) + j(\alpha(v)))u\big)\\ &\quad + 2\big(h(\alpha(v)) + j(\alpha(v)))u\big)^\top Su + u^\top Ru,\\ &= \nabla^\top V(x)\big|_{\alpha(v)} f(\alpha(v)) + L^\top(\alpha(v))L(\alpha(v))\\ &\quad + u^\top W^\top(\alpha(v))W(\alpha(v))u\\ &\quad + u^\top g^\top(\alpha(v))\nabla V(x)\big|_{\alpha(v)}\\ &\quad + 2u^\top W^\top(\alpha(v))L(\alpha(v)).\end{aligned}$$

Since

$$\nabla_v V^\top(\alpha(v(t))) = \nabla^\top V(x)\big|_{\alpha(v)} \frac{\partial \alpha(v)}{\partial v},$$

we have

$$\begin{aligned}w(u,z) &= \nabla^\top V(x)\big|_{\alpha(v)}\left[f(\alpha(v(t))) + g(\alpha(v))u\right] +\\ &\quad \big(L(\alpha(v)) + W(\alpha(v))u\big)^\top \big(L(\alpha(v)) + W(\alpha(v))u\big),\\ &= \nabla_v V^\top(\alpha(v))\dot{v} +\\ &\quad \big(L(\alpha(v)) + W(\alpha(v))u\big)^\top \big(L(\alpha(v)) + W(\alpha(v))u\big),\\ &\geq \nabla_v V^\top(\alpha(v))\dot{v} = \frac{d}{dt}V(\alpha(v)).\end{aligned}$$

Integrating the last term, we obtain

$$\int_0^t w(u(s), z(u(s), v(s)))ds \geq V(\alpha(v(t))) - V(\alpha(v(0))).$$

Therefore the system (14)is dissipative with respect to the supply rate $w(u,z)$.

For the converse the procedure is very similar.

4. PRESERVATION OF ISS, IISS AND IOSS *VIA* DISSIPATIVITY

Based on [9, 10, 14, 16] it is shown that input-to-state stability (ISS), integral input-to-state stability (iISS) and input/output to state stability (IOSS) can be preserved *via* its dissipativity characteristics. To this end let us introduce the notation and definitions needed for this section.

Defintion 6. [14, 15]

1. A continuous function $g: \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class K ($g \in K$) if
   (a) $g(0) = 0$,
   (b) $g(l) > 0 \quad \forall l \in \mathbb{R}^+$,
   (c) $g(\cdot)$ is non decreasing.

Statements 1b) and 1c) can also be replaced with the following item

1b') $g(\cdot)$ is strictly increasing so that the inverse function $g^{-1}(\cdot)$ is defined.

2. The function g is said to be of class K_∞ ($g \in K_\infty$) if $g \in K$ and $g(l) \to \infty$ when $l \to \infty$.

3. A function $g(\cdot)$ is positive definite if $g(s) > 0$ for all $s > 0$, and $g(0) = 0$.

4. A function $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a KL-function ($p \in KL$) if:

   (a) for each fixed $t \geq 0$ the function $p(\cdot, t)$ is a K-function, and

   (b) for each fixed $s \geq 0$ the function $p(s, \cdot)$ is decreasing to zero as $t \to \infty$.

Defintion 7. [14, 15] We will say that a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if it is positive definite, that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and proper, that is, $V(x) \to \infty$ as $|x| \to \infty$. This last property is equivalent to the requirement that the sets $V^{-1}([0, A])$ should be a compact subsets of $\mathbb{R}^n$, for each $A > 0$, and in the engineering literature it is usual to call such functions radially unbounded. It is well-known [14] that V is a storage function if and only if there exist functions $\underline{\alpha}, \overline{\alpha} \in K_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|).$$

The notation $\dot{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the function:

$$\dot{V}(x, u) := \nabla^\top V(x) f(x, u),$$

which provides, when evaluated at $(x(t), u(t))$, the derivative dV / dt along solutions of $\dot{x} = f(x, u)$.

Now let us consider a dynamical system as follows

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \tag{15}$$

with $f(\cdot, \cdot)$ locally Lipschitz such that $f(0, 0) = 0$, and $u(t) \in U$ where U is a set of measurable locally essentially bounded functions from $\mathbb{R}^+$ into $\mathbb{R}^m$. Let us also consider a change of coordinates, $x(t) = \alpha(v(t))$ as in Definition 1, then we can write system (15) as

$$\dot{v} = \left(\frac{\partial \alpha(v)}{\partial v} \right)^{-1} f(\alpha(v), u). \tag{16}$$

We are interested in the input-to-state mapping $(x_0, u(\cdot)) \to x(\cdot)$ stability in the following sense

Defintion 8. [15] The system (15) is ISS if:

1. For each x_0 there is a unique solution in $C^0\left(\mathbb{R}^+, \mathbb{R}^n\right)$.

2. The map $\mathbb{R}^n \times U \to C^0\left(\mathbb{R}^+, \mathbb{R}^n\right)$, $(x_0, u) \to x(\cdot)$ is continuous at $(0, 0)$.

3. There exists a nonlinear asymptotic gain $\gamma(\cdot)$ of class K such that

$$\limsup_{t \to +\infty} \left\| x\left(t, x_0, u\right) \right\| \leq \gamma\left(\|u\|_\infty \right),$$

uniformly on x_0 in any compact set and all $u \in U$, where $\|\cdot\|$ denotes the standard Euclidian norm.

Equivalently [14, 16], the system (15) is ISS if there exist some functions $\beta \in KL$ and $\gamma \in K_\infty$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma\left(\|u\|_\infty \right),$$

holds for all solutions i.e., this estimate is valid for all inputs $u(\cdot)$, all initial conditions x_0, and all $t \geq 0$.

Defintion 9. [14] A storage function $V(\cdot)$ as in Definition 7 is an ISS-Lyapunov function if there exists two class-K functions $\theta(\cdot)$ and $\chi(\cdot)$ and there exist two class-K_∞ functions $\delta_1(\cdot)$ and $\delta_2(\cdot)$ such that

$$\delta_1\left(\|x\|\right) \leq V\left(x\right) \leq \delta_2\left(\|x\|\right), \tag{17}$$

holds for all $x \in \mathbb{R}^n$ and

$$\|x\| \geq \chi\left(\|u\|\right) \Rightarrow \nabla^\top V(x) f(x, u) \leq -\theta\left(\|x\|\right), \tag{18}$$

holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Defintion 10. [14] The system (15) is iISS if there exist functions $\varphi \in K_\infty$, $\mu \in KL$ and $\nu \in K$, such that, for all $\xi \in \mathbb{R}^n$ and all $u \in U$, the solution $x\left(t, \xi, u\right)$ is defined for all $t \geq 0$, and

$$\varphi\left(\left\| x\left(t, \xi, u\right) \right\| \right) \leq \mu\left(\|\xi\|, t \right) + \int_0^t \nu\left(\|u(s)\| \right) ds, \tag{19}$$

for all $t \geq 0$.

Notice that the system (15) is iISS if and only if there exist functions $\beta \in KL$ and $\gamma_1, \gamma_2 \in K$ such that

$$\left\| x\left(t, \xi, u\right) \right\| \leq \beta\left(\|\xi\|, t \right) + \gamma_1\left(\int_0^t \gamma_2\left(\|u(s)\| \right) ds \right),$$

for all $t \geq 0$, all $\xi \in \mathbb{R}^n$ and all $u \in U$.

Defintion 11. [14] A continuously differentiable function V as in Definition 7 is called an iISS-Lyapunov function for the system (15) if there exists functions $\sigma \in K$, and a continuous positive definite function δ_3, such that

$$\delta_1\left(\|x\|\right) \leq V\left(x\right) \leq \delta_2\left(\|x\|\right),$$

for all $x \in \mathbb{R}^n$ with $\delta_1, \delta_2 \in K_\infty$, and

$$\nabla_x V^\top(x) f(x, u) \leq -\delta_3\left(\|x\|\right) + \sigma\left(\|u\|\right) = w(u(t), y(u(t), x(t))),$$

for all $x \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$.

Defintion 12. [14] A system is input/output to state stable (IOSS) if, for some $\beta \in KL$ and $\gamma_1, \gamma_2 \in K_\infty$

$$x(t) \leq \beta(|x^0|, t) + \gamma_1\left(\left\| u_{[0,t]} \right\|_\infty \right) + \gamma_2\left(\left\| y_{[0,t]} \right\|_\infty \right),$$

for all initial states and inputs, and all $t \in [0, T_{\xi,u}]$. IOSS is stronger than zero-detectability.

Defintion 13. [14] An IOSS-Lyapunov function is a smooth storage function such that

$$\nabla V(x) f(x,u) \le -\sigma_1(|x|) + \sigma_2(|u|) + \sigma_3(|y|),$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ with $\sigma_1, \sigma_2, \sigma_3 \in K_\infty$.

Using the same hypothesis of the last section, it is possible to generalize the previous results as long as a more general supply rate $w(u,y)$ with a special structure is used. We need the following lemma.

Lemma 14. Given V a in Definition 7, and $\alpha \in \mathrm{Diff}_{n,R^n}$ there exist functions $\mu_1, \mu_2 \in K_\infty$ such that

$$\mu_1(\|v(t)\|) \le V(\alpha(v(t))) \le \mu_2(\|v(t)\|),$$

for all $v \in \mathbb{R}^n$.

Proof. It is well-known [14] that given a diffeomorphism $\alpha \in \mathrm{Diff}_{n,R^n}$ there exist functions $\underline{\sigma}$, $\overline{\sigma} \in K_\infty$

$$\underline{\sigma}(r) \triangleq \min_{|v| \ge r} \|\alpha(v)\| \text{ and } \overline{\sigma}(r) \triangleq \max_{|v| \le r} \|\alpha(v)\|,$$

such that

$$\underline{\sigma}(\|v\|) \le \|\alpha(v)\| \le \overline{\sigma}(\|v\|),$$

for all $v \in \mathbb{R}^n$. It is well-known [14] that V is a storage function if and only if there exist functions $\delta_1, \delta_2 \in K_\infty$ such that

$$\delta_1(\|x\|) \le V(x) \le \delta_2(\|x\|).$$

In consequence,

$$\delta_1(\underline{\sigma}(\|v\|)) \le \delta_1(\|\alpha(v)\|) \le V(\alpha(v)),$$
$$V(\alpha(v)) \le \delta_2(\|\alpha(v)\|) \le \delta_2(\overline{\sigma}(\|v\|)).$$

Since the set K_∞ is closed under composition of functions, then

$$\mu_1(\|v\|) \le V(\alpha(v)) \le \mu_2(\|v\|),$$

where $\mu_1 = \delta_1 \circ \underline{\sigma}$ and $\mu_2 = \delta_2 \circ \overline{\sigma}$ are K_∞ functions, and for all $v \in \mathbb{R}^n$.

In Proposition 15 we present a generalization from Proposition 5 with regard to the supply rate $w(u,y)$. In this instance we do not suppose a particular structure for the supply rate $w(u,y)$ the only restriction is that it is a known function that is invariant under diffeomorphism.

Proposition 15. Consider the same hypothesis of the Proposition 5 for the supply rate $w(u,y)$ and energy storage function $V(x)$ such that

$$\frac{d}{dt} V(x(t)) \le w(u(t), y(u(t), x(t))),$$

for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$. Then for each $\alpha \in \mathrm{Diff}_{n,R^n}$ such that α preserves the structure of the supply rate $w(u(t), y(u(t), x(t)))$ (the transformed new function supply rate is denoted by $w(u(t), z(u(t), v(t)))$), and we have that

$$\frac{d}{dt} V(\alpha(v(t))) \le w(u(t), z(u(t), v(t))),$$

is satisfies for all $v \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$ where $z(u(t), v(t)) = y(u(t), \alpha(v(t)))$.

Proof. Since

$$\nabla_v V^\top(\alpha(v(t))) = \nabla_x V^\top(x)\Big|_{\alpha(v)} \frac{\partial \alpha(v)}{\partial v},$$

and taking account that the system $\dot{x} = f(x,u)$ under the coordinate change x by $\alpha(v)$ take the following form

$$\dot{v} = \left(\frac{\partial \alpha(v)}{\partial v}\right)^{-1} f(\alpha(v), u),$$

then we have that

$$\begin{aligned}\frac{d}{dt} V(\alpha(v(t))) &= \nabla_v V^\top(\alpha(v(t)))\dot{v}(t), \\ &= \nabla_x V^\top(x)\Big|_{\alpha(v)} f(\alpha(v(t)), u(t)).\end{aligned}$$

In consequence

$$\begin{aligned}\frac{d}{dt} V(\alpha(v(t))) &= \frac{d}{dt} V(x(t))\Big|_{\alpha(v)}, \\ &= \nabla_x V^\top(x)\Big|_{\alpha(v)} f(x,u)\Big|_{\alpha(v)}.\end{aligned}$$

On the other side the inequality $\frac{d}{dt} V(x(t)) \le w(u(t), y(u(t), x(t)))$ for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$, is equivalent to the following identity

$$\frac{d}{dt} V(x(t)) + \eta(x(t), u(t)) = w(u(t), y(u(t), x(t))),$$

for some continuous function $\eta(u(t), x(t)) \ge 0$ for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$. Since the substitution x by $\alpha(v)$ is an homomorphism, this preserves identities, and taking account that the last identity is forall $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$, and $\alpha : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$\begin{aligned}\frac{d}{dt} V(\alpha(v(t))) + \eta(u(t), \alpha(v(t))) &= \frac{d}{dt} V(x(t))\Big|_{\alpha(v)} \\ &\quad + \eta(u(t), x(t))\Big|_{x=\alpha(v)}, \\ &= w(u(t), y(u(t), x(t)))\Big|_{x=\alpha(v)}, \\ &= w(u(t), z(u(t), v(t)))\end{aligned}$$

is satisfies for all $v \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$. Therefore, for each $\alpha \in \mathrm{Diff}_{n,R^n}$ we have that

$$\frac{d}{dt} V(\alpha(v(t))) \le w(u(t), z(u(t), v(t))),$$

is satisfied for all $v \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$.

In the following Corollaries, we present the results on preservation of the properties ISS, iISS and IOSS.

Corollary 16. If the system (15) is ISS, then for each $\alpha \in \mathrm{Diff}_{n,R^n}$ the system (16) is also ISS.

Proof. By Theorem 5 in [14], the system $\dot{x} = f(x,u)$ is ISS, if there exists a smooth storage function V for which there exist functions $\sigma, \gamma \in K_\infty$ so that

$$\dot{V}(x,u) \le -\sigma(\|x\|) + \gamma(\|u\|),$$

for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$. In this case the structure of w is given by

$$w(u(t), y(u(t), x(t))) = -\sigma(\|x\|) + \gamma(\|u\|).$$

Using the Lemma 14 and that for each $\alpha \in \mathrm{Diff}_{n,R^n}$ there exist functions $\underline{\sigma}$, $\overline{\sigma} \in K_\infty$ so that

$$\underline{\sigma}(\|v\|) \le \|\alpha(v)\| \le \overline{\sigma}(\|v\|),$$

by arguments established in the Proposition 15, we have that

$$\dot{V}(\alpha(v), u) \ge -\sigma(\|\alpha(v)\|) + \gamma(\|u\|),$$

and as

$$-\sigma(\|\alpha(v)\|) + \gamma(\|u\|) \le -\sigma(\underline{\sigma}(\|v\|)) + \gamma(\|u\|).$$

Then

$$\dot{V}(\alpha(v), u) \ge -\sigma(\underline{\sigma}(\|v\|)) + \gamma(\|u\|),$$

where $\sigma \circ \underline{\sigma}$ is a K_∞ function. Again for the Theorem 5 in [14], the system

$$\dot{v} = \left(\frac{\partial \alpha(v)}{\partial v}\right)^{-1} f(\alpha(v), u),$$

is also ISS.

Corollary 17. If the system (15)is iISS, then for each $\alpha \in \mathrm{Diff}_{n,R^n}$ the system (16) is also iISS.

Proof. The proof is similar to Corollary 16. Using the Theorem 9 in [14], the system $\dot{x} = f(x,u)$ is iISS, if there exists a smooth storage function V for which there are a function $\gamma \in K_\infty$ and a function $gma : [0,+\infty) \to [0,+\infty)$ which is positive definite (that is, $\sigma(0) = 0$ and $\sigma(r) > 0$ for $r > 0$) such that

$$\dot{V}(x,u) \le -\sigma(\|x\|) + \gamma(\|u\|),$$

holds for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$. Now the prove is similar to the Corollary 16, but noting that the composition of functions $\sigma \circ \underline{\sigma}$ with $\underline{\sigma} \in K_\infty$ is only a positive definite function, since in this case σ is a positive definite function.

Notice that the function norm $\|\cdot\|$ is a K_∞ function. In this case the structure of w is given by

$$w(u(t), y(u(t), x(t))) = -\sigma(\|x\|) + \gamma(\|u\|),$$

with σ a positive definite function and $\gamma \in K_\infty$.

Note that the results in this section can be generalized by considering a change of variables in the space of inputs also.

Corollary 18. If the system (15) is IOSS, then for each $\alpha \in \mathrm{Diff}_{n,R^n}$ the system (16) is also IOSS.

Proof. Using the Theorem 22 in [14], the proof is similar to that of Corollary 16.

5. EXAMPLES OF PRESERVATION

In this section some examples are presented for to illustrated the results about different classes of dissipative systems.

Example 1. Consider the dissipative system in [16]

$$\begin{aligned} L\dot{x}_1 &= u - R_2 x_1 - x_2, \\ C\dot{x}_2 &= x_1 - \frac{1}{R_3} x_2, \\ y &= x_1 + \frac{1}{R_1} u. \end{aligned} \tag{20}$$

A Lyapunov function for this dissipative system is $V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$. Now taking the diffeomorphism $x = (x_1, x_2) = \alpha(v(t)) = (\arctan v_1, v_2)$, with Jacobian

$$\frac{\partial \alpha(v(t))}{\partial v} = \begin{pmatrix} \frac{1}{v_1^2+1} & 0 \\ 0 & 1 \end{pmatrix},$$

the transformed system $\dot{v}(t) = \left(\frac{\partial \alpha(v(t))}{\partial v}\right)^{-1} f(\alpha(v(t)), u(t))$ is

$$\begin{aligned} L\dot{v}_1 &= -\left(v_1^2+1\right)\left(v_2 - u + R_2 \arctan v_1\right), \\ C\dot{v}_2 &= \arctan v_1 - \frac{1}{R_3} v_2, \\ z &= \arctan v_1 + \frac{1}{R_1} u. \end{aligned} \tag{21}$$

In this case the Lyapunov function is $V(\alpha(x)) = \frac{1}{2}L\arctan^2 v_1 + \frac{1}{2}Cv_2^2$ and its derivative is $\dot{V}(\alpha(v)) = L\arctan v_1\left(\frac{\dot{v}_1}{v_1^2+1}\right) + Cv_2\dot{v}_2$. Hence

$$zu = \dot{V}(\alpha(v)) + \frac{u^2}{R_1} + R_2 \arctan^2 v_1 + \frac{v_2^2}{R_3},$$

$$zu \ge \dot{V}(\alpha(v)).$$

Example 2. Based on the system (20) with a new diffeomorphism given by $x = \alpha(v(t)) = \left(v_1^3 + v_1, v_2\right)$ and Jacobian

$$\frac{\partial \alpha(v(t))}{\partial v} = \begin{pmatrix} 3v_1^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain the transformed system

$$
\begin{aligned}
L\dot{v}_1 &= -\frac{1}{3v_1^2+1}\left(R_2\left(v_1^3+v_1\right)-u+v_2\right)\frac{1}{2},\\
C\dot{v}_2 &= v_1^3-\frac{1}{R_3}v_2,\\
z &= v_1^3+v_1+\frac{1}{R_1}u.
\end{aligned}
\tag{22}
$$

In this case the Lyapunov function is

$$V(\alpha(v))=\frac{1}{2}L\left(v_1^3+v_1\right)^2+\frac{1}{2}Cv_2^2,$$

and its derivative with respect to time

$$\dot{V}(\alpha(v))=L\left(v_1^3+v_1\right)\left(3v_1^2+1\right)\dot{v}_1+Cv_2\dot{v}_2,$$

then

$$\dot{V}(\alpha(v))=-R_2\left(v_1^3+v_1\right)^2-\frac{v_2^2}{R_3}+u\left(v_1^3+v_1\right).$$

In consequence

$$zu=\dot{V}(\alpha(v))+R_2\left(v_1^3+v_1\right)^2+\frac{v_2^2}{R_3}+\frac{u^2}{R_1},$$

$$zu\geq\dot{V}(\alpha(v)).$$

Example 3. Consider the following nonlinear system

$$
\begin{aligned}
L\dot{x}_1 &= -\frac{1}{3x_1^2+1}\left(R_2\left(x_1^3+x_1\right)-u+x_2\right),\\
C\dot{x}_2 &= x_1^3-\frac{1}{R_3}x_2,\\
y &= x_1^3+x_1+\frac{1}{R_1}u,
\end{aligned}
\tag{23}
$$

with the change of coordinates given by $x=\alpha(v(t))=\left(v_1^3+v_1,v_2\right)$, then the transformed system is

$$
\begin{aligned}
L\dot{v}_1 &= -\frac{\left(v_1^2+1\right)\left(R_2\left(\arctan^3 v_1+\arctan v_1\right)-u+v_2\right)}{3\arctan^2 v_1+1},\\
C\dot{v}_2 &= \arctan^3 v_1-\frac{1}{R_3}v_2,\\
y &= \arctan^3 v_1+\arctan v_1+\frac{1}{R_1}u.
\end{aligned}
\tag{24}
$$

The Lyapunov functionis

$$V(\alpha(v))=\frac{1}{2}L\left(\arctan^3 v_1+\arctan v_1\right)^2+\frac{1}{2}Cv_2^2,$$

and its time derivative along the trajectories of (ex2ex3a;) are

$$\dot{V}(\alpha(v))=-R_2\left(\arctan^3 v_1+\arctan v_1\right)^2+ u\left(\arctan^3 v_1+\arctan v_1\right)-\frac{v_2^2}{R_3}.$$

Therefore

$$zu=\dot{V}(\alpha(v))+\frac{u^2}{R_1}+R_2\left(\arctan^3 v_1+\arctan v_1\right)^2\frac{v_2^2}{R_3},$$

$$zu\geq\dot{V}(\alpha(v)).$$

Example 4. Consider the ISS system in [16],

$$\dot{x}=-x^3+u$$

with the diffeomorphism $x=\alpha(y)=\arctan y$, then the transformed system is given by

$$\dot{y}=-\left(y^2+1\right)\left(\arctan y\right)^3+\left(y^2+1\right)u,$$

and Lyapunov function $V=\frac{1}{2}y^2$, such that

$$\dot{V}=y\dot{y}=-\left(y^2+1\right)y\left(\arctan y\right)^3+\left(y^2+1\right)yu.$$

Notice that $-\left(y^2+1\right)y\left(\arctan y\right)^3<0$ for all $y\in\mathbb{R}$, in consequence

$$\dot{V}\leq-\left(1-\theta\right)\left(y^2+1\right)y\left(\arctan y\right)^3,\ \forall|y|\geq\left|\tan\left(\frac{|u|}{\theta}\right)^{\frac{1}{3}}\right|,$$

with $\gamma(a)=\tan\left(\frac{a}{\theta}\right)^{\frac{1}{3}}$ and $0<\theta<1$.

Example 5. Using the system from Example 4, we take the mapping $x=\alpha(y)=y^3$, note that it is not a smooth diffeomorphism. The transformed system is given by

$$\dot{y}=-\frac{y^7}{3}+\frac{y^{-2}u}{3},$$

and Lyapunov function $V=\frac{1}{2}y^2$ we obtain that

$$\dot{V}\leq-\frac{\left(1-\theta\right)}{3}y^8,\ \forall|y|\geq\left(\frac{|u|}{\theta}\right)^{\frac{1}{9}},$$

with $\gamma(a)=\left(\frac{a}{\theta}\right)^{\frac{1}{9}}$ and $0<\theta<1$.

In this last example we can see that there exist a more general type of functions than diffeomorphism such that some types of stability and dissipativity area preserved.

CONCLUSIONS

In this paper, we presented a preliminary investigation of preservation of dissipativity, input-state stability (ISS), integral input-to-state stability (iISS) and input/output to state stability (IOSS) under change of coordinates. In general, it is not clear that any class of dissipative systems can be preserved under an arbitrary change of variables, but in this note we show the preservation of some dissipative systems and forms of stability for a class of smooth diffeomorphism. Our main goal in this paper was to attract attention to preservation of dissipativity, ISS, iISS and IOSS rather than give complete recipes for preservation of this properties.

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CONFLICT OF INTEREST

None Declared.

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