

# Monitoring the Stability of Correlation Structure in Drive Rib Production Process: An MSPC Approach

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**Abstract:** The stability (or instability) of covariance structure does not ensure the stability (or instability) of correlation structure. However, these stability problems are usually tested using the same approach. The most common and widely used approach is based on likelihood ratio criterion. Unfortunately, its sensitivity is not without limitation, its computational efficiency is low and it is quite cumbersome when the number of variables is large. To overcome these obstacles in monitoring the stability of correlation structure, in this paper we propose a multivariate statistical process control approach by using vector variance as multivariate dispersion measure. First, we investigate the sampling distribution of vector variance of standardized variables and then we construct a control chart. Its sensitivity is very promising, its computational efficiency is high and it is practical even when the number of variables is large. The applicability of the proposed method in monitoring the stability of correlation structure in drive rib production process at Indonesian Aerospace Ltd. will be presented.

## INTRODUCTION

The problem of how to test the stability of correlation structure is as old as its counterpart, i.e., the problem of testing the covariance structure stability. However, unlike the former, the application of the latter in manufacturing industry came first and it is very well accepted by experts in this area. It then has been experiencing a remarkable development. See, for example, (Alt *et al.*, 1988) [1], (Wierda *et al.*, 1994) [2], (Sullivan *et al.*, 1996) [3], (Woodall *et al.*, 1999) [4], (Montgomery 2001) [5], (Montgomery 2005) [6], (Djauhari 2005) [7] and (Sullivan *et al.*, 2007) [8], and the references there in, for historical background and its development. For an early development of the method for testing the stability of covariance structure one can consult the classical literature such as (Anderson 1958) [9].

Concerning the stability of correlation structure, we witness that only since two decades ago it has been received very much attention, especially in economics studies and finance industry. See, for example, (Lee 1998) [10] and (Tang 1998) [11] who study the stability of correlation structure among stock returns, (Goetzmann *et al.*, 2002) [12] who concern about global market correlation structure, (Annaert *et al.*, 2003) [13] who focus their work on the stability of covariance and correlation structures based on small samples, and (Ragea 2003) [14] and (Da Costa Jr. *et al.*, 2005) [15] who show that the correlation stability is of vital importance in market risk management and asset management. See also (Anderson 2006) [16] for some tests, (Cooka *et al.*, 2002) [17] for a study on the stability of correlation matrices based on time series data, and (Schott 2007) [18] for a discussion of the correlation structure stability when samples are dependent.

The two stability problems mentioned above are different in nature and, using likelihood ratio criterion, it is not difficult to show that the stability (or instability) of covariance structure does not ensure the stability (or instability) of correlation structure. This is a serious limitation of any test constructed based on that criterion where covariance determinant is used to measure the dispersion of multivariate data. However, if all variables are standardized, the correlation structure can be considered as the covariance structure. Therefore, we focus our attention on how to handle the limitation of that measure. This is the first main issue of this paper.

The most commonly used statistical tests in literature to test the hypothesis of stability of correlation structure are the so-called Box's  $M$  statistic proposed by (Box 1949) [19] and Jennrich's statistic proposed by (Jennrich 1970) [20] which can be considered as a corrected Kullback's statistic. See, for example, (Annaert *et al.*, 2003) [13] for the application of Box's  $M$  statistic and Jennrich's statistic in testing that hypothesis. It is important to note that the  $M$  statistic involves the determinant of correlation matrices while Jennrich's statistic needs the inversion of the pooled correlation matrix. Furthermore, in practice, it is not rare that the number of variables  $p$  is large. See, for example, (Werner 2003) [21], (Schafer *et al.*, 2005) [22] and (Chilson *et al.*, 2006) [23] for further discussion of this case. Consequently, when  $p$  is large, the computation of the two statistics is quite cumbersome and tedious. More precisely, by using Cholesky decomposition, the computational complexity of the determinant of a correlation matrix of size  $(p \times p)$  is of order  $O(p^3)$ . Its inverse is also of that order. This order of complexity, which is too high even for a moderate value of  $p$ , is another limitation of correlation determinant as a measure of multivariate dispersion when all variables are standardized. This is the second main issue that we want to discuss in the present paper.

In what follows we focus our attention on how to handle the limitations of the likelihood ratio criterion-based test

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which depends on the use of covariance determinant as a measure of multivariate dispersion. For this purpose we use the notion of vector variance as another measure of multivariate dispersion. This measure will enable us to handle the case when  $p$  is large and, more importantly, to have a better understanding on the covariance and correlation structures. We then recall our recent result on the properties of vector variance presented in (Djauhari 2007) [24]. Based on this result, we propose a method based on what we call vector variance of standardized variables (VVSV) to monitor the stability of correlation structure. As mentioned above, handling voluminous databases or data sets with  $p$  large has received considerable attention not only in economics and finance but also in other areas. See again (Werner 2003) [21], (Schafer *et al.*, 2005) [22] and (Chilson *et al.*, 2006) [23] for the case where the sample size  $n > p$  and (Ledoit *et al.*, 2002) [25] for  $n < p$ .

Our approach is based on the notions of *vec* operator and commutation matrix, and the concept of multivariate statistical process control (MSPC). As a multivariate dispersion measure, vector variance (VV) is proposed heuristically in our recent work (Herwindiati *et al.*, 2007) [26] and analytically in our other work (Djauhari 2007) [24]. It is derived from the notion of vector covariance presented and used in (Cleroux 1987) [27], and originally introduced by (Escoufier 1973) [28] to measure the linear relationship between two random vectors. The asymptotic distribution of sample VV is investigated using the notations in (Serfling 1980) [29]. Its successful application as the stopping rule in FMCD (fast minimum covariance determinant) algorithm can be seen in (Herwindiati *et al.*, 2007) [26]. It reduces significantly the computational complexity of FMCD in data concentration step. See (Rousseeuw 1985) [30], (Rousseeuw *et al.*, 1987) [31], (Rousseeuw *et al.*, 1999) [32], (Rousseeuw *et al.*, 1999) [33], and (Hubert *et al.*, 2005) [34] for in-depth presentations and discussions on FMCD.

A comprehensive discussion on *vec* operator and commutation matrix can be found, for example, in (Muirhead 1987) [35], (Schott 1997) [36], (Schott 2001) [37], and (El Maache *et al.*, 1998) [38]. This operator simplifies the study of random matrix by means of random vector. Furthermore, in the present paper, commutation matrix is used to simplify the investigation of its parameters. Finally, the concept of MSPC is used to monitor the stability of correlation structure. This concept is standard in industry process quality control. Based on this concept, by using the approach in (Montgomery 2001 p. 532) [5] which is represented again in (Montgomery 2005 p. 511) [6] and by considering correlation matrix as the covariance matrix when all variables are standardized, monitoring the stability of correlation structure is equivalent to repeated tests of significance of the hypothesis that the process correlation matrix is equal to a particular matrix of constants. See (Wierda 1994) [2], (Sullivan *et al.*, 1996) [3] and (Woodall *et al.*, 1999) [4] for a general discussion and (Alt *et al.*, 1988) [1], (Tang *et al.*, 1996a) [39], (Tang *et al.*, 1996b) [40], (Khoo *et al.*, 2003) [41], (Khoo *et al.*, 2004) [42], and (Djauhari 2005) [7] for monitoring covariance structure.

This paper is organized as follows. The next section discusses the notion of VVSV. It is the VV when all variables are standardized. In the third section its distributional proper-

ties will be explored. Later on, in the fourth section we propose a method for monitoring the stability of correlation structure based on MSPC. The fifth section will highlight the sensitivity of the proposed method as reported in our work (Djauhari 2007) [24]. An example with real data on the production process of drive rib at Indonesian Aerospace Ltd. will be presented in the sixth section and additional remarks in the last section will close this presentation.

## VECTOR VARIANCE

Let  $X$  be a random vector which is a superposition of two random vectors  $X^{(1)}$  and  $X^{(2)}$ , each of which is of  $p$  and  $q$  dimension. We write  $X = \begin{pmatrix} X^{(1)} & X^{(2)} \end{pmatrix}^t$ . Let also,  $\mu^{(i)} = E\left(X^{(i)}\right)$  the mean of  $X^{(i)}$ ;  $i = 1, 2$ , and  $\Sigma_{ij} = E\left[\left(X^{(i)} - \mu^{(i)}\right)\left(X^{(j)} - \mu^{(j)}\right)^t\right]$  the covariance matrix of  $X^{(i)}$  and  $X^{(j)}$ ;  $i, j = 1, 2$ . Thus, the covariance matrix  $\Sigma$  of  $X$  is equal to the following partitioned matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

In order to measure the linear relationship between the two random vectors  $X^{(1)}$  and  $X^{(2)}$ , (Escoufier 1973) [28] introduced the quantity  $Tr(\Sigma_{12}\Sigma_{21})$ . Escoufier then defined it as vector covariance of those random vectors. It is the sum of all diagonal elements of  $\Sigma_{12}\Sigma_{21}$ . Accordingly, he defined  $Tr(\Sigma_{11}^2)$  and  $Tr(\Sigma_{22}^2)$  the vector variance of  $X^{(1)}$  and  $X^{(2)}$ , respectively. Those who are interested in the formal and rigorous discussion on these definitions are suggested to consult (Escoufier 1973) [28]. If  $p = q = 1$ , vector covariance is simply the square of classical covariance, and vector variance is the square of classical variance.

In what follows, we focus our attention on vector variance (VV). Suppose  $X$  is a random vector of  $p$  dimension and its covariance matrix is  $\Sigma$ . The VV of  $X$  is then equal to the sum of square of all elements of  $\Sigma$ . By using *vec* operator, the VV of the random vector  $X$  can be written as  $\|vec(\Sigma)\|^2$ . See (Muirhead 1982) [35], (Schott 1997) [36], (Schott 2001) [37] and (El Maache *et al.*, 1998) [38] for the details of this operator. In the present paper, we define vector variance of standardized variables. Let  $Z$  be the random vector where its  $k$ -th component is the standardized form of the  $k$ -th component of  $X$ ;  $k = 1, 2, \dots, p$ . The covariance matrix  $P$  of  $Z$  is the correlation matrix of  $X$  and we call  $\|vec(P)\|^2$  the vector variance of standardized variables (VVSV). A more general presentation and discussion on vector variance can be found in our recent work (Djauhari 2007) [24]. See also (Herwindiati *et al.*, 2007) [26] for its application in robust estimation of location and scatter.

In the rest of the paper we exploit the sample version of VVSV and we investigate its sampling distribution. Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  drawn from a  $p$ -variate distribution with covariance matrix  $\Sigma$ . The sample covariance matrix is

$$S = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^t$$

where  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  is the sample mean vector. The sample

correlation matrix is thus  $R = D^{-1/2}SD^{-1/2}$  where  $D$  is a diagonal matrix; its  $i$ -th diagonal element is equal to the  $i$ -th diagonal element of  $S$ .

VVSV or VV, like correlation determinant or covariance determinant (CD) which is also known as generalized variance (GV), is a multivariate dispersion measure. However, the computational complexity of VV or, equivalently, VVSV is much lower than that of the latter. The former is of order  $O(p^2)$  while the latter, as mentioned in the previous section, is of order  $O(p^3)$ . See (Djauhari 2007) [24]. This is

a reason why, in the present paper, we propose VVSV-based statistic instead of Box's  $M$  statistic and Jennrich's statistic. It is also due to this advantage that in (Herwindiati *et al.*, 2007) [26] we use VV to increase the computational efficiency in data concentration step of FMCD algorithm. More precisely, we use minimum vector variance (MVV) as the stopping rule to substitute the minimum covariance determinant (MCD) which, as mentioned above, has high computational complexity. However, in that work, our approach is more heuristic than analytical. An analytical approach which investigates the properties of sample vector variance  $\|vec(S)\|^2$  is given in our work (Djauhari 2007) [24]. In that work we have showed that this statistic is not only apt for large  $p$  but also for the condition where the covariance matrix is singular. We have also showed that, when  $n$  and  $p$  are small or when  $p$  is large but  $n$  is not too large, it is more sensitive than sample CD or, equivalently, sample GV to small shift of covariance structure. Their sensitivities are similar to large shift when  $n$  and  $p$  are large.

Due to those commendable properties of vector variance, then we propose to use VVSV when the correlation structure stability is of our interest, especially when the database or data set is of high dimension. For that purpose, in the next section we discuss the distributional properties of sample VVSV.

**SAMPLING DISTRIBUTION**

Let  $Z$  be a random vector where its  $k$ -th component is the standardized form of the  $k$ -th component of  $X$ ;  $k = 1, 2, \dots, p$ . The covariance matrix  $P$  of  $Z$  is the so-called correlation matrix of  $X$  and  $\|vec(P)\|^2$  is the VVSV. We define

$\|vec(R)\|^2$  or, equivalently,  $Tr(R^2)$  the sample VVSV. To

investigate the asymptotic distribution of  $\|vec(R)\|^2$  we first recall the result given by (Schott 2007) [18]. Let  $K_{pp}$  be the

commutation matrix of size  $p^2 \times p^2$ , i.e.,  $K_{pp} = \sum_{i=1}^p \sum_{j=1}^p N_{ij} \otimes N_{ij}^t$  and  $N_{ij}$  is a  $(p \times p)$  matrix having all

elements equal 0 except its  $(i,j)$ -th element equals 1. Let also

$\Lambda_p = \sum_{i=1}^p (e_i e_i^t \otimes e_i e_i^t)$  where  $e_i$  is the  $i$ -th column of  $I_p$ . In

(Schott 2007, p. 1993) [18] the following result concerning the asymptotic distribution of  $vec(R)$  is given.

**Theorem**

Let  $X_1, X_2, \dots, X_n$  be a random sample drawn from  $p$ -variate normal distribution  $N_p(\mu, \Sigma)$ . Then,

$$\sqrt{n-1} \{vec(R) - vec(P)\} \xrightarrow{d} N_{p^2}(0, \Gamma) \text{ where}$$

a.  $\Gamma = 2M_p \Phi M_p$  with  $M_p = \frac{1}{2}(I_{p^2} + K_{pp})$ ;

b.  $\Phi = B^t (P \otimes P) B$  with  $B = I_{p^2} - (I_p \otimes P) \Lambda_p$ .

Based on this result, now we investigate the asymptotic distribution of sample VVSV. For this purpose, we use the following result in (Djauhari 2007) [24]. Let the sequence of random vectors  $\{X_n\} \xrightarrow{p} c$ ,  $c$  be a vector of constants of  $p$  dimension, and  $X_n \xrightarrow{d} N_p(c, \Sigma)$ . Let also  $u$  be a real valued function of  $X_n$ , where  $u'$  exist and  $u'(X_0) \neq 0$  for all  $X_0$  in the neighborhood of  $c$ . Then, the variable random

$$Y_n = u(X_n) \xrightarrow{d} N(\mu_Y, \sigma_Y^2) \dots\dots\dots (1)$$

where

a.  $\mu_Y \rightarrow u(c)$ , and

b.  $\sigma_Y^2 \rightarrow \left( \frac{\partial u(c)}{\partial X_n} \right)^t \Sigma \left( \frac{\partial u(c)}{\partial X_n} \right)$ .

The above theorem and the result (1) have a direct consequence which is formulated in the following proposition.

**Proposition 1.** If in (1) we define  $X_n = vec(R)$ , then

$$\sqrt{n-1} \{u(vec(R)) - u(vec(P))\} \xrightarrow{d} N(0, \sigma^2) \text{ where } \sigma^2$$

$$\rightarrow \left( \frac{\partial u(vec(P))}{\partial R} \right)^t \Gamma \left( \frac{\partial u(vec(P))}{\partial R} \right)$$

From this proposition we get the following main result of this paper.

**Proposition 2.** If in Proposition 1 we define random variable  $u(\text{vec}(R)) = \|\text{vec}(R)\|^2$ , then we have  $\sqrt{n-1} \left\{ \|\text{vec}(R)\|^2 - \|\text{vec}(P)\|^2 \right\} \xrightarrow{d} N(0, \sigma^2)$  and  $\sigma^2 \rightarrow 4(\text{vec}(P))^t \Gamma(\text{vec}(P))$ .

The proof is straight forward because the partial derivative of  $\|\text{vec}(R)\|^2$  with respect to  $R$  at  $P$  is equal to  $2(\text{vec}(P))$ . This result will be exploited in the next sections to monitor the stability of correlation structure based on  $m$  independent samples.

**PROPOSED MSPC APPROACH**

Suppose  $m$  independent samples are available, each of which is of size  $n_1, n_2, \dots$ , and  $n_m$  and drawn from a  $p$ -variate normal distribution with positive definite covariance matrix  $\Sigma_1, \Sigma_2, \dots$ , and  $\Sigma_m$ . Let  $P_i$  be the  $i$ -th correlation matrix and  $R_i$  be its sample version. In this section we discuss the use of MSPC approach for testing the stability of correlation structure based on these samples.

We are interested in testing the hypothesis  $H_0: P_1 = P_2 = \dots = P_m$  (= say  $P_0$ ) versus  $H_1: P_i \neq P_j$  for at least an  $i \neq j$ . In this circumstance, Montgomery points out that testing the stability of process correlation structure,  $H_0$  versus  $H_1$ , is equivalent to a repeated tests of significance of the hypothesis that the process correlation matrix is equal to a particular matrix of constants. See (Montgomery 2001, p. 532) [5] and (Montgomery 2005, p. 511) [6]. Thus, in practice, that procedure is customarily represented by means of a control chart where its control limits are

$$\begin{aligned} \text{UCL} &= \mu + z_{\alpha/2} \cdot \sigma, \text{ and} \dots\dots\dots (2) \\ \text{LCL} &= \max \{0, \mu - z_{\alpha/2} \cdot \sigma \} \end{aligned}$$

with  $\mu$  and  $\sigma^2$  are given in Proposition 2, and  $z_{\alpha/2}$  is the  $(1 - \alpha/2)$ -th quantile of the standard normal distribution.

**$P_0$  IS KNOWN**

In this case, the control limits in (2) become,

$$\begin{aligned} \text{UCL} &= \mu_0 + \frac{z_{\alpha/2}}{\sqrt{n-1}} \cdot \sigma_0 \dots\dots\dots (3) \\ \text{LCL} &= \max \{0, \mu_0 - \frac{z_{\alpha/2}}{\sqrt{n-1}} \cdot \sigma_0 \} \end{aligned}$$

where

- a.  $\mu_0 = \|\text{vec}(P_0)\|^2$ , and
- b.  $\sigma_0^2 = 4(\text{vec}(P_0))^t \Gamma_0(\text{vec}(P_0))$ , with  $\Gamma_0 = 2M_p \Phi_0 M_p$ ,  $\Phi_0 = B_0^t (P_0 \otimes P_0) B_0$ , and  $B_0 = I_{p^2} - (I_p \otimes P_0) \Lambda_p$ .

If we plot  $\|\text{vec}(R_i)\|^2; i = 1, 2, \dots, m$ , in that chart, according to Proposition 2, we conclude that the correlation structure is stable if there is no  $k$  such that  $\|\text{vec}(R_k)\|^2$  is outside the interval (LCL, UCL).

**$P_0$  IS UNKNOWN**

In general,  $P_0$  is unknown. In this case, the control limits LCL and UCL in (2) must be estimated. If  $\hat{\mu}$  and  $\hat{\sigma}^2$  are the estimators of  $\mu = \|\text{vec}(P_0)\|^2$  and  $\sigma^2$ , respectively, the control chart is then defined by the following estimate of control limits (we use the same notation for control limits and their estimates)

$$\begin{aligned} \text{UCL} &= \hat{\mu} + \frac{z_{\alpha/2}}{\sqrt{n-1}} \cdot \hat{\sigma} \dots\dots\dots (4) \\ \text{LCL} &= \max \{0, \hat{\mu} - \frac{z_{\alpha/2}}{\sqrt{n-1}} \cdot \hat{\sigma} \} \end{aligned}$$

We plot  $\|\text{vec}(R_i)\|^2$  for all  $i = 1, 2, \dots, m$  in that chart and we conclude that the process signals unstable correlation structure at sample  $k$  if  $\|\text{vec}(R_k)\|^2$  is less than LCL or greater than UCL.

In the following sub-section the estimates  $\hat{\mu}$  and  $\hat{\sigma}^2$  of  $\mu$  and  $\sigma^2$ , respectively, will be discussed.

**PARAMETER ESTIMATION**

Let us denote  $R_{pooled}$  the pooled correlation matrix,

i.e.,  $R_{pooled} = \frac{1}{N} \sum_{i=1}^m n_i R_i$ , where  $N = \sum_{i=1}^m n_i$ . If all sample

sizes are equal, then  $R_{pooled} = \bar{R}$  is the average of  $m$  correlation matrices. To estimate the process correlation matrix we use Montgomery's approach in estimating the process covariance matrix. See (Montgomery 2001, p. 532) [5] and also (Montgomery 2005, p. 511) [6]. We use then  $R_{pooled}$  as an estimator of  $P_0$ . Accordingly,

- a.  $\hat{\mu} = \|\text{vec}(R_{pooled})\|^2$  and
- b.  $\hat{\sigma}^2 = 4(\text{vec}(R_{pooled}))^t \hat{\Gamma}(\text{vec}(R_{pooled}))$ , where  $\hat{\Gamma} = 2M_p \hat{\Phi} M_p$ ,  $\hat{\Phi} = \hat{B}^t (R_{pooled} \otimes R_{pooled}) \hat{B}$  and  $\hat{B} = I_{p^2} - (I_p \otimes R_{pooled}) \Lambda_p$ .

**ABOUT SENSITIVITY**

Let us consider correlation matrix as the covariance matrix of standardized variables. Then, the sensitivity of the

Table 1 A. Diagonal Elements of Sample Covariance Matrices

Sample	s(1,1)	s(2,2)	s(3,3)
1	1.96E-04	3.17E-02	2.60E-04
2	1.19E-03	2.56E-02	2.87E-05
3	2.35E-03	9.44E-03	2.36E-05
4	5.15E-04	2.12E-02	3.40E-05
5	8.00E-04	2.05E-02	4.16E-05
6	1.06E-03	1.17E-02	7.03E-05
7	3.81E-04	4.70E-03	8.23E-05
8	1.90E-04	1.78E-03	6.93E-05
9	5.17E-04	3.41E-02	5.17E-05
10	1.29E-03	2.03E-02	3.27E-05
11	1.51E-03	3.86E-03	1.33E-04
12	1.17E-03	6.61E-03	2.94E-04
13	8.69E-05	8.78E-03	1.48E-04
14	9.40E-04	2.41E-03	5.58E-06
15	2.72E-04	2.88E-02	2.57E-05
16	3.07E-03	9.69E-02	1.77E-05
17	5.17E-04	3.41E-02	5.17E-05
18	1.29E-03	2.03E-02	3.27E-05
19	9.17E-07	2.26E-02	1.40E-05
20	7.59E-04	2.71E-02	2.12E-04
21	5.61E-04	3.13E-02	5.17E-05
22	4.74E-03	2.47E-02	1.90E-05

procedure proposed in the previous section can be considered as the sensitivity of the procedure of testing the stability of covariance structure based on vector variance. In (Djauhari 2007) [24] we have compared the power of vector variance-based test with likelihood ratio criterion-based test in testing the hypothesis  $H_0: \Sigma = \Sigma_0$  versus  $H_1: \Sigma = k\Sigma_0$  where  $\Sigma_0$  is a specified positive definite matrix and  $k > 1$ . This is equivalent to testing the hypothesis  $H_0: \Sigma = I_p$  versus  $H_1: \Sigma = kI_p$ .

We have showed in that article that, when the sample size  $n$  and the number of variables  $p$  are small or when  $p$  is large but  $n$  is not too large, vector variance-based test is more sensitive than likelihood ratio criterion-based test to small shift of covariance structure. Their sensitivities are similar to large shift when  $n$  and  $p$  are large. In the next section we illustrate how the proposed method in the previous section works by using real data and compare it with the methods of Box's  $M$  statistic and Jennrich's statistic.

**A CASE STUDY**

In this case study,  $P_0$  is unknown. Then, we use the proposed VVSV chart (4) for testing the stability of correlation structure. The process that we want to investigate its correlation structure stability is the production process of drive rib,

Table 1 B. Non-Diagonal Elements of Sample Covariance Matrices

Sample	s(1,2)	s(1,3)	s(2,3)
1	-7.17E-04	-5.34E-05	-1.44E-03
2	-1.93E-03	-1.23E-04	5.24E-04
3	-9.39E-04	-1.41E-04	-1.08E-04
4	-1.82E-03	-4.53E-05	5.91E-04
5	-7.70E-04	-7.93E-05	-5.13E-05
6	-1.28E-03	5.75E-05	-4.74E-04
7	-5.28E-04	8.71E-05	1.11E-04
8	3.95E-05	-6.45E-05	-1.22E-04
9	-1.93E-03	-9.25E-05	3.43E-04
10	-1.79E-03	-9.75E-06	-2.46E-04
11	-9.03E-04	-3.26E-04	2.64E-04
12	-1.99E-03	-1.67E-04	1.62E-04
13	-2.98E-04	5.17E-05	2.20E-04
14	-8.49E-04	4.33E-05	-8.56E-05
15	-1.80E-03	6.88E-06	-3.15E-04
16	-5.08E-03	-1.68E-05	-7.97E-04
17	-1.93E-03	-9.25E-05	3.43E-04
18	-1.79E-03	-9.75E-06	-2.46E-04
19	5.69E-05	1.25E-06	7.50E-05
20	-5.54E-04	9.33E-05	-4.16E-04
21	-2.13E-03	-9.60E-05	3.29E-04
22	-4.83E-03	-1.30E-04	2.02E-04

a component of aircraft wing, at Indonesian Aerospace Ltd. located in Bandung, the capital of West Java. As a supplier of an international aircraft industry, the process quality is of primary concern at Indonesian Aerospace Ltd. It must fulfill the requirement. The goal is to ensure that there is no shift of correlation structure during the process. Due to its confidentiality, the name of the aircraft is kept undeliverable.

For drive rib1-inboard (left wing) production process,  $m = 22$  independent samples were collected during the production process, each of which is of size  $n = 4$ , and the number of important quality characteristics is  $p = 3$ . The sample covariance matrices are presented in Table 1 (A,B), where  $s(i,j)$  is the covariance between the  $i$ -th and the  $j$ -th variables.

Based on these data, in this section we discuss how to test the stability of correlation structure of that process. First, we use the MSPC approach mentioned in the previous section. Then we compare the result with that given by Box's  $M$  statistic and Jennrich's statistic.

**MSPC APPROACH**

From Table 1 we compute the corresponding correlation matrix and the results are presented in Table 2 where  $r(i,j)$  is the correlation coefficient between the  $i$ -th and the  $j$ -th variables. In this table only the non-diagonal elements of  $R$  are presented. All diagonal elements equal 1.

**Table 2. Non-Diagonal Elements of Sample Correlation Matrices**

Sample	r(1,2)	r(1,3)	r(2,3)
1	-0.2879	-0.2368	-0.5018
2	-0.3498	-0.6633	0.6113
3	-0.1994	-0.5973	-0.2291
4	-0.5515	-0.3420	0.6956
5	-0.1901	-0.4349	-0.0555
6	-0.3652	0.2108	-0.5235
7	-0.3945	0.4919	0.1788
8	0.0679	-0.5626	-0.3469
9	-0.4604	-0.5661	0.2586
10	-0.3483	-0.0475	-0.3021
11	-0.3746	-0.7292	0.3686
12	-0.7148	-0.2847	0.1160
13	-0.3413	0.4564	0.1936
14	-0.5632	0.5970	-0.7376
15	-0.6429	0.0822	-0.3655
16	-0.2941	-0.0719	-0.6086
17	-0.4604	-0.5661	0.2586
18	-0.3483	-0.0475	-0.3021
19	0.3952	0.3489	0.1333
20	-0.1222	0.2326	-0.1740
21	-0.5076	-0.5640	0.2589
22	-0.4455	-0.4314	0.2946

Based on data in Table 2 we first compute the estimates  $\hat{\mu}$  and  $\hat{\sigma}^2$ , and then determine LCL and UCL. Later on we compute the VVSV of each sample. From that table we obtain the pooled sample correlation matrix, written in triangular form,

$$R_{pooled} = \begin{pmatrix} 1 & & \\ -0.3156 & 1 & \\ -0.1752 & -0.0394 & 1 \end{pmatrix}$$

and the VVSV of the pooled correlation matrix is  $\|vec(R_{pooled})\|^2 = 3.2637$ . This is the estimate  $\hat{\mu}$ . To compute the estimate  $\hat{\sigma}^2$ , we need the following matrices, each of which is of size  $p^2 \times p^2$ ,

- a.  $\hat{B} = I_{p^2} - (I_p \otimes R_{pooled})\Lambda_p$ ,
- b.  $\hat{\Phi} = \hat{B}^t (R_{pooled} \otimes R_{pooled})\hat{B}$ , and
- c.  $\hat{\Gamma} = 2M_p \hat{\Phi} M_p$ ,

An algebraic manipulation on those matrix operations will give us,

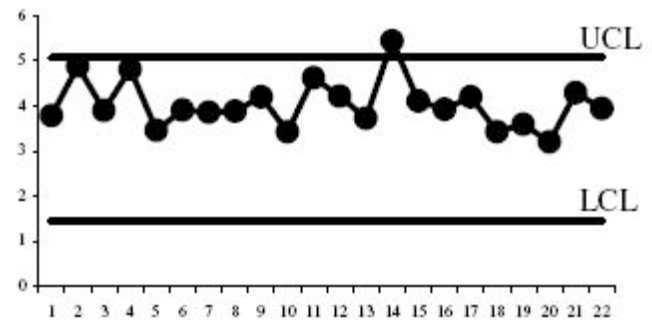
$$\hat{\sigma}^2 = 4 \left( vec(R_{pooled}) \right)^t \hat{\Gamma} \left( vec(R_{pooled}) \right) = 2.5462.$$

Thus, for the level of significance  $\alpha = 5\%$ , the estimated control limits in (4) are  $UCL = 5.0694$  and  $LCL = 1.4580$ . Now we compute the VVSV of each sample. The result is in Table 3.

**Table 3. Sample VVSV**

Sample	VVSV	Sample	VVSV
1	3.7815	12	4.2108
2	4.8721	13	3.7245
3	3.8980	14	5.4353
4	4.8097	15	4.1074
5	3.4568	16	3.9242
6	3.9038	17	4.1987
7	3.8590	18	3.4297
8	3.8828	19	3.5914
9	4.1987	20	3.1986
10	3.4297	21	4.2855
11	4.6159	22	3.9427

In Fig. (1) we present the VVSV chart where all sample VVSVs in Table 3 are plotted. The vertical axis is the value of sample VVSV and the horizontal axis is the sample number. From this figure we learn that at sample 14, VVSV is greater than UCL. Thus, sample 14 signals a change in correlation structure. We then reject the hypothesis of stability of correlation structure.



**Fig. (1).** VVSV chart.

**BOX'S M STATISTIC APPROACH**

To test the hypothesis of the stability of correlation structure, (Tang 1998) [11], (Annaert *et al.*, 2003) [13], and (Da Costa Jr. *et al.*, 2005) [15], among others, use Box's *M* statistic

$$M = N \ln |R_{pooled}| - \sum_{i=1}^m n_i \ln |R_i|.$$

If the stability is true, then  $eM \xrightarrow{d} F_{a,d}$  where

- a.  $a = \frac{p(p+1)(m-1)}{2}$ ;
- b.  $b = \frac{2p^2 + 3p - 1}{6(p+1)(m-1)} \left[ \sum_{i=1}^m \frac{1}{n_i} - \frac{1}{N} \right]$ ;
- c.  $c = \frac{(p-1)(p+2)}{6(m-1)} \left[ \sum_{i=1}^m \frac{1}{n_i^2} - \frac{1}{N^2} \right]$ ;
- d.  $d = \frac{(a+2)}{(c-b^2)}$ ;
- e.  $e = \frac{(1-b-\frac{a}{d})}{a}$ .

The stability is rejected if  $eM > F_{\alpha;a,d}$ , the  $(1-\alpha)$ -th quantile of  $F$  distribution with  $a$  and  $d$  degrees of freedom. For  $m = 22, n = 4$ , and  $p = 3$ , we obtain the summary of important quantities  $a, b, c, d$  and  $e$  needed to compute the critical point and the statistic  $eM$ .

**Table 4. Important Quantities**

$a$	$b$	$c$	$d$	$e$
126	0.2831	0.1091	4421.990	0.0055

From this table, for significance level  $\alpha = 5\%$ , the critical point  $F_{\alpha;a,d} = 1.2199$ . On the other hand, the 22 correlation matrices in Table 2 give us  $M = 41.5567$  and hence we obtain  $eM = 0.2270$ . Thus,  $eM < F_{\alpha;a,d}$ , which means that the hypothesis of the stability of correlation structure is accepted.

**JENNRICH'S STATISTIC APPROACH**

Jennrich in (Jennrich 1970) [20] proposes to test the hypothesis of the stability of correlation structure by using the following statistic,

$$J = \sum_{i=1}^m \left\{ \frac{1}{2} \| \text{vec}(Z_i) \|^2 - \Delta_i^t H^{-1} \Delta_i \right\}$$

where

- a.  $Z_i = \sqrt{n_i} R_{pooled}^{-1} (R_i - R_{pooled})$ ;
- b.  $\Delta_i$  is the column vector where its elements are equal to the diagonal elements of  $Z_i$ ;
- c.  $H = I_p + R_{pooled} * R_{pooled}^{-1}$  where  $*$  is the Hadamard product of two matrices. See (Schott 1997) [36] for the notion of this product.

Under the hypothesis that the correlation structure is stable, (Jennrich 1970) [20] shows that  $J \xrightarrow{d} \chi_{df}^2$  where the degree of freedom  $df = \frac{(m-1)p(p-1)}{2}$ . The hypothesis of stability is then rejected if  $J > \chi_{\alpha;df}^2$ , the  $(1-\alpha)$ -th quantile of chi-squared distribution with  $df$  degree of freedom. Further computation based on data in Table 2 gives us  $J = 37.8530$  and, for significance level  $\alpha = 5\%$ , the critical point is  $\chi_{\alpha;df}^2 = 82.5287$ . Thus,  $J < \chi_{\alpha;df}^2$ . We conclude that the hypothesis of the stability of correlation structure is accepted.

We learn from the above example that Jennrich's statistic and Box's  $M$  statistic might give the same result which is different from the result given by MSPC procedure (4).

**ARL COMPARISON**

In the previous section, we have mentioned the sensitivity of vector variance-based test compared with that of likelihood ratio criterion-based test. In what follows, in order to compare the ARL of VVSV and that of Jennrich's statistic, their power  $(1 - \beta)$  will be studied under  $H_0: P = P_0$  and  $H_1: P = P_{10}$  where  $P_{10} = I_p + k(P_0 - I_p)$ ;  $P_0$  is given and  $0 < k < 1$ . Let us denote  $ARL_1(C)$  the ARL of control chart C when  $H_1$  is true. It is equal to  $1/(1 - \beta)$  with  $\beta = P(\text{accept } H_0 | H_1 \text{ is true})$ . After a long and tedious mathematical derivation,  $ARL_1(VVSV)$  is given by the following proposition.

**Proposition 3.** Let  $D_Q$  be a diagonal matrix where its  $k$ -th diagonal element is equal to the  $k$ -th diagonal element of a matrix  $Q$ . Then,  $\beta = P(LZ < Z < UZ)$  where  $Z \sim N(0,1)$  and

- a.  $LZ = \frac{A - \mu - z_{\alpha/2} B}{\sigma}$  and  $UZ = \frac{A - \mu + z_{\alpha/2} B}{\sigma}$ ;
- b.  $\mu = \| \text{vec}(P_0) \|^2$  and  $\sigma^2 = \frac{8}{(n-1)} \left( \text{Tr}(P_0^4) - 2\text{Tr}(D_{P_0^2} P_0^3) + \text{Tr}(D_{P_0^2} P_0)^2 \right)$ ;
- d.  $A = p(1-k^2) + k^2 \text{Tr}(P_0^2)$ ;
- e.  $B^2 = \frac{8}{(n-1)} \left( \text{Tr}(P_{10}^4) - 2\text{Tr}(D_{P_{10}^2} P_{10}^3) + \text{Tr}(D_{P_{10}^2} P_{10})^2 \right)$ .

In Proposition 4 we present the value of  $\beta$  from which we can derive  $ARL_1(J)$ , i.e., the ARL of Jennrich's statistic when  $H_1$  is true.

**Proposition 4.**  $\beta = P(J < b)$  where  $J \xrightarrow{d} \chi_{\frac{1}{2}p(p-1)}^2$

and

- a.  $b = \chi^2_{1-\alpha; \frac{1}{2}p(p-1)} - \frac{1}{2} \|\text{vec}(A)\|^2 - \Delta_1^t \cdot H_1^{-1} \cdot \Delta_1$ ;
- b.  $A = \sqrt{n} \cdot P_{10}^{-1} \cdot (P_0 - P_{10})$ ;
- c.  $\Delta_1$  is the vector where its  $k$ -th component is the  $k$ -th diagonal element of  $A$ ;
- d.  $H_1 = I_p + P_{10} * P_{10}^{-1}$ ;  $*$  is the Hadamard product.

Those who are interested in the proofs of the above propositions are pleased to contact the authors.

By using the last two propositions, based on simulation experiment for several values of  $p$  and  $n$ , and  $k$  runs from 0.1 until 1 in Step 0.1, in Tables (5 – 7) we present the value of  $(1 - \beta)$  of VVSV and Jennrich’s statistic ( $J$ ), respectively. To obtain those tables, we generate random data from  $p$ -variate normal distribution with zero mean vector and correlation matrix  $P_0$  where all non-diagonal elements are equal to 0.5. For each selected value of  $p$ ,  $n$  and  $k$ , we use 1000 replication and the final value of  $(1 - \beta)$  is defined as the average of all replications. In this experiment, we use  $n = 5$  and 15 for  $p = 2$  and 5;  $n = 50$  and 100 for  $p = 15$ ; with  $\alpha = 5\%$ .

**Table 5. Values of  $(1 - \beta)$  for  $p = 2$**

$k$	$n = 5$		$n = 15$	
	VVSV	$J$	VVSV	$J$
0.1	0.8338	0.0929	0.9024	0.3745
0.2	0.6723	0.0820	0.7949	0.2379
0.3	0.5214	0.0736	0.6724	0.1660
0.4	0.3877	0.0671	0.5377	0.1232
0.5	0.2765	0.0619	0.4014	0.0958
0.6	0.1905	0.0578	0.2778	0.0775
0.7	0.1288	0.0546	0.1787	0.0651
0.8	0.0877	0.0522	0.1093	0.0568
0.9	0.0628	0.0506	0.0680	0.0518
1	0.05	0.05	0.05	0.05

We learn from those tables that, in general, VVSV is better than Jennrich’s statistic in the sense that:

- a. If  $p$  is small,  $ARL_1(VVSV)$  is less than  $ARL_1(J)$ .
- b. If  $p$  is large and  $k$  is small,  $ARL_1(VVSV)$  and  $ARL_1(J)$  have similar performance. But, if  $p$  and  $k$  are large, the former is less than the latter.

**THE STABILITY OF COVARIANCE STRUCTURE**

In what follows we discuss the use of MSPC approach for testing the stability of covariance structure. First, we use the VV chart, i.e., the chart constructed based on VV as multivariate dispersion measure. Second, the chart based on GV. The results will show that, as mentioned in Introduction section, the stability (instability) of covariance structure does not necessarily ensure the stability (instability) of correlation

**Table 6. Values of  $(1 - \beta)$  for  $p = 5$**

$k$	$n = 5$		$n = 15$	
	VVSV	$J$	VVSV	$J$
0.1	0.9310	0.3964	0.9821	1
0.2	0.8425	0.2091	0.9542	0.9766
0.3	0.7328	0.1322	0.9064	0.5925
0.4	0.6035	0.0958	0.8250	0.2989
0.5	0.4632	0.0762	0.6981	0.1667
0.6	0.3270	0.0647	0.5286	0.1062
0.7	0.2117	0.0576	0.3443	0.0761
0.8	0.1276	0.0533	0.1882	0.0604
0.9	0.0758	0.0508	0.0899	0.0525
1	0.05	0.05	0.05	0.05

**Table 7. Values of  $(1 - \beta)$  for  $p = 15$**

$k$	$n = 50$		$n = 100$	
	VVSV	$J$	VVSV	$J$
0.1	1	1	1	1
0.2	1	1	1	1
0.3	1	1	1	1
0.4	0.9992	0.9737	1	1
0.5	0.9944	0.5892	1	1
0.6	0.9640	0.2565	0.9987	0.6672
0.7	0.8344	0.1244	0.9735	0.2610
0.8	0.5253	0.0746	0.7709	0.1081
0.9	0.1835	0.0552	0.2892	0.0609
1	0.05	0.05	0.05	0.05

structure. Those who are interested in a more general discussion on the material of this section are suggested to see (Djauhari 2005) [7] and (Djauhari *et al.*, 2008) [43].

We start with VV chart. Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from a random vector  $X$  which is distributed as a  $p$ -variate normal distribution  $N_p(\mu, \Sigma)$ . We

recall  $\|\text{vec}(\Sigma)\|^2$  the vector variance (VV) of the random vector  $X$ . In (Djauhari 2007) [24] we have showed the following proposition which gives us the asymptotic distribution of sample VV.

**Proposition 5.**  $\|\text{vec}(S)\|^2 \xrightarrow{d} N(\theta, \eta^2)$  where  $\theta = \|\text{vec}(\Sigma)\|^2$ ,  $\eta^2 = \frac{4}{n-1} \text{vec}(\Sigma)^t \Gamma \text{vec}(\Sigma)$  and  $\Gamma = (I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)$ .



The application of this proposition is seemingly complicated because the expression of the variance involves matrix multiplications of high dimension even for moderate value of  $p$ . However, Proposition 6 shows that it can be computed in a simple manner. The proof, using the properties of commutation matrix and  $vec$  operator, is derived in (Djauhari 2007) [24].

**Proposition 6.**  $\eta^2 = \frac{8}{n-1} \left\| vec \left( \Sigma^2 \right) \right\|^2$ .

**MSPC APPROACH**

Based on Propositions 5 and 6, as in the fourth section, in what follows we present the use of MSPC approach for monitoring the stability of covariance structure. For that purpose we first estimate the parameters  $\theta$  and  $\eta^2$  based on  $m$  independent samples described in the fifth section.

Let  $S_i$  be the sample covariance matrix of the  $i$ -th sample and  $S_{pooled}$  be the pooled covariance matrix, i.e.,

$S_{pooled} = \frac{1}{N-m} \sum_{i=1}^m (n_i - 1) S_i$ . In our case all sample sizes are equal, say  $n$ . Accordingly,  $S_{pooled}$  is equal to the average  $\bar{S}$  of  $S_i$ ;  $i = 1, 2, \dots, m$ . In (Djauhari 2007) [24] we have derived that the statistics  $\hat{\theta} = \left\{ 1 - \frac{2}{m(n-1)+2} \right\} \left\| vec \left( \bar{S} \right) \right\|^2$  and  $\hat{\eta}^2 = \frac{8}{(n-1)} \left\{ 1 + \frac{12}{m(n-1)} + \frac{12}{\{m(n-1)\}^2} \right\}^{-1} \left\| vec \left( \bar{S}^2 \right) \right\|^2$  are asymptotically unbiased estimators of  $\theta$  and  $\eta^2$ , respectively. By using these estimates, the control chart that we are seeking is defined by the following estimate of control limits.

$UCL = \hat{\theta} + z_{\alpha/2} \hat{\eta}$ , and ..... (5)

$LCL = \max \left( 0, \hat{\theta} - z_{\alpha/2} \hat{\eta} \right)$ .

The stability of covariance structure is then monitored by plotting the sample VVs,  $\left\| vec \left( S_i \right) \right\|^2$ ;  $i = 1, 2, \dots, m$ , in that chart. We call it VV chart. For this purpose, we compute the sample VV for all samples. The results are in Table 8.

Furthermore, a simple algebraic manipulation gives us the following matrices  $\bar{S}$  and  $\bar{S}^2$ , written in triangular form,

$\bar{S} = \begin{pmatrix} 1.06E-03 & & \\ -1.53E-03 & 2.22E-02 & \\ -5.02E-05 & -5.16E-05 & 7.71E-05 \end{pmatrix}$ ,

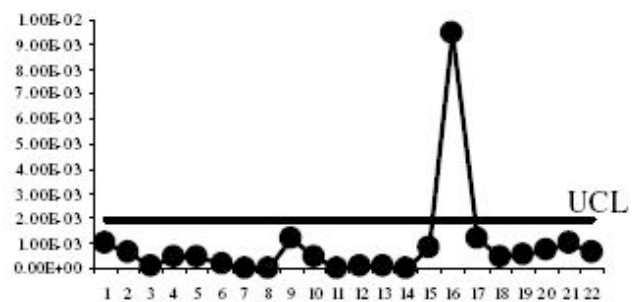
**Table 8. Sample VV**

Sample	VV	Sample	VV
1	1.01E-03	12	5.32E-05
2	6.64E-04	13	7.74E-05
3	9.65E-05	14	8.17E-06
4	4.57E-04	15	8.39E-04
5	4.23E-04	16	9.46E-03
6	1.41E-04	17	1.17E-03
7	2.28E-05	18	4.22E-04
8	3.26E-06	19	5.11E-04
9	1.17E-03	20	7.35E-04
10	4.22E-04	21	9.91E-04
11	1.92E-05	22	6.81E-04

$\bar{S}^2 = \begin{pmatrix} 3.49E-06 & & \\ -3.57E-05 & 4.96E-04 & \\ 2.19E-08 & -1.07E-06 & 1.11E-08 \end{pmatrix}$ ,

and we get  $\left\| vec \left( \bar{S} \right) \right\|^2 = 4.99E-04$  and  $\left\| vec \left( \bar{S}^2 \right) \right\|^2 = 2.48E-$

07. These statistics are used to compute the estimates  $\hat{\theta}$  and  $\hat{\eta}^2$  of  $\theta$  and  $\eta^2$ , respectively. We obtain  $\hat{\theta} = 4.84E-04$  and  $\hat{\eta}^2 = 5.59E-07$ . Thus, the estimated control limits in (5) are  $UCL = 1.95E-03$  and  $LCL = 0$ . These control limits and the value of  $\left\| vec \left( S_k \right) \right\|^2$ ;  $k = 1, 2, \dots, m$ , are plotted in Fig. (2), where the vertical axis is the value of sample VV and the horizontal axis is the sample number.



**Fig. (2).** VV chart.

We learn from this figure that a signal of the instability of covariance structure occurs at the sixteenth sample and not at fourteenth sample as indicated in Fig. (1). We see that the instability of covariance structure does not mean, in the same sense, the instability of correlation structure.

In what follows we investigate the stability of covariance structure based on GV. A worst situation might be signaled if we use this measure. Look at Table 9 which presents the value of sample GVs.

Table 9. Sample GV

Sample	GV	Sample	GV
1	8.70E-10	12	1.00E-09
2	3.03E-10	13	6.50E-11
3	2.60E-10	14	3.53E-12
4	1.33E-10	15	9.77E-11
5	5.20E-10	16	2.70E-09
6	5.45E-10	17	4.87E-10
7	7.37E-11	18	6.66E-10
8	1.37E-11	19	2.15E-13
9	4.87E-10	20	3.96E-09
10	6.66E-10	21	4.59E-10
11	3.04E-10	22	1.43E-09

Based on the improved generalized variance (IGV) chart, proposed in our work (Djauhari 2005) [7], we have UCL = 2.42E-09 and LCL = 0. These estimated control limits and the value of sample GVs are plotted in Fig. (3). The vertical axis is the value of sample GV and the horizontal axis is the same as in the two previous figures.

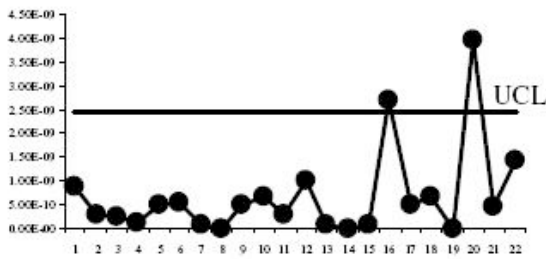


Fig. (3). IGV chart.

This figure signals a worst situation compared with Fig. (2). There are two signals of instability in this figure while there is only one in Fig. (2). This is not surprising because VV and GV are different measures. Moreover, the VV chart is in general more sensitive than the IGV chart to the shift of covariance structure. See (Djauhari 2007) [24] for further discussion on this matter.

**ADDITIONAL REMARKS**

We define VVSV as a measure of data concentration when all variables are standardized. By using *vec* operator and the property of commutation matrix, its asymptotic sampling distribution is investigated. Sample VVSV converges in distribution to a normal distribution. We have proved, see Proposition 2, that the random variable

$\sqrt{n-1} \left\{ \|vec(R)\|^2 - \|vec(P)\|^2 \right\}$  converges in law to a normal distribution  $N(0, \sigma^2)$  where the variance  $\sigma^2$  is asymptotically equal to  $4 \left( vec(P) \right)^t \Gamma \left( vec(P) \right)$ . We see that the

computation of  $\sigma^2$  involves neither matrix inversion nor determinant. It only involves some matrix operations.

Based on that result, by using repeated tests suggested in (Montgomery 2001) [5] and (Montgomery 2005) [6], we propose to test the stability of correlation structure based on control chart (3) if  $P_0$  is known or (4) if  $P_0$  is unknown. All computations and drawings in the previous sections are done using MS Excel. This shows the simplicity of the proposed method. However, it is still in our investigation to reduce the computational complexity of  $\hat{\sigma}^2$  which involves numerous matrices of high size even for moderate value of  $p$ . On the other hand, when  $p$  is large, the use of Box’s  $M$  statistic is quite cumbersome because it involves the computation of the determinants  $|\bar{R}|$  and  $|R_i|$ . It is so with Jennrich’s statistic which needs the inversion of  $\bar{R}$ . These obstacles are what we intend to eliminate by using MSPC approach proposed in (3) or (4).

The example in the fifth section shows that the proposed method gives us a different conclusion than those given by Box’s  $M$  statistic and Jennrich’s statistic. Fig. (1) summarizes that the correlation structure is not stable whereas Box’s  $M$  statistic and Jennrich’s statistic give us the opposite conclusion. This difference is not surprising because the proposed method and the latter two methods use different measure of multivariate dispersion. However, as mentioned in A Case Study section,  $ARL_1(VVSV)$  is in general less than  $ARL_1(J)$ . Thus, in general, VVSV is more sensitive than Jennrich’s statistic to the shift of correlation structure.

The VVSV chart in Fig. (1) presents the history of correlation structure from sample to sample given by (4) whereas the VV chart in Fig. (2) shows the history of covariance structure from sample to sample based on (5). In Fig. (1), the fourteenth point lies outside the interval (LCL, UCL). This means that the correlation structure is not stable. On the other hand, Fig. (2) indicates that the covariance structure is not stable at sixteenth sample. This strengthens our statement in Introduction section that the stability (or instability) of covariance structure does not necessarily ensure the stability (or instability) of correlation structure. Signal of a worst situation might occur when one uses generalized variance chart as indicated by Fig. (3).

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