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Equivalent Transformation Pairs of a *q*-ary Sequence and its Complex Polyphase Form and their Application

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Abstract: The Golay complementary sequences have been studied for more than five decades since Golay first discovered those sequences. However, few studies on the correlation function of transformation form of q-ary complementary sequences. New notions of equivalent transformation on q-ary sequences and equivalent transformation pair of a q-ary sequence and its complex polyphase form are put forward. A theorem on equivalent transformations of q-ary sequences is proposed and proved. Two theorems of equivalent transformation pairs of a q-ary sequence and its complex polyphase form are presented and proved. Finally, Constructions of Golay complementary pairs based on the above theorems and examples are given. These new notions and new theorems are the basis for various constructions of Golay complementary pairs.

Keywords: Aperiodic autocorrelation functions, equivalent transformation pair, equivalent transformation, golay complementary pair, q-ary sequence.

1. INTRODUCTION

A set of equal length sequences is called a aperiodic complementary set if the sum of the aperiodic autocorrelation functions of the sequences involved is zero except for a zero-shift term (*i.e.*, the sum is a delta function). Such a binary complementary pair, which is called a Golay complementary pair, was originally considered by Golay in connection with his study of infrared spectrometry in 1949 [1]. The mathematical properties of aperiodic complementary pairs were studied by the same author later in 1961 [2]. Tseng and Liu [3] studied binary aperiodic complementary sets of sequences with more than two sequences involved. R. Sivaswamy [4] investigated multiphase complementary sets, which form a class of generalized complementary sets.

Golay complementary pairs were subsequently used in radar, synchronization, channel estimation, and so on [5-7]. In particular, modern application of Golay complementary pairs is in multicarrier communications, which have recently attracted much attention in wireless applications. Orthogonal frequency-division multiplexing (OFDM) is a method of transmitting data simultaneously over multiple equally-spaced carrier frequencies, using Fourier transform processing for modulation and demodulation [8, 9]. The method has been employed as a multiplexing and a multiple access technique in several wireless communication standards such as IEEE802.11 Wireless LAN [10], and 3GPP-LTE [11]. OFDM offers many well-documented advantages for multicarrier transmission at high data rates, particularly in mobile applications. However, multicarrier communications have the major drawback of the high peak-to-average power ratio (PAPR) of transmitted signals [12]. A constructive way for peak power control in multicarrier communications is to use complementary sequences for subcarriers such that the sequences provide low peak-to-mean envelope power ratio (PMEPR) of at most 2 for transmitted signals [13], where the PAPR of the signals is bounded by the PMEPR. Later on, Davis and Jedwab [14] made a major theoretical advance, announcing that the large sets of binary length 2m Golay complementary pairs over, can be obtained from certain second-order cosets of the classical first order Reed-Muller code. Davis and Jedwab in [15], were able to obtain, at least for small numbers of carriers, a range of binary, quaternary, and octary OFDM codes with good error-correcting capabilities, efficient encoding and decoding, high code rates, and enjoying tightly controlled PMEPR. In 2000, Paterson generalized this result by replacing with ZH [16], where H is an arbitrary even integer. Due to this correlation property, Golay sequences have also been proposed to construct Hadamard matrix for direct sequence code division multiple access (DS-CDMA) system [17]. Li and Chu [18] found 1024 new quaternary Golay sequence of length 16 that do not occur as a single coset of the first-order Reed-Muller code in the second-order Reed-Muller code. Fiedler, Jedwab, and Parker [19] identified a framework of constructions of all known Golay complementary sequences of length 2m by a matrix structure and an explicit Boolean expression. Please refer to [20] more details on QAM Golay complementary sequences. The utilization of Golay sequences in DS-CDMA and OFDM are based on the property that the sum of out-of-phase autocorrelation of the pair equals to zero. However, synchronization and detection of the signal is equivalent to computing its own autocorrelation. In this case, investigation of the autocorrelation of single sequence is of our interest. More recently, Guang Gong *et al.* have presented three constructions of Golay sequences with large zero autocorrelation zone (ZACZ) of length approximately a half, a quarter or one eighth of their periods [21]. The large zero odd autocorrelation zone of some Golay sequences and 4^q-QAM Golay sequences have been reported in [22]. Thus, it is important issue to construct Golay sequences with larger ZACZ. However, few studies on the correlation function of transformation form of q-ary sequences.

The rest of this letter is organized as follows. The new concepts of equivalent transformation on q-ary sequences and equivalent transformation pair of a q-ary sequence and its complex polyphase form are introduced in Section II. A theorem on equivalent transformations of q-ary sequences is proposed and proved in Section III. Two theorems of equivalent transformation pairs of a q-ary sequence and its complex polyphase form are presented and proved in Section IV. Constructions of Golay complementary pairs based on the theorems and examples are given in Section V. Concluding remarks are given in Section in Section VI.

2. PRELIMINARIES

A discrete-time signal is called a q-ary sequence or a multilevel sequence if it has q possible discrete values over Z_q ($Z_q = \{0,1,2,\cdots,q-1\}$), where q is an integer and $q \geq 2$. In order to convert a q-ary sequence into a practical signal with useful properties, it is necessary to apply an appropriate transformation to the sequence elements. There are three kinds of transformation that can be used in practice: bipolar integer transformation, bipolar real transformation, and complex polyphase transformation. Complex polyphase transformation is defined as

$$x_n = \xi^{a_n}$$

where i is imaginary unit, and $\xi = i2\pi / q$. This is, $\{x_n\}$ is the complex polyphase sequence of a q-ary sequence $\{a_n\}$.

Based on the identity $(\xi^{a_n})^* = \xi^{-a_n}$, the aperiodic autocorrelation function of a q-ary sequence $\{a_n\}$ of length N, is defined as

$$C_{\boldsymbol{a}}(\tau) = \sum_{n=0}^{N-1-\tau} \xi^{a_n - a_{n+\tau}}, \quad 0 \leq \tau \leq N-1 \tag{1}$$

or

$$C_{\boldsymbol{a}}(\tau) = \sum_{n=0}^{N-1-\tau} x_n x_{n+\tau}^*, \quad 0 \leq \tau \leq N-1$$

(2)

where b^* denotes the complex conjugate of b.

A pair of q-ary sequences ($\boldsymbol{a}$; $\boldsymbol{b}$) of length N is called a Golay complementary pair, if

$$C_{\boldsymbol{a}}(\tau) + C_{\boldsymbol{b}}(\tau) = \begin{cases} 2N, & \tau = 0, \\ 0, & 1 \leq \tau \leq N-1. \end{cases}$$

(3)

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are called as Golay complementary sequences.

Lemma 1([23, Lemma 4]): Let ($\boldsymbol{a}$; $\boldsymbol{b}$) is a q-ary Golay pair, and $-\bar{\boldsymbol{a}} = (-a_{N-1},\cdots,-a_1,-a_0)$. Then $C_{\boldsymbol{a}}(\tau) = C_{-\bar{\boldsymbol{a}}}(\tau)$ for all $0 \leq \tau < N$, and $(-\bar{\boldsymbol{a}},\boldsymbol{b})$ is also a q-ary Golay complementary pair.

In [23], If $\boldsymbol{a}$ is a binary sequence, then $\bar{\boldsymbol{a}} = -\bar{\boldsymbol{a}}$, where $\bar{\boldsymbol{a}} = (a_{N-1},\cdots,a_1,a_0)$, since $-1 \equiv 1 (\mathrm{mod}\, 2)$ and $-0 \equiv 0 (\mathrm{mod}\, 2)$. If ($\boldsymbol{a}$; $\boldsymbol{b}$) is a binary Golay complementary pair, then ($\bar{\boldsymbol{a}}$; $\boldsymbol{b}$) is also a binary Golay complementary pair [2, General Property 3], and (-$\bar{\boldsymbol{a}}$; $\boldsymbol{b}$) is also a binary Golay complementary pair [23, lemma4]. The reason is that a binary sequence over {0, 1}, its reversal, and negative reversal have the same autocorrelation function. In general, this is not true for polyphase Golay complementary pairs.

Definition 1: For $\boldsymbol{a}$ is a q-ary sequences, we define the transformation f as an equivalent transformation on q-ary sequences based on aperiodic correlation, if $C_{\boldsymbol{a}}(\tau) = C_{f(\boldsymbol{a})}(\tau)$.

Definition 1 is also suitable for the complex polyphase sequence of a q-ary sequence.

Definition 2: For two transformations f and g, we define them as an equivalent transformation pair, if $C_{\boldsymbol{a}}(\tau) = C_{f(\boldsymbol{a})}(\tau)$, $C_{\boldsymbol{x}}(\tau) = C_{g(\boldsymbol{x})}(\tau)$ and $g(x_n) = \xi^{f(a_n)}$, where $\boldsymbol{x}$ is the complex polyphase sequence of a q-ary sequence $\boldsymbol{a}$.

According to the above argument, let $-\bar{\boldsymbol{a}} = f(\boldsymbol{a})$, then f is an equivalent transformation, this is $f(a_n) = -a_{N-n-1}$.

3. THEOREM OF EQUIVALENT TRANSFORMATION

Equivalent transformations are the basis for various constructions of Golay complementary pairs. In Golay complementary sequence design, we hope to find more than equivalent transformations. Lemma 1 can be seen as a theorem on equivalent transformations. In this section, another theorem on equivalent transformations is proposed and proved.

Theorem 2: Let $\boldsymbol{a} = (a_0, a_1, \cdots, a_{N-1})$ be a q-ary sequences of length N. The following transformations preserve the aperiodic auto-correlation function of the original sequence

$$f(a_n) = c + a_n,\ c \in Z_q \text{ and } q \geq 2 \tag{4}$$

This is, f is an equivalent transformation of the q-ary sequences $\boldsymbol{a}$.

Proof:

$$\begin{aligned} C_{f(\boldsymbol{a})}(\tau) &= \sum_{n=0}^{N-1-\tau} \xi^{f(a_n) - f(a_{n+\tau})} \\ &= \sum_{n=0}^{N-1-\tau} \xi^{a_n - a_{n+\tau}} \\ &= C_{\boldsymbol{a}}(\tau) \end{aligned}$$

The proof is complete.

Theorem 2 is also suitable for the complex polyphase sequence of a q-ary sequence.

4. THEOREMS OF EQUIVALENT TRANSFORMATION PAIR

In sequence design, there are mainly two kinds of sequence design. One is that q-ary sequences is designed over Z_q, and another is that the complex polyphase form of q-ary sequence is designed over $\{\xi^k, k = 0,1,2,\cdots, q, \xi = i2\pi / q\}$. The above-mentioned issue is mainly focused on q-ary sequences. Now, we will our main attention to the complex polyphase sequence of q-ary sequence.

Theorem 3: Let $\boldsymbol{a} = (a_0, a_1, \cdots, a_{N-1})$ be a q-ary sequences of length N, and $\boldsymbol{x} = (x_0, x_1, \cdots, x_{N-1})$ be the complex polyphase form of $\boldsymbol{a}$. Two transformations f and g form an equivalent transformation pair, if

$$f(a_n) = c + a_n,\ g(x_n) = \xi^c x_n, c \in Z_q.$$

(5)

Proof:

By Theorem 2, then we have

$$C_{f(\boldsymbol{a})}(\tau) = C_{\boldsymbol{a}}(\tau).$$

$$\begin{aligned} C_{g(\boldsymbol{x})}(\tau) &= \sum_{n=0}^{N-1-\tau} g(x_n) g(x_{n+\tau})^* \\ &= \sum_{n=0}^{N-1-\tau} (\xi^c x_n)(\xi^c x_{n+\tau})^* \\ &= \left|\xi^c\right| \sum_{n=0}^{N-1-\tau} (x_n)(x_{n+\tau})^* \\ &= C_{\boldsymbol{x}}(\tau). \end{aligned}$$

$$\begin{aligned} \xi^{f(a_n)} &= \xi^{c+a_n} = \xi^c \xi^{a_n} \\ &= g(x_n). \end{aligned}$$

The proof is complete.

Theorem 4: Let $\boldsymbol{a} = (a_0, a_1, \cdots, a_{N-1})$ be a q-ary sequences of length N, and $\boldsymbol{x} = (x_0, x_1, \cdots, x_{N-1})$ be the complex polyphase form of $\boldsymbol{a}$. Two transformations f and g form an equivalent transformation pair, if

$$f(a_n) = -a_{N-n-1},\ g(x_n) = x_{N-n-1}^* \tag{6}$$

Proof:

By Lemma 1, then we have

$$C_{f(\boldsymbol{a})}(\tau) = C_{\boldsymbol{a}}(\tau).$$

Taking $t = N - (n+\tau) - 1$,

$$\begin{aligned} C_{g(\boldsymbol{x})}(\tau) &= \sum_{n=0}^{N-1-\tau} g(x_n) g(x_{n+\tau})^* \\ &= \sum_{n=0}^{N-1-\tau} x_{N-n-1}^* x_{N-(n+\tau)-1} \\ &= \sum_{t=0}^{N-1-\tau} (x_t)(x_{t+\tau})^* \\ &= C_{\boldsymbol{x}}(\tau). \end{aligned}$$

$$\begin{aligned} \xi^{f(a_n)} &= \xi^{-a_{N-n-1}} = \left(\xi^{a_{N-n-1}}\right)^* \\ &= \left(x_{N-n-1}\right)^* = g(x_n). \end{aligned}$$

The proof is complete.

5. CONSTRUCTIONS OF GOLAY COMPLEMENTARY PAIRS BASED ON THEOREMS 3 AND 4

Based on Theorems 3, theorem 4 and lemma 1, we may obtain the following corollaries.

Corollary 5: Let ($\boldsymbol{a}$; $\boldsymbol{b}$) is a q-ary Golay complementary pair, then $(-\tilde{\boldsymbol{a}}, \boldsymbol{b})$ and $(\boldsymbol{a} + \boldsymbol{1}\cdot c, \boldsymbol{b})$ are q-ary Golay complementary pairs.

For example, if c=3 and q=4, $\boldsymbol{a}$= (0, 2, 2, 2, 1, 1, 3, 1), $\boldsymbol{b}$= (0, 2, 2, 2, 3, 3, 1, 3), then

$-\tilde{\boldsymbol{a}}$ = (3, 1, 3, 3, 2, 2, 2, 0),

$\boldsymbol{a} + \boldsymbol{1}\cdot c$ = (3, 1, 1, 1, 0, 0, 2, 0).

By calculating, we have

$C_{\boldsymbol{a}}(\tau) = C_{-\tilde{\boldsymbol{a}}}(\tau) = C_{\boldsymbol{a}+\boldsymbol{1}\cdot c}(\tau)$ =(8, i, 2i, i, 0, -i, 2i, -i)

$C_{\boldsymbol{b}}(\tau)$ = (8, -i, -2i, -i, 0, i, -2i, i).

Corollary 5: Let ($\boldsymbol{x}$; $\boldsymbol{y}$) is a complex polyphase complementary pair**,** $\boldsymbol{x}$ and $\boldsymbol{y}$ are complex polyphase form of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, then $(\tilde{\boldsymbol{x}}^*, \boldsymbol{y})$ and $(\xi^c \boldsymbol{x}, \boldsymbol{y})$ are complex polyphase Golay complementary pairs.

For example, if c=3 and q=4**,** $\boldsymbol{x}$= (1,-1,-1,-1, i, i, -i, i), $\boldsymbol{b}$= (1,-1,-1,-1,-i, -i, i, -i), then

$\tilde{\boldsymbol{x}}^*$ = (-i, i, -i, -i, -1,-1,-1, 1),

$\xi^c \boldsymbol{x}$ = (-i, i, i, i, 1, 1,-1,1).

By calculating, we have

$C_{\boldsymbol{x}}(\tau) = C_{\tilde{\boldsymbol{x}}^*}(\tau) = C_{\xi^c \boldsymbol{x}}(\tau)$ = (8, i, 2i, i, 0, -i, 2i, -i)

$C_{\boldsymbol{y}}(\tau)$ = (8, -i, -2i, -i, 0, i, -2i, i).

CONCLUSION

The new notions of equivalent transformation on q-ary sequences and equivalent transformation pair of a q-ary sequence and its complex polyphase form are put forward.

A theorem on equivalent transformations of q-ary sequences is proposed and proved. Two theorems of equivalent transformation pairs of a q-ary sequence and its complex polyphase form are presented and proved. Finally,

Constructions of Golay complementary pairs based on the above theorems and examples are given. Finding more equivalent transformations on q-ary sequences and equivalent transformation pairs are our future work.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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