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The Solubility of the Group of the Form *ABA*†

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Abstract: In this paper, we show that if A and B are abelian subgroups of coprime orders and A is self normalizing then $G = ABA$ possesses a normal complement to A. The proof presented here is direct and elementary.

1. INTRODUCTION

In [1] Gorenstein and Herstein have shown that if $G = ABA$ where A and B are both cyclic subgroups of relatively prime orders, then G is soluble, moreover the Sylow p-subgroups of G, for odd p, are abelian and the Sylow 2-subgroups of G are either abelian or isomorphic to the Quaternion group. Furthermore if $N_G(A) = A$ then G contains a normal complement to A. In general, Gorenstein [2] has proved that if $G = ABA$ where A and B are both cyclic subgroups and $N_G(A) = A$, then G is soluble. In this paper, we show that if A and B are abelian subgroups of coprime orders and A is self normalizing then $G = ABA$ possesses a normal complement to A. The proof presented here is direct and elementary.

2. PRELIMINARIES

Theorem 2.1 (H.Wielandt). *Let H be an abelian Hall subgroup of a group G. Then there is a normal complement to H in G if and only if no two distinct elements of H are conjugate in G.*

Proof. See [4, Corollary 10.18].

Theorem 2.2. *Let G be a group which possesses a nilpotent Hall π-subgroup H. Then every π-subgroup of G is contained in a conjugate of H. In particular, all Hall π-subgroups of G are conjugate.*

Proof. See [3, 9.1.10]

By using the Wieldant's Theorem, we can generalize the Frattini Argument and Burnside normal p-complement Theorem.

Proposition 2.3. *Let K be a normal subgroup of G and suppose that H is a nilpotent Hall π-subgroup of K then*

$$G = N_G(H)K.$$

Proof. let $g \in G$. Then H^g is a nilpotent Hall π-subgroup of K. By Theorem 2.2 H^g, H are conjugate in K so there exists $k \in K$ such that $H^{gk} = H$, thus $gk \in N_G(H)$. Hence $g \in N_G(H)K$.

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Proposition 2.4. *Let H be a nilpotent Hall π-subgroup of G then the following hold*

(1) *Any two elements of $Z(H)$ which are conjugate in G are conjugate in $N_G(H)$.*

(2) *If $H \leq Z(N_G(H))$ then H has a normal complement.*

Proof. (1) Choose x, x^g in $Z(H)$ where $g \in G$. Now H, $H^{g^{-1}} \leq C_G(x)$, so by Theorem 2.2 there exists $y \in C_G(x)$ such that $H^y = H^{g^{-1}}$. Therefore $H^{yg} = H$, thus $yg \in N_G(H)$ and $x^{yg} = x^g$.

(2) Choose x, x^g in $H = Z(H)$. By (1) there exists $y \in N_G(H)$ such that $x = x$. But $x \in Z(N_G(H))$, so $x^g = x$. Since no distinct elements of H are conjugate then by Theorem 2.1 H has normal complement.

3. *ABA*-GROUPS

Theorem 3.1. *Let G be a group that contains abelian subgroups A and B with the following properties*

(1) *$G = ABA$.*

(2) *A and B have coprime orders.*

(3) *A is its own normalizer.*

Then A is a Hall subgroup and G possesses a normal complement to A.

Proof. Assume the theorem is false. Suppose G is a counter example of minimal order.

Set

$$\pi = \{p \in \pi(A) : O_p(A) \text{ is not normal in } G\}$$

$$\sigma = \{p \in \pi(A) : O_p(A) \text{ is normal in } G\}.$$

It is obvious that $A = O_\pi(A) \times O_\sigma(A)$. For the sake of clarity, we break up the proof into a sequence of steps.

Step 1. *If* $A_0 \leq A$, then $N_G(A_0) = A(N_G(A_0) \cap B)A$.

Proof. It is clear.

Step 2. *$O_\pi(A)$ is a Hall Subgroup of G and G has a normal π-complement.*

Proof. Choose a Sylow p-subgroup P of G such that $O_p(A) \leq P$. By Step 1, we have $N_G(O_p(A)) = A(N_G(O_p(A)) \cap B)A$.

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Since $N_G(O_p(A)) < G$ and by the minimality of G, there exists $K \trianglelefteq N_G(O_p(A))$ such that $N_G(O_p(A)) = AK$ and $A \cap K = 1$.

Now A is a Hall subgroup of $N_G(O_p(A))$, so $O_p(A) \in Syl_p(N_G(O_p(A)))$. We have

$$N_P(O_p(A)) \leq O_p(A).$$

Therefore $N_P(O_p(A)) = O_p(A)$, thus $O_p(A) = P$. In Particular, $O_p(A) \in Syl_p(G)$. Now $O_\pi(A)$ is a Hall subgroup of G.

By Step 1,

$$N_G(O_\pi(A)) = AN_B(O_\pi(A))A.$$

Let $g \in N_B(O_\pi(A))$. Now $O_\sigma(A)$ is a normal subgroup of G, so we have

$$g \in N_G(A) = A.$$

Therefore

$$N_G(O_\pi(A)) = A.$$

Since $O_\pi(A) \leq Z(N_G(O_\pi(A))) = A$ and $O_\pi(A)$ is a Hall subgroup of G then by Proposition 2.4 (2) G has a normal complement to $O_\pi(A)$.

Step 3. $O_\sigma(A)$ is contained in $Z(G)$.

Proof. We have $O_\pi(A) \leq C_G(O_\sigma(A))$. Since $C_G(O_\sigma(A))$ is a normal subgroup of G and $O_\pi(A)$ is a nilpotent Hall subgroup of G then by Proposition 2.3, we have that

$$G = N_G(O_\pi(A))C_G(O_\sigma(A)).$$
$$= AC_G(O_\sigma(A)).$$
$$= C_G(O_\sigma(A)).$$

Hence $O_\sigma(A) \leq Z(G)$.

Step 4. Every σ-element of G is contained in $O_\sigma(A)$.

Proof. Let g be a σ-element of G. Then

$$g = a_\pi a_\sigma a_\pi a'_\sigma$$

for some $a_\pi, a'_\pi, a'_\sigma \in A$ and $b \in B$. Since $O_\sigma(A) \leq Z(G)$ then $g = a_\pi a_\sigma a'_\sigma b a'_\pi$. By Step 3, we have

$$G = O_\pi(A)O_{\pi'}(G).$$

So $a_\sigma a'_\sigma b \in O_{\pi'}(G)$. Set

$$\bar{G} = G/O_{\pi'}(G).$$

Therefore

$$\bar{g} = \overline{a_\pi a_\pi}.$$

But $\bar{g}$ is σ-element, so

$$a_\pi a'_\pi \in O_{\pi'}(G) \cap O_\pi(A) = 1.$$

Thus $a_\pi = a'^{-1}_\pi$. Hence $g = \left(a_\sigma b a'_\sigma\right)^{a_\pi^{-1}}$ Since the orders of A and B are relatively prime and by Step 3 we deduce that $b = 1$. Hence

$$g = a_\pi a_\sigma a_\pi^{-1} = a_\sigma a'_\sigma.$$

Step 5. $O_\sigma(A)$ *is a Hall subgroup and G has a normal σ-complement*.

Proof. By Steps 4 and 5, $O_\sigma(A)$ is a Hall normal abelian subgroup of G. By Proposition 2.4, $O_\sigma(A)$ has a normal σ-complement in G.

Finally, $O_\pi(A)$ and $O_\sigma(A)$ are Hall subgroups and have normal complements in G. By taking the intersection of the two complements, we get a normal complement to A in G.

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