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On the Mittag-Leffler Property

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Abstract: Let C be a category with strong monomorphic strong coimages, that is, every morphism f of C factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism and this factorization is universal. We define the notion of strong Mittag-Leffler property in pro-C. We show that if $f : X \to Y$ is a level morphism in pro-C such that $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, then X has the strong Mittag-Leffler property provided f is an isomorphism. Also, if $f : X \to Y$ is a strong epimorphism of pro-C and X has the strong Mittag-Leffler property, we show that Y has the strong Mittag-Leffler property. Moreover, we show that this property is invariant of isomorphisms of pro-C.

Keywords: Pro-categories, strong Mittag-Leffler property, categories with strong monomorphic strong coimages.
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1. INTRODUCTION

In [1], J. Dydak and F. R. Ruiz del Portal generalized the notion of Mittag-Leffler property to arbitrary balanced categories with epimorphic images. They obtained several results.

In [2], the author defined the notion of categories with strong monomorphic strong coimages. C is a category with strong monomorphic strong coimages if every morphism f of C factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism and this factorization is universal among such factorization. In this paper, we define the notion of strong Mittag-Leffler property in pro-C. We show that if $f : X \to Y$ is a level morphism in pro-C such that $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, then X has the strong Mittag-Leffler property provided f is an isomorphism (Theorem 3.2). Also, if $f : X \to Y$ is a strong epimorphism of pro-C and X has the strong Mittag-Leffler property, we show that Y has the strong Mittag-Leffler property (Corollary 3.6). Moreover, we show that this property is invariant of isomorphisms of pro-C (Corollary 3.5).

2. PRELIMINARIES

First we recall some basic facts about pro-categories. The main reference is [3] and for more details see [4].

Let C be an arbitrary category. Loosely speaking, the pro-category pro-C of C is the universal category with inverse limits containing C as a full subcategory. An object of pro-C is an inverse system in C, denoted by $X = \left(X_\alpha, p_\alpha^\beta, A\right)$, consisting of a directed set A, called the *index set*, of C objects X_α for each $\alpha \in A$, called the *terms* of X and of C morphisms $p_\alpha^\beta : X_\beta \to X_\alpha$ for each related pair $\alpha < \beta$, called the *bonding morphisms* of X. A morphism of two objects $f : X = \left(X_\alpha, p_\alpha^\beta, A\right) \to Y = \left(Y_{\alpha'}, p_{\alpha'}^{\beta'}, A'\right)$ consists of a function $\varphi : A' \to A$ and of morphisms $f_{\alpha'} : X_{\varphi(\alpha')} \to Y_{\alpha'}$ in C one for each $\alpha' \in A'$ such that whenever $\alpha' < \beta'$, then there is $\gamma \in A$, $\gamma > \varphi(\alpha'), \varphi(\beta')$ for which $f_{\alpha'} p_{\phi(\alpha')}^{\gamma} = p_{\alpha'}^{\beta'} f_{\beta'} p_{\phi(\beta')}^{\gamma}$. From now onward, the index set A of an object X of pro-C will be denoted by $I(X)$ and the bonding morphisms by $p(X)_\alpha^\beta$ for each $\alpha < \beta$.

If P is an object of C and X is an object of pro-C, then a morphism $f : X \to P$ in pro-C is the direct limit of Mor $\left(X_\alpha, P\right)$, $\alpha \in I(X)$ and so f can be represented by $g : X_\alpha \to P$. Note that the morphism from X to X_α represented by the identity $X_\alpha \to X_\alpha$ is called the *projection morphism* and denoted by $p(X)_\alpha$.

If X and Y are two objects in pro-C with identical index sets, then a morphism $f : X \to Y$ is called a *level morphism* if for each $\alpha < \beta$, the following diagram commutes.

$$
\begin{tikzcd}
X_\beta \arrow[r, "f_\beta"] \arrow[d, "p(X)_\alpha^\beta"'] & Y_\beta \arrow[d, "p(Y)_\alpha^\beta"] \\
X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha
\end{tikzcd}
$$

Theorem 2.1. For any morphism $f : X \to Y$ of pro-C there exists a level morphism $f' = X' \to Y'$ and isomorphisms $i : X \to X'$, $j : Y' \to Y$ such that $f = j \circ f' \circ i$ and $I(X')$ is a

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cofinite directed set. Moreover, the bonding morphisms of X' (respectively, Y') are chosen from the set of bonding morphisms of X (respectively, Y).

Recall that a morphism $f : X \to Y$ of C is called a *monomorphism* if $f \circ g = f \circ h$ implies $g = h$ for any two morphisms $g, h : Z \to X$. A morphism $f : X \to Y$ of C is called an *epimorphism* if $g \circ f = h \circ f$ implies $g = h$ for any two morphisms $g, h : Y \to Z$.

If f is a morphism of C, then its domain will be denoted by $D(f)$ and its range will be denoted by $R(f)$. Hence, $f : D(f) \to R(f)$.

Next, we recall definitions of strong monomorphism and strong epimorphism and state some of their basic results obtained. The main reference is [3].

Definition 2.2. A morphism $f : X \to Y$ in pro-C is called a *strong monomorphism* (*strong epimorphism,* respectively) if for every commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ P \arrow[r, "g"] & Q \end{tikzcd}$$

with P, Q objects in C, there is a morphism $h : Y \to P$ such that $h \circ f = a$ ($g \circ h = b$, respectively).

Note that if X and Y are objects of C, then $f : X \to Y$ is a strong monomorphism (strong epimorphism, respectively) if and only if f has a left inverse (a right inverse, respectively).

The following result presents the relation between monomorphisms and strong monomorphisms and between epimorphisms and strong epimorphisms.

Lemma 2.3. If f is a strong monomorphism (strong epimorphism, respectively) of pro-C, then f is a monomorphism (epimorphism, respectively) of pro-C.

Lemma 2.4. If $g \circ f$ is a strong monomorphism (strong epimorphism, respectively), then f is a strong monomorphism (g is a strong epimorphism, respectively).

The following theorem is a characterization of isomorphisms in pro-C.

Theorem 2.5. Let $f : X \to Y$ be a morphism in pro-C. The following statements are equivalent.

(i) f is an isomorphism.

(ii) f is a strong monomorphism and an epimorphism.

Theorem 2.6. Suppose that,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ Z \arrow[r, "g"] & T \end{tikzcd}$$

is a commutative diagram in pro-C. If f is an epimorphism and g is a strong monomorphism, then there is a unique morphism $h : Y \to Z$ such that $h \circ f = a$ and $g \circ h = b$.

In [2], the author defined the notion of categories with strong monomorphic strong coimages as follows.

Definition 2.7. C is a *category with strong coimages* if every morphism f of C factors as $f = u \circ g$ so that g is a strong epimorphism and this factorization is universal among such factorization, that is, given another factorization $f = v \circ h$ with h being a strong epimorphism there is $t : D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$.

Definition 2.8. C is a *category with strong monomorphic strong coimages* if it is a category with strong coimages and u in the universal factorization $f = u \circ g$ is a strong monomorphism.

We will need the following results which are proved in [2]. For this paper to be self-contained we include the results with their proofs.

Lemma 2.9. Let C be any category. Then the following conditions on C are equivalent:

(i) C is a category with strong monomorphic strong coimages.

(ii) Any morphism f factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism. Given another factorization $f = v \circ h$ with h being a strong epimorphism and v a strong monomorphism, there is an isomorphism $t : D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$.

Proof. (i) $\Longrightarrow$ (ii) Any morphism f factors as $f = u \circ g$ such that g is a strong epimorphism and u is a strong monomorphism. Assume that f have another factorization $f = v \circ h$ with h being a strong epimorphism and v a strong monomorphism, there is $t : D(v) \to D(u)$ such that $t \circ h = g$ and $u \circ t = v$. Since g is a strong epimorphism, t is a strong epimorphism and hence an epimorphism by Lemma 2.3. Since v is a strong monomorphism, t is a strong monomorphism. Thus, t is an isomorphism by Theorem 2.5.

(ii) $\Longrightarrow$ (i) Any morphism f factors as $f = u \circ g$ such that g is a strong epimorphism and u is a strong monomorphism. Now we show that f is universal. Assume that $f = v \circ h$ is another factorization with h a strong epimorphism. The morphism v can be factored as $v = b \circ a$ where a is a strong epimorphism and b is a strong monomorphism, there is an isomorphism c such that $c \circ a \circ h = g$ and $u \circ c = b$. Let $t = c \circ a$. Hence, the result holds.

By Lemma 2.9, any morphism f of a category C with strong monomorphic strong coimages has a unique, up to isomorphism, factorization into a composition $f = u \circ g$ where g is a strong epimorphism and u is a strong monomorphism. We write this unique factorization as $f = SM(f) \circ SE(f)$.

The range of $SE(f)$ will be called the *image* of f and denoted by $\mathrm{im}(f)$.

Theorem 2.10. Let C be a category with strong monomorphic strong coimages. Let $f : X \to Y$ be a level morphism in pro-C. Then there exist level morphisms $g : X \to Z$ and $h : Z \to Y$ such that $g_\alpha = SE(f_\alpha)$, $h_\alpha = SM(f_\alpha)$ for each $\alpha \in I(X)$ and $f = h \circ g$. Moreover, if f is an isomorphism in pro-C, then both h and g are isomorphisms.

Proof. First note that we have $f_\alpha \circ p(X)_\alpha^\beta = p(Y)_\alpha^\beta \circ f_\beta$ for $\beta > \alpha$. Since C is a category with strong monomorphic strong coimages, we have,

$$SM(f_\alpha) \circ SE(f_\alpha) \circ p(X)_\alpha^\beta = p(Y)_\alpha^\beta \circ SM(f_\beta) \circ SE(f_\beta).$$

This implies that the following diagram,

$$\begin{tikzcd} X_\beta \arrow[r, "SE(f_\beta)"] \arrow[d, "SE(f_\alpha)\circ p(X)_\alpha^\beta"'] & \mathrm{im}(f_\beta) \arrow[d, "p(Y)_\alpha^\beta \circ SM(f_\beta)"] \\ \mathrm{im}(f_\alpha) \arrow[r, "SM(f_\alpha)"] & Y_\alpha \end{tikzcd}$$

is commutative in pro-C with $SE(f_\beta)$ a strong epimorphism and $SM(f_\alpha)$ a strong monomorphism. Thus, there is a unique morphism $v : \mathrm{im}(f_\beta) \to \mathrm{im}(f_\alpha)$ by Theorem 2.6. Put $Z_\alpha = \mathrm{im}(f_\alpha)$ and $p(Z)_\alpha^\beta = v$. Thus, Z is an object of pro-C. Further, put $g_\alpha = SE(f_\alpha)$ and $h_\alpha = SM(f_\alpha)$ for each $\alpha \in I(X)$. Hence, $f = h \circ g$. Obviously, g is a strong epimorphism and h is a strong monomorphism. Note that if f is an isomorphism, hence a strong monomorphism by Theorem 2.5, then g is a strong monomorphism by Lemma 2.4 and thus it is an isomorphism by Theorem 2.5. Also, if f is an isomorphism, then h is a strong epimorphism and thus it is an isomorphism. Hence, the result holds.

We denote g by $SE(f)$ and h by $SM(f)$. Therefore, we write f as $SM(f) \circ SE(f)$.

3. STRONG MITTAG-LEFFLER PROPERTY

Definition 3.1. Let C be a category with strong monomorphic strong coimages. An object X of pro-C has the *strong Mittag-Leffler property* if for every $\alpha \in I(X)$ there is $\beta > \alpha$ such that $SE\left(p(X)_\alpha^\beta\right) \circ p(X)_\beta^\gamma$ is a strong epimorphism for all $\gamma > \beta$.

We write $SE\left(p(X)_\alpha^\beta\right) \circ p(X)_\beta^\gamma = SE\left(p(X)_\alpha^\gamma\right)$.

Note that if C is a category with strong monomorphic strong coimages and X is an object of pro-C such that $p(X)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, then X has the strong Mittag-leffler property. Indeed, $SE\left(p(X)_\alpha^\beta\right) \circ p(X)_\beta^\gamma$ is a strong epimorphism for all $\gamma > \beta$.

Theorem 3.2. Let C be a category with strong monomorphic strong coimages. Let $f : X \to Y$ be a level morphism in pro-C such that $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$. If f is an isomorphism in pro-C, then X has the strong Mittag-Leffler property.

Proof. Assume that f is an isomorphism in pro-C. Then for each α, there exists $\beta > \alpha$ and a morphism $r : Y_\beta \to X_\alpha$ in C such that the following diagram commutes.

$$\begin{tikzcd} X_\beta \arrow[r, "f_\beta"] \arrow[d, "p(X)_\alpha^\beta"'] & Y_\beta \arrow[d, "p(Y)_\alpha^\beta"] \arrow[dl, "r"'] \\ X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \end{tikzcd}$$

That is, $p(X)_\alpha^\beta = r \circ f_\beta$ and $p(Y)_\alpha^\beta = f_\alpha \circ r$. Since $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, we have f_α is a strong epimorphism by Lemma 2.4. For any $\gamma > \beta$, we have,

$$r \circ p(Y)_\beta^\gamma \circ f_\gamma = r \circ f_\beta \circ p(X)_\beta^\gamma = p(X)_\alpha^\beta \circ p(X)_\beta^\gamma = p(X)_\alpha^\gamma$$

Note that,

$$r \circ p(Y)_\beta^\gamma \circ f_\gamma = SM(r) \circ SE(r) \circ p(Y)_\beta^\gamma \circ f_\gamma$$

Therefore,

$$\begin{aligned} SE\left(p(X)_\alpha^\gamma\right) &= SE(r) \circ p(Y)_\beta^\gamma \circ f_\gamma = SE(r) \circ f_\beta \circ p(X)_\beta^\gamma \\ &= SE\left(p(X)_\alpha^\beta\right) \circ p(X)_\beta^\gamma \end{aligned}$$

Hence, X has the strong Mittag-Leffler property.

We need the following special case of Theorem 2.10.

Lemma 3.3. Let C be a category with strong monomorphic strong coimages. If Y is an object of pro-C and $e : I(Y) \to I(Y)$ is an increasing function, then there exist level morphisms $g : X \to Z$ and $h : Z \to Y$ such that $g_\alpha = SE\left(p(Y)_\alpha^{e(\alpha)}\right)$ and $h_\alpha = SM\left(p(Y)_\alpha^{e(\alpha)}\right)$ for all $\alpha \in I(Y)$. Moreover, both g and h are isomorphisms.

Proof. Let Y be an object of pro-C and $e : I(Y) \to I(Y)$ be an increasing function. Let $X_\alpha = Y_{e(\alpha)}$ and $p(X)_\alpha^\beta = p(Y)_{e(\alpha)}^{e(\beta)}$ for all $\alpha, \beta \in I(Y)$ with $\beta > \alpha$. Let $f : X \to Y$ be defined by $f_\alpha = p(Y)_\alpha^{e(\alpha)}$ for each $\alpha \in I(Y)$. Now if we continue as in the proof of Theorem 2.10, then the result holds.

Theorem 3.4. Let C be a category with strong monomorphic strong coimages. If Y has the strong Mittag-Leffler property and $I(Y)$ is cofinite, then there is a level morphism $h : Z \to Y$ in pro-C such that $p(Z)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, each h_α is a strong monomorphism and h is an isomorphism in pro-C.

Proof. Suppose that Y have the strong Mittag-Leffler property and $I(Y)$ is cofinite. Then for every $\alpha \in I(Y)$, there is $\beta > \alpha$ such that $SE\left(p(Y)_\alpha^\beta\right) \circ p(Y)_\beta^\gamma = SE\left(p(Y)_\alpha^\gamma\right)$ is a strong epimorphism for all $\gamma > \beta$. If we switch from β to $\omega < \gamma$, then we have,

$$SE\left(p(Y)_\alpha^\omega\right) \circ p(Y)_\omega^\gamma = SE\left(p(Y)_\alpha^\beta\right) \circ p(Y)_\beta^\omega \circ p(Y)_\omega^\gamma$$
$$= SE\left(p(Y)_\alpha^\beta\right) \circ p(Y)_\beta^\gamma$$

It is a strong epimorphism. But,

$$SM\left(p(Y)_\alpha^\omega\right) \circ SE\left(p(Y)_\alpha^\omega\right) \circ p(Y)_\omega^\gamma = p(Y)_\alpha^\omega \circ p(Y)_\omega^\gamma = p(Y)_\alpha^\gamma$$

Thus, $SE\left(p(Y)_\alpha^\omega\right) \circ p(Y)_\omega^\gamma = SE\left(p(Y)_\alpha^\gamma\right)$ is a strong epimorphism for $\gamma > \omega$. Therefore, by induction on the number of predecessors $n(\alpha)$ of $\alpha \in I(Y)$, we can construct an increasing function $e : I(Y) \to I(Y)$ such that $SE\left(p(Y)_\alpha^{e(\alpha)}\right) \circ p(Y)_{e(\alpha)}^\gamma = SE\left(p(Y)_\alpha^\gamma\right)$ is a strong epimorphism for all $\gamma > e(\alpha)$. Using Lemma 3.3, there exist level morphisms $g : X \to Z$ and $h : Z \to Y$ such that $g_\alpha = SE\left(p(Y)_\alpha^{e(\alpha)}\right)$ and $h_\alpha = SM\left(p(Y)_\alpha^{e(\alpha)}\right)$ for all $\alpha \in I(Y)$. h is an isomorphism. Note that $p(Z)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$ since,

$p(Z)_\alpha^\beta \circ SE\left(p(Y)_\beta^{e(\beta)}\right) = SE\left(p(Y)_\alpha^{e(\alpha)}\right) \circ p(Y)_{e(\alpha)}^{e(\beta)}$ is a strong epimorphism. Hence, the theorem is proved.

Corollary 3.5. Let C be a category with strong monomorphic strong coimages. If X is isomorphic to Y in pro-C and Y has the strong Mittag-Leffler property, then X has the strong Mittag-Leffler property.

Proof. Assume that $f : X \to Y$ is an isomorphism in pro-C and Y have the strong Mittag-Leffler property. Using Theorem 2.1, we can see that Y' has the strong Mittag-Leffler property if and only if Y has the strong Mittag-Leffler property since the bonding morphisms of Y' are chosen from the bonding morphisms of Y. Therefore, we may assume that $p(Y')_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$ by Theorem 3.4. There exists a level isomorphism $f' : X' \to Y'$ by Theorem 2.1. Thus, X' has the strong Mittag-Leffler property by Theorem 3.2. Hence, X has the strong Mittag-Leffler property.

Corollary 3.6. Let C be a category with strong monomorphic strong coimages. If $f : X \to Y$ is a strong epimorphism of pro-C and X has the strong Mittag-Leffler property, then Y has the strong Mittag-Leffler property.

Proof. Assume that $f : X \to Y$ is a strong epimorphism of pro-C and X have the strong Mittag-Leffler property. Using Theorem 3.4, we may assume that f is a level morphism of pro-C such that each $p(X)_\alpha^\beta$ is a strong epimorphism. Note that there exist level morphisms $SE(f) : X \to Z$ and $SM(f) : Z \to Y$ such that each $SE(f)_\alpha$ is a strong epimorphism and each $SM(f)_\alpha$ is a strong monomorphism and $f = SM(f) \circ SE(f)$ by Theorem 2.10. Since $SE(f)_\alpha \circ p(X)_\alpha^\beta = p(Z)_\alpha^\beta \circ SE(f)_\beta$ is a strong epimorphism, we have $p(Z)_\alpha^\beta$ is a strong epimorphism by Lemma 2.4. Since f is a strong epimorphism, we have $SM(f)$ is a strong epimorphism and thus it is an isomorphism by Theorem 2.5. Hence, Y has the strong Mitttag-Leffler property by Theorem 3.2.

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