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Several Mathematical Methods Applied in Image Compression and Restoration

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Abstract: Several mathematical methods are discussed in this paper, which are applied in image compression and restoration. Singular value decomposition (SVD) is used in compressing image. Conjugate gradients (CG) method and truncated Singular value decomposition (TSVD) regularization method are applied in image restoration. From the experience results we can see that those methods are effective in image compression and image restoration.

Keywords: Conjugate gradient, image compression, image restoration, regularization method, singular value decomposition.

1. INTRODUCTION

If an image to be transmitted has m×n pels, we need to transmit m×n data, which often has huge data quantity. So we expect to transmit relatively less date, which can be used to restore original image, this process realizes image compression, which is very important in a information society. Moreover, images are often blurred by outside condition, so more and more people pay attention to effective image restoration, nowadays image restoration technology [1] has been used in radio astronomy, secondary planet remote sensing, physic Imaging, industry vision and so on.

2. SINGULAR VALUE DECOMPOSITION APPLIED IN IMAGE COMPRESSION

2.1. Basic Theory About Singular Value Decomposition

Lemma1: (SVD about matrix) Let $A \in R^{m\times n}$, then there exist orthogonal matrix $U \in R^{m\times m}$ and $V \in R^{n\times n}$ such that

$A = U\Sigma V^T$ (or $A = U\Sigma V^H$).

$$\text{where } \Sigma = \begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} \tag{1}$$

and $\Sigma_1 = diag(\sigma_1, \sigma_2, \cdots, \sigma_r)$, whose diagonal elements are arranged in the following form [2]:

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = rank(A).$

When SVD is applied in image compression, we often need a low order matrix to approach a disturbed matrix or a matrix contained noise. The following lemma will give out appraisement standard about approach quality.

Lemma2: Let the matrix's $A \in R^{m\times n}$ SVD given by $A = \sum_{i=1}^{p} \sigma_i u_i v_i^T$, and $p = rank(A)$.

If $k < p$, and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, then the approach quality can be measured by the following spectrum norm and Frobenius norm;

$$\min_{rank(B)=k} \| A - B \|_{spec} = \| A - A_k \|_{spec} = \sigma_{k+1}$$

$$\min_{rank(B)=k} \| A - B \|_F = \| A - A_k \|_F = \sqrt{\sum_{i=k+1}^{q} \sigma_i^2} \tag{2}$$

where $q = \min\{m, n\}$.

2.2. Compressing Ratio and Reconstruction Formula

We can use a matrix which has m×n element to denote m×n original pels to be transmitted, then SVD is applied in the matrix. If we choice k large singular value in the SVD result and corresponding left and right vectors to the k large singular value to approach original image, we can use $k(m+n+1)$ numerical value to replace m×n image data. The $k(m+n+1)$ new data to be selected are the former k singular values of matrix A, the former k columns of m×m matrix U which contain left singular vectors and the former k columns of n×n matrix V which contain right singular vectors. Ratio is called image compressing ratio. Clearly, the quantity k of large singular values should satisfy the condition $k(m+n+1) < mn$ or $k < \frac{mn}{m+n+1}$. Therefore, in the process of image transmission, we need only transmit $k(m+n+1)$ data related to singular values and singular vectors to replace m×n original data. In the sink, when we incept

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singular values, left singular vectors and right singular vectors, we can make use of truncated Singular value decomposition (TSVD) formula to reconstruct original image. As the singular values downward arranged, many small singular values have no influence on the image restoration, so we can drop the small singular values [3, 4] to reduce the storage request:

$$\rho = \frac{mn}{k(m+n+1)} \tag{3}$$

$$\hat{A} = \sum_{i=1}^{k} \sigma_i u_i v_i^T \tag{4}$$

2.3. Peak Value SNR

As a standard, Peak value SNR(whose unit is decibel) which uses the following formula [5] to carry out calculation is often used to scale image quality:

$$PSNR = 10\log_{10}\left(\frac{255^2}{RMSE}\right), RMSE = \sqrt{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} - \hat{a}_{ij})^2}{mn}} \tag{5}$$

2.4. Experimentation Result

In the Fig. (**2**), the image (1) is the original input image which contains128×128 pels, a 128×128 matrix can be used to denote the image (1), the distribution of the 128 singular values about the denotation matrix is shown in the Fig. (**1**), clearly, the singular values turn to zero gradually, thereout, a sense estimate about the image quality corresponding different k will be displayed.

The following images are reconstructed by selecting different quantity of singular values, it is clearly to see from the images that we are not necessarily to utilize all of the singular values to take part in the calculation.

In the Fig. (**2**) the compression ratios of image (2) to image (6) are 12.75, 6.38, 3.19, 1.06, 0.64, it is clearly to see that less singular values whose quantity is k means large compression ratio and poor image reconstruction. Whereas, too many singular values whose quantity is also denoted by k means small compression ratio and slow transmission of images. Therefore, it is necessary for us to select fit compression ratio, which can give attention to both transmission efficiency and reconstruction quality when deal with different kinds of images.

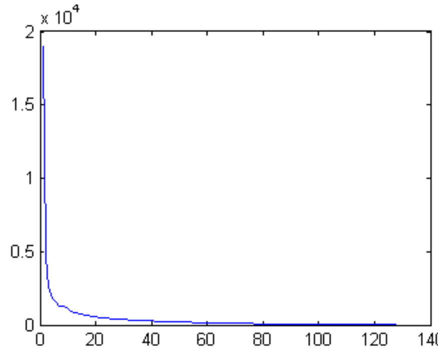


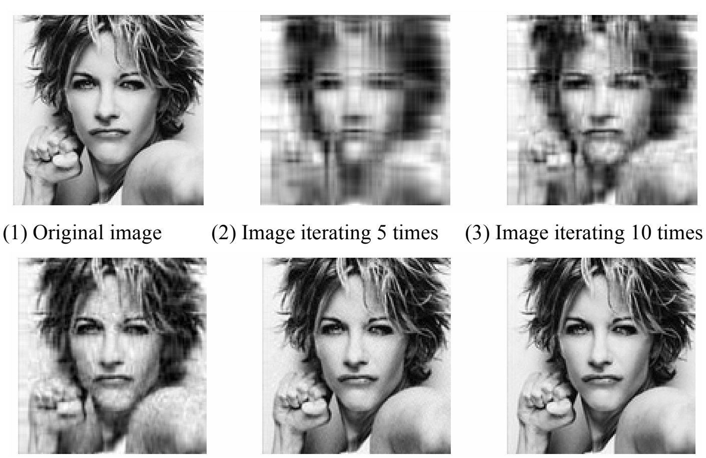
Fig. (1). The distribution of the singular values about the original image.

In the Fig. (**2**) the Peak value SNR of image (2) to image (6) are 32.98, 34.33, 36.17, 41.82, 48.64, it is clearly to see that along with the increasing of k, the Peak value SNR ($PSNR$) are also increased and higher quality reconstructed images can be gotten, but the compression ratio of images decrease clearly.

In the Fig. (**2**) the Frobenius norms of image (2) to image (6) are 4186.50, 3066.82, 2006.88, 542.16, 85.11, it is clearly to see that along with the increasing of k, the Frobenius norms are decreased, higher quality reconstructed images can be gotten and the compression ratio of images decrease clearly.

3. IMAGE RESTORATION

The first kinds of Fredholm integral equation (IFK) is usually used in image restoration, which has the following expression;



(1) Original image (2) Image iterating 5 times (3) Image iterating 10 times

(4) Image iterating 20 times (5) Image iterating 60 times (6) Image iterating 100 times

Fig. (2). The results of different singular values used for image compression.

$$g(x,y) = \int_a^b \int_a^b k(x,y;s,t) f(s,t)\,dsdt \tag{6}$$

where the function $g(x,y)$ is the image which can be observed, the function $f(s,t)$ is original image, the function $k(x,y;s,t)$ is blurring operator. It is not a good choice to use the formula (1) to analyse the problem directly, when the space domain of function $k(x,y;s,t)$ do not change, it has the following expression:

$$k(x,y;s,t) = k(x-s)\omega(y-t) \tag{7}$$

Based on the formula (7), we can obtain the discretization of the formula (1), which can be expressed by the following formulas;

$$\begin{aligned} g &= Kf \\ K &= \bar{H} \otimes H \end{aligned} \tag{8}$$

where g, f denote two $1 \times N^2$ column vectors which piled by poor quality image and original image. K is the blurring operator (a $N^2 \times N^2$ matrix), as to the case the space domain do not change, K is a block circulation matrix, which can be expressed by the form of Kronecker product, $\bar{H}$ and H are the Toeplitz matrixes.

IFK is the continuous model of image restoration, it is one kinds of equations which are very sensitive to disturber data, which is also means that the discretization of IFK is seriously ill-posed. We can not find effective result of equation (8) if we adopt normal method; furthermore the K is usually a large matrix, so iterating methods are often used in obtaining the approximate value of f. In this paper, conjugate gradients (CG) method and regularization method are applied in image restoration.

We firstly construct the right side g when the real solution f and blurring operator K are given, which will be taken as the research object needed to be cleared away. The most familiar blurring equation is the gauss pulse function, whose expression is given by the following formula (9), and the space domain of the function do not change, moreover it has convolution kernel which has divided form, and has the same blurring degree to the image which will be dealt with in the x and y directions:

$$h(x,y) = \frac{1}{2\pi\sigma^2} e^{\left(-\frac{x^2+y^2}{2\sigma^2}\right)} \tag{9}$$

The gauss pulse function can be described by the following band Toeplitz matrix:

$$(T_\sigma)_{ij} = \begin{cases} e^{\left(-\frac{1}{2}\left(\frac{i-j}{\sigma}\right)^2\right)} & i-j<band \\ 0 & \text{others} \end{cases} \tag{10}$$

Only the element of the matrix T whose distance to the diagonal is band-1 are nonzero, $K2\left(\pi\sigma^2\right)^{-1} \mathrm{T} \otimes \mathrm{T}$, K is symmetry positive sparse matrix. The function of σ is to control the shape of the gauss pulse function, along with the increase of the σ's values the ill-posed problems [6, 7] become more and more difficult to deal with.

3.1. The Application of Conjugate Gradients (CG) Method in Image Restoration

We let $N = 128$, band=6, σ=10, here the matrix K's condition number is infinite. In Fig. (**3**), the blurring image (2) can be obtained by blurring original image (1). At this time the problem is to how to effectively restore original image, when the blurring image and blurring operator are given. K is a seriously ill-posed matrix, if we solve the problem by using direct method, a restoration image can be given by the following image (3) in Fig. (**3**), which can not clearly denote the original image.

Now conjugate gradients (CG) method will be used in image restoration:

Firstly, let initial vector $f_0 = 0$ which will take part in iteration, $p_0 = r_0 = g - Kf_0 = g$, p_k is the search direction, r_k is the survival difference, conjugate gradients (CG) method is given as follows:

Let: $f_0 = 0, p_0 \equiv r_0 \equiv g,$

$i = 0$

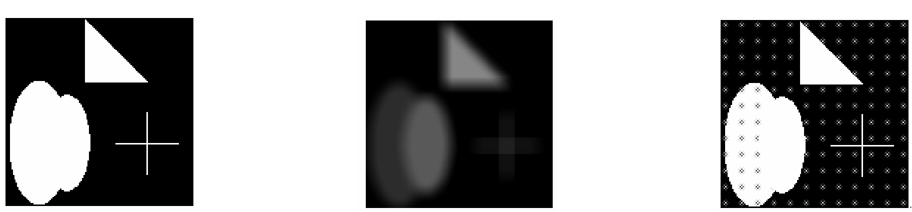
when $r_i \neq 0$ do as follows:

(1) $\alpha_i = \dfrac{r_i^T r_i}{p_i^T K p_i}$

(2) $f_{(i+1)} = f_i + \alpha_i p_i$

(3) $r_{i+1} = r_i - \alpha_i A p_i$

(4) $\beta_i = \dfrac{r_{i+1}^T r_{i+1}}{r_i^T r_i}$



（1）Original image （2）Blurring image （3）Restoration image obtained by direct method

Fig. (3). The direct method applied in image restoration.

(5) $p_{i+1} = r_{i+1} + \beta_{i+1} p_i$

(6) $i = i + 1$

End the 3d step.

Conjugate gradients (CG) method [8] is applied in the above ill-posed problem, which has taken the effect of regularization, the following image in the Fig. (**4**) is the result which has iterated 5000 times.

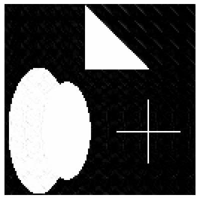


Fig. (4). The restoration image obtained by using CG method.

The relative error between image (3) and the original image (1) in the Fig. (**3**) is 0.23451, between image in the Fig. (**4**) and the original image (1) in the Fig. (**3**) is 0.02591, which are easy to calculate. It is clearly to see that the restoration image obtained by using CG method can clearly denote the original image.

3.2. Truncated Singular Value Decomposition (TSVD) Regularization Methodapplied in Singular Value Decomposition (SVD)

Pretreatment techniques are often used in speedup convergent speed, first of all let us consider the following pretreatments [9]:

To deal with matrix K by using singular value decomposition $K = U\Sigma V^T$, let the pretreatment operator $P = U\Sigma V^T$, that the solution of formula (8) is:

$$f = V\Sigma^{-1}U^T g \quad (11)$$

As to the ill-posed problems, it is clearly that using this method can not get fitness solution, and the method will spread blurring effect, so it is difficult to realize image restoration [10].

If we consider using the TSVD to regularize the pretreatment operator, then solution will has the following form:

$$f = V\Sigma^{+}U^T g$$
$$\Sigma^{+} = diag(\frac{1}{\sigma_1},\cdots\frac{1}{\sigma_k},0,\cdots 0) \quad (12)$$

Here the pretreatment operator is $P = U\Sigma^{+}V^T$, and the pretreatment operator is singular, when $g \neq 0$ we can not ensure that $f \neq 0$, but the pretreatment conjugate gradients (CG) method will stop when $f = 0$.

To use the truncated singular value decomposition (TSVD) regularization method, defining the pretreatment operator $P_\tau = U\Sigma_\tau V^T$, where;

$\Sigma_\tau = diag(\sigma_1,\cdots\sigma_k,1,\cdots 1)$,

at this case it is easy to calculate the solution:

$$f = V\Sigma_\tau^{-1}U^T g \quad (13)$$

As to the pretreatment system $P^{-1}K$ of ①, all of the singular values are near by the 1 and we can not distinguish signal space from noise space; But the pretreatment system of ③ is $P_\tau^{-1}K = V\Delta V^T$, $\Delta = diag(1,\cdots,1,\sigma_{k+1},\cdots,\sigma_n)$ whose large singular values are near by the 1, and corresponding to the signal space small singular values correspond to noise space.

It is easy to see that the method ③ is a feasible method by the above analyse, but the difficulty to restore image by using the method truncated singular value decomposition (TSVD) is how to determine a suitable regularization parameter, the following three methods are in common use.

Picard condition: The values corresponding to the points which make the Fourier coefficients become balanced can be taken as regularization parameters.

L-curve criterion: The inflexion of L-curve is the point which can best balance the solution $\| f^\alpha \|$ and the surplus item $\| g - Kf^\alpha \|$, and the value of the inflexion can be taken as regularization parameter, which is also the largest curvature of L-curve.

Generalized discrepancy principle (GCV):

$$\alpha_G = \arg\min \frac{\left\| Ax_\alpha - b \right\|_2}{trace(I - AA^{\#})}$$

The $trace(A)$ denotes the sum of the elements on the matrix A's diagonal, $A^{\#} = (AA^T + \alpha_G{}^2 I)^{-1} A^T$, and α_G is the regularization parameter.

Let $N = 20$, $band = 6$, $\sigma = 50$, now the condition number of the matrix K is $8.88284535536596 \times 10^{16}$. Generalized discrepancy principle is applied to select regularization parameter in this paper. The abscissa of the following Fig. (**5**) is 1 to N^2, and the ordinate is the values of double logarithm which correspond to GCV's function value.

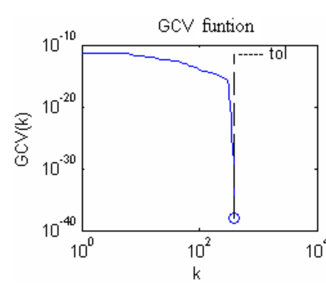
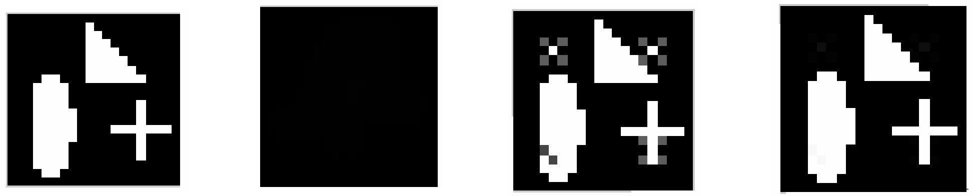


Fig. (5). The double logarithm of the GCV's function value.

The regularization parameter 2.689026×10-15 can be obtained through calculation, then the truncated singular value decomposition (TSVD) regularization method is applied, and the results is shown as following:

As shown in Fig. (**6**), the relative error between image (3) and original image (1) is 0.26134, and the relative error be-



(1) Original image (2) Blurring image (3) Restoration image by direct method (4) Restoration image by TSVD regularization method

Fig. (6). The experimental result.

tween image (4) and original image (1) is 0.00997, it is clearly to see that TSVD regularization method is feasible to restore image.

CONCLUSION

Singular value decomposition (SVD) has a better effect in compressing image, which can give attention to both compressing ratio and image quality. We can select different compressing ratios according to different requests.

Conjugate gradients (CG) method can effectively restore image, but it has a slowly convergent speed, we need look for more effectively pretreatment methods.

Truncated Singular value decomposition (TSVD) regularization method can also effectively restore image, but it is difficult for us to select regularization parameter.

The mathematic methods discussed above provide new paths in Image Processing.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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REFERENCES

[1] X. Zhang, *Matrix Analyse and Application*, Beijing: Qinghua University Press, 2004, pp.48-96.

[2] M.M. Siddeq, and M.A. Rodrigues, “A Novel Image compression algorithm for high resolution 3D reconstruction,” *3D Research*, vol. 5, no. 2, pp. 35-52, March 2014.

[3] S. Juliet, E.B. Rajsingh, and K. Ezra, “A novel image compression method for medical images using geometrical regularity of image structure,” *Signal, Image and Video Processing*, vol. 22, no. 6, pp. 178-191, June 2014.

[4] S. Juliet, E.B. Rajsingh, and K. Ezra, “A novel medical image compression using Ripplet transform,” *Journal of Real-Time Image Processing*, vol. 4, no. 3, pp. 75-87, March 2013.

[5] K.K. Hasan, U.K. Ngah, and M.F. M.Salleh, “Efficient hardware-based image compression schemes for wireless sensor networks: A survey”, *Wireless Personal Communications*, vol. 12, no. 8, pp. 1415-1436, August 2014.

[6] P. Novati, and M.R. Russo, “Adaptive arnoldi tikhonov regularization for image restoration”, *Numerical Algorithms*, vol. 65, no. 3, pp. 745-757, March 2014.

[7] Z. Zuo, W.D. Yang, X. Lan, L. Liu, J. Hu, and L. Yan, “Adaptive nonconvex nonsmooth regularization for image restoration based on spatial information,” *Circuits, Systems, and Signal Processing*, vol. 33, no. 2, pp. 2549-2564, February 2014.

[8] A. Langer, S. Osher, and C.B. Schönlieb, “Bregmanized domain decomposition for image restoration,” *Journal of Scientific Computing*, vol. 54, no. 2, pp. 549-576, February 2013.

[9] E.J. Palomo, E. Domínguez, R.M. Luque-Baena, and J. Muñoz, “Image compression and video segmentation using hierarchical self-organization”, *Neural Processing Letters*, vol. 37, no. 1, pp. 69-87 January 2013.

[10] V.K. Bairagi, A.M. Sapkal, and M.S. Gaikwad, “The role of transforms in image compression,” *Journal of the Institution of Engineers (India): Series B*, vol. 94, no. 2, pp. 135-140, February 2013.

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