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WAVELET BASIS IN THE SPACE $C^\infty[-1,1]$

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ABSTRACT. We show that the polynomial wavelets suggested by T.Kilgor and J.Prestin in [12] form a topological basis in the space $C^\infty[-1,1]$.

During the last twenty years wavelets have found a lot of applications in mathematics, physics and engineering. Our interest in wavelets is related to their ability to represent a function, not only in the corresponding Hilbert space, but also in other function spaces with perhaps quiet different topology. Wavelets form unconditional Schauder bases in Lebesgue spaces ([16], [8], see also [3] and [11]) and in the Hardy space ([23], [16]). Weighted spaces $L^p(w)$, $H^p(w)$ were considered in [4], [5]. For the multidimensional case, see also [19]. Wavelet topological bases were found in Sobolev spaces ([9], [2]) and in their generalizations, as in Besov ([1],[10]) and Triebel-Lizorkin ([14]) spaces. The list is far from being complete. Using "multiresolution analysis" of the space of continuous functions, Girgensohn and Prestin constructed in [6] (see also [18], [15] and [13]) a polynomial Schauder basis of optimal degree in the space $C[-1,1]$. Here we show that the polynomial wavelets suggested in [12] form a topological basis in the space $C^\infty[-1,1]$. As far as we know this is the first (but we are sure not the last!) example when wavelets form a topological basis in non-normed Fréchet space. Since the space is nuclear, the basis is absolute.

1. **Polynomial wavelets.**

T.Kilgor and J.Prestin suggested in [12] the following wavelets constructed from the Chebysev polynomials. Let Π_n denote the set of all polynomials of degree at most n. For $n \in \mathbb{N}_0 := \{0,1,2,\cdots\}$ and $|\,x\,| \le 1$ let $T_n(x) = \cos(n \cdot \arccos x)$ be the Chebyshev polynomial of the first kind. Let $\omega_0(x) = 1 - x^2$ and for $n \in \mathbb{N}_0$ let

$$\omega_{n+1}(x) = 2^{n+1}(1-x^2)\,T_1(x)\,T_2(x)\,T_4(x)\cdots T_{2^n}(x).$$

The scaling functions are given by the condition

$$\varphi_{j,k}(x) = \frac{\omega_j(x)}{\omega_j'(\cos\frac{k\pi}{2^j})(x-\cos\frac{k\pi}{2^j})},\ \ k = 0,1,\cdots,2^j,\ \ j \in \mathbb{N}_0.$$

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Now the Kilgor-Prestin wavelets are defined as

$$\psi_{j,k}(x) = \frac{T_{2^j}(x)}{2^j\,(x - x_{j,k})}\,[2\omega_j(x) - \omega_j(x_{j,k})], \;\; k = 0, 1, \cdots, 2^j - 1, \;\; j \in \mathbb{N}_0$$

with $x_{j,k} = \cos\frac{(2k+1)\pi}{2^{j+1}}$.

Then (see [12] for more details) the subspaces $W_{-1} := \Pi_1$ and $W_j := span\{\psi_{j,k},\, k = 0, 1, \cdots, 2^j - 1\} = span\{T_{2^j+1}, T_{2^j+2}, ..., T_{2^{j+1}}\}$, $j \in \mathbb{N}_0$ give the decomposition

$$\Pi_{2^{j+1}} = W_{-1} \oplus W_0 \oplus \cdots \oplus W_j \tag{1}$$

which is orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{-1/2}\,dx.$$

By H we denote the corresponding Hilbert space. Let $\varepsilon_{j,n}$ take the value 1 for $1 \le n \le 2^j - 1$ and $\varepsilon_{j,0} = \varepsilon_{j,2^j} = 1/2$.

Lemma 1. *(Lemma 2.2 in [12]) The wavelets can be written as*

$$\psi_{j,k}(x) = 2^{1-j} \sum_{n=2^j+1}^{2^{j+1}} T_n(x)T_n(x_{j,k})\varepsilon_{j+1,n}.$$

Let us express the Chebyshev polynomials in terms of the system $\{\psi_{j,k}\}$.

Lemma 2. *If* $2^j + 1 \le n \le 2^{j+1}$ *for* $j \in \mathbb{N}_0$ *then*

$$T_n = \sum_{k=0}^{2^j-1} T_n(x_{j,k})\psi_{j,k}.$$

Proof:

Since the decomposition (1) is orthogonal, we get $T_n = \sum_{k=0}^{2^j-1} d_k^{(n)}\psi_{j,k}$. To find $d_k^{(n)}$ we can use the following interpolational property of wavelets ([12], (2.4))

$$\psi_{j,k}(x_{j,m}) = \delta_{m,k} \text{ for } m, k = 0, 1, ..., 2^j - 1.$$

Hence, $d_k^{(n)} = T_n(x_{j,k})$. $\square$

Lemma 3. *Any function* $f \in H$ *can be represented in the form*

$$f = \frac{1}{\pi}\langle f, T_0\rangle T_0 + \frac{2}{\pi}\langle f, T_1\rangle T_1 + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}$$

where

$$c_{j,k} = \frac{2}{\pi}\sum_{n=2^j+1}^{2^{j+1}} \langle f, T_n\rangle T_n(x_{j,k})$$

and convergence is considered with respect to the Hilbert norm.

Proof: The Chebyshev polynomials form a Hilbert basis in the space H. Since $\langle T_n, T_n\rangle = \pi/2$ for $n \geq 1$ and $\langle T_0, T_0\rangle = \pi$, we have $f = \frac{1}{\pi}\langle f, T_0\rangle T_0 + \frac{2}{\pi}\langle f, T_1\rangle T_1 + \frac{2}{\pi}\sum_{j=0}^{\infty}\sum_{n=2^j+1}^{2^{j+1}}\langle f, T_n\rangle T_n$. By using Lemma 2 and changing the order of summation we get the desired result. $\Box$

Lemma 4. *For any* $j \in \mathbb{N}_0$ *the matrices* $X_j = (T_n(x_{j,k}))_{n=2^j+1,\, k=0}^{2^{j+1},\ 2^j-1}$ *and* $Y_j = 2^{1-j}\ (T_n(x_{j,k})\, \varepsilon_{j+1,n})_{k=0,\ \ n=2^j+1}^{2^j-1,\ \ 2^{j+1}}$ *are not singular.*

Proof: We get the matrix Y_j if we transpose X_j, then multiply the last column by $1/2$ and take the common coefficient 2^{1-j}. Let us multiply the $p-$th row of Y_j by the $q-$th column of X_j :

$$2^{1-j}\sum_{n=2^j+1}^{2^{j+1}} T_n(x_{j,p})T_n(x_{j,q})\varepsilon_{j+1,n} = \psi_{j,p}(x_{j,q}) = \delta_{p,q}.$$

Therefore, $Y_j \cdot X_j = I$ and both matrices are not singular.

Since $\det(Y_j) = 2^{-j}\det(X_j)$, we get $\det(X_j) = \pm 2^{j/2}$ and $\det(Y_j) = \pm 2^{-j/2}$. $\Box$

Remark. If we multiply the $p-$th row of X_j by the $q-$th column of Y_j, then we get the orthogonality property (1.141) from [21].

2. Wavelet Schauder basis in $C^\infty[-1,1]$.

Topology τ of the space $C^\infty[-1,1]$ of all infinitely differentiable functions on $[-1,1]$ can be given by the system of norms

$$|f|_p = \sup\{|f^{(i)}(x)| : |\,x\,| \leq 1,\, i \leq p\},\ p \in \mathbb{N}_0.$$

The first basis in $C^\infty[-1,1]$, namely the Chebyshev polynomials, was found by Mityagin ([17], L.25). And what is more, by the Dynin-Mityagin theorem ([17], T.9), every topological basis of nuclear Fréchet space is absolute. In our case we see that the series $\frac{1}{\pi}\langle f, T_0\rangle T_0 + \frac{2}{\pi}\sum_{n=1}^{\infty}\langle f, T_n\rangle T_n$ converges to $f \in C^\infty[-1,1]$ in the topology τ. The convergence is absolute, that is for any $p \in \mathbb{N}_0$ the series $\sum_{n=0}^{\infty}|\langle f, T_n\rangle| \cdot |T_n|_p$ converges. Furthermore, if $\{e_n, \xi_n\}$ is a biorthogonal system with the total (that is $\xi_n(f) = 0, \forall n \Longrightarrow f = 0$) over $C^\infty[-1,1]$ sequence of functionals and for every $p \in \mathbb{N}_0$ there exist $q \in \mathbb{N}_0$ and $C > 0$ such that

$$|\,e_n|_p \cdot |\,\xi_n|_{-q} \leq C \qquad \text{for all} \quad n,$$

then (e_n) is a Schauder basis in $C^\infty[-1,1]$.

Here and subsequently, $|\cdot|_{-q}$ denotes the dual norm: for a bounded linear functional ξ let $|\,\xi|_{-q} = \sup\{|\,\xi(f)| : \,|f|_q \leq 1\}$.

Theorem 1. *The system* $\{T_0, T_1, (\psi_{j,k})_{j=0,\,k=0}^{\infty,\ 2^j-1}\}$ *is a topological basis in the space* $C^\infty[-1,1]$.

Proof: We suggest two proofs of the theorem.
The 1^{st} *proof* is similar in spirit to the arguments of Mityagin in [17], L.25. Let $\xi_0(f) = \frac{1}{\pi}\langle f, T_0\rangle$, $\xi_1(f) = \frac{2}{\pi}\langle f, T_1\rangle$ and for $j \in \mathbb{N}_0$, $0 \leq k \leq 2^j - 1$ let

$\xi_{j,k}(f) = c_{j,k}$, where $c_{j,k}$ are given in Lemma 3. Then $\xi_{j,k}(\psi_{i,l}) = 0$ if $i \neq j$, as is easy to see. For the wavelets and functionals of the same level we get

$$\xi_{j,k}(\psi_{j,l}) = \frac{2}{\pi} \sum_{n=2^j+1}^{2^{j+1}} T_n(x_{j,k})\ 2^{1-j} \sum_{m=2^j+1}^{2^{j+1}} T_m(x_{j,l})\varepsilon_{j+1,m}\langle T_m(\cdot), T_n(\cdot)\rangle =$$

$$= 2^{1-j} \sum_{n=2^j+1}^{2^{j+1}} T_n(x_{j,k})T_n(x_{j,l})\varepsilon_{j+1,n} = \psi_{j,l}(x_{j,k}) = \delta_{l,k}.$$

Therefore the functionals $\{\xi_0, \xi_1, (\xi_{j,k})_{j=0,\,k=0}^{\infty,\ 2^j-1}\}$ are biorthogonal to the system $\{T_0, T_1, (\psi_{j,k})_{j=0,\,k=0}^{\infty,\ 2^j-1}\}$. Let us check that this sequence of functionals is total over $C^\infty[-1,1]$. Suppose that $\xi_{j,k}(f) = 0$ for all j and k. For fixed j we get the system of 2^j linear equations $\langle f, T_n\rangle T_n(x_{j,k}) = 0,\ n = 2^j+1, \cdots, 2^{j+1}$ with unknowns $\langle f, T_n\rangle$. By Lemma 4 the system has only the trivial solution. Together with $\xi_0(f) = \xi_1(f) = 0$ it follows that $\langle f, T_n\rangle = 0$ for all n. But the Chebyshev polynomials form a basis in $C^\infty[-1,1]$ and so $f = 0$. Thus it is enough to check the Dynin-Mityagin condition. Let us fix $p \in \mathbb{N}_0$. For Chebyshev polynomials we have (see e.g.[21])

$$|\, T_n\,|_m = T_n^{(m)}(1) = \frac{n^2\,(n^2-1)\,(n^2-2^2)\cdots(n^2-(m-1)^2)}{1\cdot 3\cdot 5\cdots(2m-1)}\,. \qquad (2)$$

By Lemma 1,

$$|\,\psi_{j,k}\,|_p \le 2^{1-j}\ \sup_{m\le p}\ \sum_{n=2^j+1}^{2^{j+1}} |\,T_n\,|_m \le 2^{1-j}\,2^j\,|\,T_{2^{j+1}}\,|_p \le 2^{(j+1)2p+1}. \qquad (3)$$

On the other hand, by orthogonality

$$\langle f, T_n\rangle = \int_0^\pi f(\cos t)\,\cos nt\,dt = \int_0^\pi [f(\cos t) - Q(\cos t)]\,\cos nt\,dt$$

for any polynomial $Q \in \Pi_{n-1}$. As in [7] we can take the polynomial $Q = Q_{n-1}$ of best approximation to f on $[-1,1]$ in the norm $|\cdot|_0$. By the Jackson theorem (see e.g. [20], T.1.5) for any $q \in \mathbb{N}_0$ there exists a constant C_q such that for any $n > q$

$$|\,f - Q_{n-1}\,|_0 \le C_q\,n^{-q}\,|\,f\,|_q.$$

Therefore, $|\langle f, T_n\rangle| \le \pi\,C_q\,n^{-q}\,|\,f\,|_q$ and for $2^j > q$ we get

$$|\,\xi_{j,k}|_{-q} \le 2\,C_q 2^j\,(2^j)^{-q}.$$

Taking into account (3) we see that the values $q = 2p+1$ and $C = 4^{p+1}C_q$ will give us the desired conclusion.

In the 2^{nd} *proof* we introduce the operator A first on the basis (T_n) and then by linearity. Let $AT_0 = T_0$, $AT_1 = T_1$ and $AT_n = \psi_{j,k}$ for $n = 2^j+k+1$, where $j \in \mathbb{N}_0$, $k = 0, 1, \cdots, 2^j-1$. Let us show that for any $p \in \mathbb{N}_0$ there exist $q \in \mathbb{N}_0$ and $C > 0$ such that

$$|\,\psi_{j,k}|_p \le C|\,T_{2^j+k+1}|_q \qquad \text{for all}\ \ j \text{ and } k. \qquad (4)$$

For the left side we already have the bound (3). Also, from (2) we obtain

$$|\,T_{2^j+k+1}|_q \geq |\,T_{2^j}|_q \geq \frac{1}{1\cdot 3\cdot 5\cdots(2q-1)}\,(2^{2j}-q^2)^q.$$

Clearly, the value $q = p+1$ provides the inequality (4) for large enough j. Hence there exists C depending only on p that ensures the result for all j and k.

From (4) we deduce that the operator

$$A : C^\infty[-1,1] \longrightarrow C^\infty[-1,1] : f = \sum_0^\infty \xi_n\, T_n \mapsto \sum_0^\infty \xi_n\, AT_n$$

is well defined and continuous. If $Af = 0$, then for any $j \in \mathbb{N}_0$ we have $\sum_{k=0}^{2^j-1} \xi_{2^j+k}\, \psi_{j,k} = 0$. Lemma 4 implies $\xi_{2^j+k} = 0$. Therefore, $ker A = 0$. In the same way, one can easily show that A is surjective. Therefore the operator A is a continuous linear bijection. By the open mapping theorem, A is an isomorphism. Thus the system $\{T_0, T_1, (\psi_{j,k})_{j=0,\,k=0}^{\infty,\;2^j-1}\}$ is a topological basis and what is more, it is equivalent to the classical basis $(T_n)_0^\infty$ (see e.g. [22] for the definition of equivalent bases).

□

Remark. Since $\{T_0, T_1, (\psi_{j,k})_{j=0,\,k=0}^{\infty,\;2^j-1}\}$ is a block-system with respect to the basis $(T_n)_0^\infty$, one can suggest also a third proof based on a generalization of Corollary 7.3 from [22], Ch.1 for the case of countably normed space.

References

[1] Almedia A. Wavelet bases in generalized Besov spaces. J Math Anal Appl 2005; 304: 198-211.

[2] Boigelot Ch. Wavelet in Sobolev spaces. Bull Soc Roy Sci Liége 2000; 69: 157-246.

[3] Daubechies I. Ten Lectures on Wavelets: Philadelphia 1992.

[4] García-Cuerva J.and Kazarian K. Calderon-Zygmund operators and unconditional bases of weighted Hardy spaces. Studia Math 1994; 109: 257-276.

[5] García-Cuerva J.and Kazarian K. Spline wavelet bases of weighted L^p spaces, $1 \leq p < \infty$. Proc Amer Math Soc 1995; 123: 433-439.

[6] Girgensohn R. and Prestin J. Lebesgue constraints for an orthogonal polynomial Schauder basis. J Comput Anal Appl 2000; 2: 159-175.

[7] Goncharov A. Spaces of Whitney functions with basis. Math Nachr 2000; 220: 45-57.

[8] Gripenberg G. Wavelet bases in $L^p(\mathbb{R})$. Studia Math 1993; 106: 175-187.

[9] Gripenberg G. Unconditional bases of wavelets for Sobolev spaces. SIAM J Math Anal 1993; 24: 1030-1042.

[10] Haroske D. and Triebel H. Wavelet bases and entropy numbers in weighted function spaces. Math. Nachr 2005; 278: 108-132.

[11] Hernández E. and Weiss G. A First Course on Wavelets: CRC Press 1996.

[12] Kilgore T. and Prestin J. Polynomial wavelets on the interval. Constr Approx 1996; 12: 95-110.

[13] Kilgore T., Prestin J. and Selig K. Orthogonal algebraic polynomial Schauder bases of optimal degree. J Fourier Anal Appl 1996; 2: 597-610.

[14] Kyriazis G. and Petrushev P. New bases for Triebel-Lizorkin and Besov spaces. Trans Amer Math Soc 2002; 354: 749-776.

[15] Lorentz R.A. and Sahakian A.A. Orthogonal trigonometric Schauder bases of optimal degree for $C(K)$. J Fourier Anal Appl 1994; 1: 103-112.

[16] Meyer Y. Ondelettes et opérateurs. Paris: Hermann 1990.

[17] Mitjagin B.S. Approximate dimension and bases in nuclear spaces. Russian Math Surveys 1961; 16: 59-127.

[18] Offin D. and Oskolkov K. A note on orthonormal polynomial bases and wavelets. Constr Approx 1993; 9: 319-325.

[19] Pompe W. Unconditional biorthogonal wavelet bases in $L^p(\mathbb{R}^d)$. Colloq Math 2002; 92: 19-34.

[20] Rivlin T.J. An introduction to the approximation of functions: Blaisdell Publishing Co 1969.

[21] Rivlin T.J. The Chebyshev Polynomials. New York: Wiley 1974.

[22] Singer I. Bases in Banach spaces. I. Berlin: Springer-Verlag 1970.

[23] Strömberg J.O. A modified Franklin system and higher-order spline systems on $\mathbb{R}^n$ as unconditional bases for Hardy spaces, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II. Chicago: Wadsworth 1983.

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