Generalized Statistical Convergence in Probabilistic Normed Spaces

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Abstract. In this paper we define the concepts of λ-statistical convergence and λ-statistically Cauchy in probabilistic normed space and prove some interesting results. Furthermore, we display an example such that our method of convergence is stronger than the usual convergence in probabilistic normed spaces.

Keywords and phrases: Statistical convergence; λ-statistical convergence; t-norm; probabilistic normed space.

1. Introduction and Background

The idea of statistical convergence was first introduced by Fast [2] and then studied by various authors, e.g. Šalát [11], Fridy [4], Connor [1] and many others and in normed spaces by Kolk [6]. The concept of statistical convergence for double and multiple sequences was studied by Mursaleen - Edely [10] and Moricz [8]. Recently Karakus [5] has studied the concept of statistical convergence in probabilistic normed spaces.

In this paper we will study the concepts of λ-statistical convergence and λ-statistical Cauchy in probabilistic normed spaces.

Firstly, we recall some definitions which form the background of the present work which are taken from Freedman, Sember and Raphael [3] and Steinhaus [14].

Definition 1.1. Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the *asymptotic density of* K denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n : |x_k - \ell| > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - \ell| > \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = \ell$.

For the following definition we refer to Mursaleen [9].

Definition 1.2. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 0.$$

Let $K \subseteq \mathbb{N}$. The number

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|$$

is said to be the λ-*density* of K.

If $\lambda_n = n$ for every n then every λ-density is reduced to asymptotic density.

A sequence $x = (x_k)$ is said to be λ-*statistically convergent* to ℓ if for every $\epsilon > 0$, the set $N(\epsilon)$ has λ-density zero, where

$$N(\epsilon) = \{k \in I_n : |x_k - \ell| > \epsilon\},$$

and $I_n = [n - \lambda_n + 1, n]$. In this case we write $st_\lambda\text{-}\lim x = \ell$

For the following concepts, we refer to Menger [7], Schweizer - Sklar ([12,13]).

Definition 1.3. A *triangular norm* (*t-norm*) is a continuous mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $([0,1], T)$ is an abelian monoid with unit one and for all $a, b, c \in [0,1]$:

(*i*) $T(c,d) \geq T(a,b)$ if $c \geq a$ and $d \geq b$;

Definition 1.4. A function $f : \Re \longrightarrow \Re_0^+$ is called a *distribution function* if it is non-decreasing and left-continuous with $\inf_{t \in \Re} f(t) = 0$ and $\sup_{t \in \Re} f(t) = 1$.
By D, we denote the set of all distribution functions.

Definition 1.5. Let X be a real linear space and $\nu : X \to D$. Then the *probabilistic norm* or ν-*norm* is a t-norm satisfying the following conditions:

(*i*) $\nu_x(0) = 0$;

(*ii*) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$;

(*iii*) $\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \Re \setminus \{0\}$;

(*iv*) $\nu_{x+y}(s+t) \geq T(\nu_x(s), \nu_y(t))$ for all $x, y \in X$ and $s, t \in \Re_0^+$.

where ν_x means $\nu(x)$ and $\nu_x(t)$ is the value of ν_x at $t \in \Re$.

Remark 1.1. We can say that t-norm is a binary operation $*$ given by $T(a,b) = a * b$.

Space $(X, \nu, *)$ is called a *probabilistic normed space* (*PN-space*), and by a PN-space X we mean the triplet $(X, \nu, *)$.

Definition 1.6. A sequence $x = (x_k)$ is said to be *convergent* to ℓ in PN-space X, that is, $x_k \xrightarrow{\nu} \ell$ if for every $\epsilon > 0$ and $\theta \in (0,1)$, there is a positive integer k_0 such that $\nu_{x_k-\ell}(\epsilon) > 1-\theta$ whenever $k \geq k_0$.
In this case we write $\nu\text{-}\lim x = \ell$.

Definition 1.7. A sequence $x = (x_k)$ is called *Cauchy sequence* in PN-space X, if for every $\epsilon > 0$ and $\theta \in (0,1)$, there is a positive integer k_0 such that $\nu_{x_k-x_j}(\epsilon) > 1-\theta$ for all $j, k \geq k_0$.

Definition 1.8. A sequence $x = (x_k)$ is said to be *bounded* in PN-space X, if there is $r \in \Re^+$ such that $\nu_x(r) > 1-\theta,\ 0 < \theta < 1$.
We denote by ℓ_∞^ν the space of bounded sequences in PN-space.

2. Main Results

First, we define the following concepts:

Definition 2.1. A sequence $x = (x_k)$ is said to be λ-*statistically convergent* to ℓ in PN-space X if for every $\epsilon > 0$ and $\theta \in (0,1)$,

$$\delta_\lambda(\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1-\theta\}) = 0$$

or equivalently

$$\delta_\lambda(\{k \in I_n : \nu_{x_k-\ell}(\epsilon) > 1-\theta\}) = 1$$

and we say that x is $\lambda(PN)$-*statistically convergent.* In this case we write $x_k \xrightarrow{\nu} \ell(S_\lambda)$ or $st_\lambda(PN)\text{-}\lim x = \ell$, and denote the set of all $\lambda(PN)$-statistically convergent sequences by $(S_\lambda)_\nu$.

Definition 2.2. A sequence $x = (x_k)$ is said to be λ-*statistically Cauchy* in PN-space X if for every $\epsilon > 0$ and $\theta \in (0,1)$, there exist a number $N = N(\epsilon)$ such that

$$\delta_\lambda(\{k \in I_n : \nu_{x_k-x_N}(\epsilon) \leq 1-\theta\}) = 0.$$

Theorem 2.1. If a sequence $x = (x_k)$ is λ-statistically convergent in PN-space X then $st_\lambda(PN)$-limit is unique.

Proof. Suppose that $st_\lambda(PN)\text{-}\lim x = \ell_1$ and $st_\lambda(PN)\text{-}\lim x = \ell_2$. Let $\epsilon > 0$ and $\theta > 0$. Choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) \geq 1-\theta$. Then we define the following sets as

$$K_1(\gamma, \epsilon) = \{k \in I_n : \nu_{x_k-\ell_1}(\epsilon) \leq 1-\gamma\},$$

$$K_2(\gamma, \epsilon) = \{k \in I_n : \nu_{x_k-\ell_2}(\epsilon) \leq 1-\gamma\}.$$

So that we have $\delta_\lambda(K_1(\gamma, \epsilon)) = 0$ and $\delta_\lambda(K_2(\gamma, \epsilon)) = 0$ for all $\epsilon > 0$. Now let

$$K_3(\gamma, \epsilon) = K_1(\gamma, \epsilon) \cap K_2(\gamma, \epsilon).$$

It follows that $\delta_\lambda(K_3(\gamma, \epsilon)) = 0$, which implies

$$\delta_\lambda(I\!N \setminus K_3(\gamma, \epsilon)) = 1.$$

If $k \in \mathbb{N} \setminus K_3(\gamma, \epsilon)$, we have

$$\nu_{\ell_1-\ell_2}(\epsilon) = \nu_{(\ell_1-x_k)+(x_k-\ell_2)}(\epsilon/2+\epsilon/2) \geq \nu_{x_k-\ell_1}(\epsilon/2)*\nu_{x_k-\ell_2}(\epsilon/2) > (1-\gamma)*(1-\gamma) \geq 1-\theta.$$

Since $\theta > 0$ was arbitrary, we get $\nu_{\ell_1-\ell_2}(\epsilon) = 1$ for all $\epsilon > 0$, which gives $\ell_1 = \ell_2$. Hence $st_\lambda(PN)$-limit is unique.

This completes the proof of the theorem.

Theorem 2.2. If ν- $\lim x = \ell$ then $st_\lambda(PN)$- $\lim x = \ell$. But converse does not hold.

Proof. Let ν- $\lim x = \ell$. Then for every $\theta \in (0,1)$ and $\epsilon > 0$, there is a number $k_0 \in \mathbb{N}$ such that $\nu_{x_k-\ell}(\epsilon) > 1-\theta$ for all $k \geq k_0$. Hence the set $\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1-\theta\}$ has natural density zero and hence

$$\delta_\lambda(\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1-\theta\}) = 0.$$

For converse, we consider the following example:

Example 2.1. Define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} k & ; \quad \text{for } \; n - [\sqrt{\lambda_n}] + 1 \leq k \leq n \\ 0 & ; \quad \text{otherwise} \end{cases} .$$

Let for $\epsilon > 0, 0 < \theta < 1$

$$K_n(\theta, \epsilon) = \{k \in I_n : \nu_{x_k}(\epsilon) \leq 1-\theta\}.$$

Then

$$\delta_\lambda(K_n(\theta,\epsilon)) = \nu\text{-}\lim \frac{\sqrt{\lambda_n}}{\lambda_n} = 0 \text{ as } n \longrightarrow \infty$$

implies that $x_k \xrightarrow{\nu} 0(S_\lambda)$, while it is obvious that $x_k \overset{\nu}{\not\longrightarrow} 0$.

This completes the proof of the theorem.

Theorem 2.3. Let $(X, \nu, *)$ be a PN-space and $x = (x_k)$ be a sequence. Then $st_\lambda(PN)$- $\lim x = \ell$ if and only if there exists a subset $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and ν- $\lim_{n\to\infty} x_{k_n} = \ell$.

Proof. Suppose that $st_\lambda(PN)$- $\lim x = \ell$. Then, for any $\epsilon > 0$ and $r \in \mathbb{N}$, let

$$K(r,\epsilon) = \left\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1 - \frac{1}{r}\right\},$$

and

$$M(r,\epsilon) = \left\{k \in I_n : \nu_{x_k-\ell}(\epsilon) > 1 - \frac{1}{r}\right\}.$$

Then $\delta_\lambda(K(r,\epsilon)) = 0$ and

$$M(1,\epsilon) \supset M(2,\epsilon) \supset ...M(i,\epsilon) \supset M(i+1,\epsilon) \supset ... \tag{2.3.1}$$

and

$$\delta_\lambda(M(r,\epsilon)) = 1, \; r = 1, 2, \cdots. \tag{2.3.2}$$

Now we have to show for $k \in M(r,\epsilon)$, $x = (x_k)$ is ν-convergent to ℓ. Suppose that the sequence $x = (x_k)$ is not ν-convergent to ℓ. Therefore there is $\theta > 0$ such that the set

$$\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1 - \theta\}$$

has infinitely many terms. Let

$$M(\theta,\epsilon) = \{k \in I_n : \nu_{x_k-\ell}(\epsilon) > 1 - \theta\} \text{ and } \theta > \frac{1}{r} \; (r = 1, 2, \cdots).$$

Then

$$\delta_\lambda(M(\theta,\epsilon)) = 0 \tag{2.3.3}$$

by (2.3.1) we have

$$M(r,\epsilon) \subset M(\theta,\epsilon).$$

Hence $\delta_\lambda(M(r,\epsilon)) = 0$ which contradicts (2.3.2). Therefore $x = (x_k)$ is ν-convergent to ℓ.

Conversely, suppose that there exists a subset $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and $\nu\text{-}\lim_{k \in K} x_k = \ell$. Then there exists $N \in \mathbb{N}$ such that for every $\theta \in (0,1)$ and $\epsilon > 0$

$$\nu_{x_k-\ell}(\epsilon) > 1 - \theta, \text{ for all } k \geq N.$$

Now

$$\begin{aligned} M(\theta,\epsilon) &= \{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1 - \theta\} \\ &\subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, \cdots\}. \end{aligned}$$

Therefore $\delta_\lambda(M(\theta,\epsilon)) \leq 1 - 1 = 0$. Hence $st_\lambda(PN)\text{-}\lim x = \ell$.

This completes the proof of the theorem.

Theorem 2.4. In PN-space X, x is λ-statistically convergent if and only if it is λ-statistically Cauchy.

Proof. Let x be λ-statistically convergent to ℓ in PN-space X, i.e., $st_\lambda(PN)\text{-}\lim x = \ell$. Then for every $\epsilon > 0$ and $\theta \in (0,1)$, we have

$$\delta_\lambda(\{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1 - \theta\}) = 0.$$

Choose a number $N = N(\epsilon)$ such that $\nu_{x_N-\ell}(\epsilon) \leq 1 - \theta$. Now let

$$A(\theta,\epsilon) = \{k \in I_n : \nu_{x_k-x_N}(\epsilon) \leq 1 - \theta\}$$

$$B(\theta,\epsilon) = \{k \in I_n : \nu_{x_k-\ell}(\epsilon) \leq 1 - \theta\}$$

$$C(\theta,\epsilon) = \{k = N \in I_n : \nu_{x_N-\ell}(\epsilon) \leq 1 - \theta\}.$$

Then $A(\theta,\epsilon) \subseteq B(\theta,\epsilon) \cup C(\theta,\epsilon)$ and therefore

$$\delta_\lambda(A(\theta,\epsilon)) \leq \delta_\lambda(B(\theta,\epsilon)) + \delta_\lambda(C(\theta,\epsilon)).$$

Hence x is λ-statistically Cauchy.

Conversely, let x be λ-statistically Cauchy but not λ-statistically convergent. Then there exists N such that the set $A(\theta, \epsilon)$ has natural density zero. Hence the set

$$E(\theta, \epsilon) = \{k \in I_n : \nu_{x_k - x_N}(\epsilon) > 1 - \theta\}$$

has nutural density 1, that is, $\delta_\lambda(E(\theta, \epsilon)) = 1$. In particular, we can write

$$\nu_{x_k - x_N} \leq 2\nu_{x_k - \ell} < \epsilon \tag{2.4.1}$$

if $\nu_{x_k - \ell} < \epsilon/2$. Since x is not λ-statistically convergent, the set $B(\theta, \epsilon)$ has natural density 1, i.e.,

$$\delta_\lambda(\{k \in I_n : \nu_{x_k - \ell}(\epsilon) > 1 - \theta\}) = 0.$$

Therefore by (2.4.1), we have

$$\delta_\lambda(\{k \in I_n : \nu_{x_k - x_N}(\epsilon) > 1 - \theta\}) = 0,$$

i.e, the set $A(\theta, \epsilon)$ has natural density 1 which is contradiction. Hence x is λ-statistically convergent.

This completes the proof of the theorem.

Conclusion. The idea of probabilistic norm is very useful to deal with the convergence problems of sequences of fuzzy real numbers.The main purpose of the present paper is to generalize the results on statistical convergence proved by Karakus [5]. We have introduced a wider class of λ-statistically convergent sequences in PN-space to deal with the sequences which are not covered in [5].

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Received: July 15, 2008 Revised: August 25, 2008 Accepted: October 27, 2008