The Open Mechanics Journal, 2008, *2*, 73-76

Small−Strain Hypoplasticity and Polyelastic Materials

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Abstract: This papers addresses the behaviour of hypoplastic equations near the reference configuration. By an approximation process, a general small–strain hypoplastic equation is derived. Two particular equations are examined. It is shown that they can represent solids exhibiting different Young's moduli in compression and in extension.

1. INTRODUCTION

Hypoplasticity is a relatively young theory of the mechanical behaviour of soils; it appeared about thirty years ago as a generalization of hypoelasticity. Since then many hypoplastic models have been proposed and applied to different types of soils in situations of interest in engineering. In references [1-11], among others, the reader can have a broad idea of the main research lines in the field.

The objective of this paper is to start the study of the behaviour of hypoplastic equations near the reference configuration, an aspect not yet investigated. This may be an interesting point as far as small–strain problems are concerned. It must be clearly stated that hypoplasticity is applicable to deformations of any magnitude, including situations of failure; however, the practical importance and the greater simplicity of small–strain problems suggests a study of this kind.

After a presentation of the fundamentals of hypoplasticity, we will derive, by an approximation process, the hypoplastic equation for deformations in the vicinity of the reference configuration. Two particular equations, special cases of the obtained general small–strain hypoplastic equation, will receive attention in the rest of the paper. One of them is the equation of the so–called bielastic material [12, 13]; the other one is new. In these equations the Cauchy stress $\mathbf{T}$ is related to the infinitesimal strain tensor $\mathbf{E}$ through a non-linear positively homogeneous function of degree one. It will be shown that there are sets of strain tensors in which the materials defined by these equations behave as a linear elastic material, in a certain sense.

2. DEFINITION AND GENERAL PROPERTIES OF HYPOPLASTICITY

In this section we will review fundamental notions in hypoplasticity (concepts and the standard notation of modern continuum mechanics will be employed) [14, 15].

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Consider a continuously differentiable function $\mathbf{F}:R\rightarrow Lin^{+}$, delivering, at any instant t, the deformation gradient $\mathbf{F}(t)$ at a material point P of a body (R is the set of all real numbers and Lin^{+} is the set of all second order tensors with positive determinant). The stretching tensor and the spin tensor at P, at each instant, are defined, respectively, by $\mathbf{D}(t) = sym(\dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t))$ and $\mathbf{W}(t) = skw(\dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t))$ (sym and skw refer to the symmetric and skew-symmetric parts of a tensor, and the superposed dot indicates time derivative). Suppose in addition that, at t=0, the Cauchy stress at P is $\mathbf{T}_0$.

For hypoplastic materials, the corresponding Cauchy stress $\mathbf{T}(t)$, at P, at each instant t, is given by the solution of an ordinary differential equation:

$$\dot{\mathbf{T}} = \mathbf{h}(\mathbf{T},\mathbf{D}) - \mathbf{TW} + \mathbf{WT} \tag{1}$$

with the initial condition $\mathbf{T}(0)=\mathbf{T}_0$. The function $\mathbf{h}:Sym \times Sym \rightarrow Sym$ has the following properties (Sym is the set of all symmetric second–order tensors):

a) $\mathbf{h}$ is isotropic.

b) For each $\mathbf{T}$, the function $\mathbf{D} \mapsto \mathbf{h}(\mathbf{T},\mathbf{D})$ is positively homogeneous of degree one, *i.e.*,

   b.1) $\mathbf{h}(\mathbf{T}, a\mathbf{D})=a\mathbf{h}(\mathbf{T}, \mathbf{D})$, for any $\mathbf{D}\in Sym$ and for any $a\geq 0$, $a\in R$.

   b.2) $\mathbf{h}(\mathbf{T}, a\mathbf{D})\neq a\mathbf{h}(\mathbf{T}, \mathbf{D})$, for any $\mathbf{D}\neq\mathbf{0}$ in Sym and for any $a<0$, $a\in R$.

Therefore $\mathbf{h}$ is non-linear in $\mathbf{D}$, in contrast with hypoelasticity [14].

c) $\mathbf{h}$ has such smoothness properties so as to ensure existence and uniqueness of solution of initial value problems.

Usually the hypoplastic equation is written in a more compact form with the introduction of $\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{WT} + \mathbf{TW}$ (the Jaumann derivative or corotational rate of $\mathbf{T}$, [14]):

$$\overset{\circ}{\mathbf{T}} = \mathbf{h}(\mathbf{T},\mathbf{D}) \tag{2}$$

From the above properties of $\mathbf{h}$ we can derive important mechanical characteristics of the hypoplastic material. From

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property *a* we conclude that the constitutive equation of hypoplasticity obeys the principle of material frame-indifference [14]; it is thus an appropriate theory for finite deformations. Part 1 of property *b* tells that the behaviour of a hypoplastic material is rate-independent and part 2 shows that the hypoplastic equation is able to represent plastic (irreversible) deformation, since $\mathbf{h}(\mathbf{T},-\mathbf{D})\neq-\mathbf{h}(\mathbf{T},\mathbf{D})$.

In most hypoplastic models, the function **h** has the form:

$$\mathbf{h}(\mathbf{T},\mathbf{D}) = \mathbf{L}(\mathbf{T})(\mathbf{D}) + \mathbf{N}(\mathbf{T})\|\mathbf{D}\| \quad (3)$$

where, $\|\mathbf{D}\| = \sqrt{tr\mathbf{D}^2}$ is the Euclidean norm of **D** (*tr* indicates the trace function), and the functions **L** and **N** associate to each $\mathbf{T}\in Sym$, the fourth-order tensor $\mathbf{L}(\mathbf{T})$ and the symmetric second-order tensor $\mathbf{N}(\mathbf{T})$, respectively.

3. THE HYPOPLASTIC EQUATION NEAR THE REFERENCE CONFIGURATION

With the aim of studying the response of hypoplastic equations near the reference configuration (at t=0), we write

$$\mathbf{T}(t) = \mathbf{T}_0 + \dot{\mathbf{T}}(0)t + o(t), \quad (4)$$

as $t\rightarrow 0$.

From equation (1): $\dot{\mathbf{T}}(0) = \mathbf{h}(\mathbf{T}_0,\mathbf{D}_0) + \mathbf{W}_0\mathbf{T}_0 - \mathbf{T}_0\mathbf{W}_0$ (the subscript "0" refers to values calculated at t=0). Taking into account that $\mathbf{D}_0 = \dot{\mathbf{E}}_0$ e $\mathbf{W}_0 = \dot{\mathbf{R}}_0^*$ (**E** is the infinitesimal strain tensor and **R*** is the infinitesimal rotation tensor), we get

$$\mathbf{T}(t)=\mathbf{T}_0+\mathbf{h}(\mathbf{T}_0,\mathbf{E}(t))+\mathbf{R}^*(t)\mathbf{T}_0-\mathbf{T}_0\mathbf{R}^*(t)+o(t). \quad (5)$$

Finally, if we neglect $o(t)$, an approximate equation for small strains from the reference configuration is obtained:

$$\mathbf{T}=\mathbf{T}_0+\mathbf{h}(\mathbf{T}_0,\mathbf{E})+\mathbf{R}^*\mathbf{T}_0-\mathbf{T}_0\mathbf{R}^*. \quad (6)$$

As **h** is hypoplastic, a non-linear behaviour is represented. Note that equation (6) reduces to $\mathbf{T}=\mathbf{T}_0+\mathbf{h}(\mathbf{T}_0,\mathbf{E})$ when $\mathbf{T}_0$ is spherical.

In the next section we will restrict our attention to equations of the form

$$\mathbf{T}=\varphi(\mathbf{E}), \quad (7)$$

obtained from equation (6) with $\mathbf{T}_0=\mathbf{0}$ and $\varphi(\mathbf{E})=\mathbf{h}(\mathbf{0},\mathbf{E})$. Clearly, φ is a non-linear positively homogeneous function of degree 1.

Let us now consider a set of strain tensors in which the following properties hold for a material defined by equation (7): for any $\mathbf{E}_1$, $\mathbf{E}_2$ in the set, and for any $\beta\geq 0$; 1) $\mathbf{E}_1+\mathbf{E}_2$ and $\beta\mathbf{E}_1$ are in the set, 2) $\varphi(\mathbf{E}_1+\mathbf{E}_2)=\varphi(\mathbf{E}_1)+\varphi(\mathbf{E}_2)$ and $\varphi(\beta\mathbf{E}_1)=\beta\varphi(\mathbf{E}_1)$. In this case we say that the material behaves as a linear elastic one in that set. It is easy to conclude that a material defined by equation (7) behaves as a linear elastic one in any set composed by the positive multiples of a given strain tensor; in general, in each different set of this kind, the material may behave as a different linear elastic material. Hence, it can be called polyelastic.

In particular, these materials exhibit different linear elastic behaviour in compression and extension. This phenomenon can be seen, for instance, in small–strain experiments with soils and some metals [16, 17]. Since, for $\beta\geq 0$, $\varphi(\beta\mathbf{E})=\beta\varphi(\mathbf{E})$ and, for $\mathbf{E}\neq\mathbf{0}$, $\varphi(-\mathbf{E})\neq-\varphi(\mathbf{E})$, then a linear stress-strain curve is obtained, for example, in a uniaxial compression or extension test, but the stiffness is different in compression and in extension.

4. TWO PARTICULAR EQUATIONS

In the rest of the paper we will investigate two particular instances of equation (7). In both equations $\varphi(\mathbf{E})$ has the form $\varphi(\mathbf{E})=\mathbf{f}(\mathbf{E})+g(\mathbf{E})\mathbf{I}$, where $\mathbf{f}(\mathbf{E})=a\,tr(\mathbf{E})\mathbf{I}+b\mathbf{E}$ (**I** is the second–order identity tensor, and *a* and *b* are constants) and *g* is a scalar-valued non-linear positively homogeneous function. Thus, the materials to be introduced below are defined by equations of the form:

$$\mathbf{T}=\mathbf{f}(\mathbf{E})+g(\mathbf{E})\mathbf{I}, \quad (8)$$

Firstly, it is important to remark that the deviatoric parts of **T** and **E** (indicated below by the subscript "d") are related here as in linear elasticity:

$$\mathbf{T}_d=b\mathbf{E}_d \quad (9)$$

(the function *g* has no influence), so *b* can be identified with twice the classic shear modulus. On the other hand, the spherical part of **T**, symbolized by $\mathbf{T}_s$, depends on $g(\mathbf{E})$:

$$\mathbf{T}_s=[(a+b/3)tr\mathbf{E}+g(\mathbf{E})]\mathbf{I}. \quad (10)$$

Two particular cases of equation (8), obtained by special choices of the function *g*, namely, $g_X(\mathbf{E}) = c\|\mathbf{E}\|$ and $g_Y(\mathbf{E}) = c|tr\mathbf{E}|$, will receive attention in the rest of the paper:

$$\mathbf{T} = [a\,tr(\mathbf{E}) + c\|\mathbf{E}\|]\mathbf{I} + b\mathbf{E}, \quad (11)$$

$$\mathbf{T} = [a\,tr(\mathbf{E}) + c|tr\mathbf{E}|]\mathbf{I} + b\mathbf{E}, \quad (12)$$

where *c* is a material constant, $\|\mathbf{E}\| = \sqrt{tr\mathbf{E}^2}$ is the Euclidean norm of **E** and $|tr\mathbf{E}|$ is the absolute value of $tr\mathbf{E}$. The materials defined by equations (11) and (12) will be called materials X and Y, respectively.

We can deduce from equation (12) that in the sets $S^n=\{\mathbf{E}\ /\ tr\mathbf{E}\leq 0\}$ and $S^p=\{\mathbf{E}\ /\ tr\mathbf{E}\geq 0\}$, material Y behaves as linear elastic materials (in the sense explained in the last section) with Lamé moduli $\lambda=a-c$ and $\lambda=a+c$, respectively, and $\mu=b/2$ in both. This is the so–called bielastic material [12, 13].

As for material X, given $r>0$, consider the conical surfaces in strain-space $C_r^n=\{\mathbf{E}\ /\|\mathbf{E}\| = r|tr\mathbf{E}|$ and $tr\mathbf{E}\leq 0\}$ and $C_r^p=\{\mathbf{E}\ /\|\mathbf{E}\| = r|tr\mathbf{E}|$ and $tr\mathbf{E}\geq 0\}$. In C_r^n and in C_r^p material X does not behave as a linear elastic material, but the stress-strain relation is that of a linear elastic material with $\lambda=a-cr$ and $\lambda=a+cr$, respectively, and $\mu=b/2$ in both.

5. SOME SIMPLE DEFORMATIONS RESTRICTIONS ON MATERIAL CONSTANTS

In order to study some simple deformations, we employ in this section a cartesian coordinate system, associated to an orthonormal basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$). The coordinates (x_1, x_2, x_3) identifies the position occupied in the deformed configuration by the particle that in the reference configuration was at the position (X_1, X_2, X_3).

a) Pure shear

Consider the pure shear defined by $x_1=X_1+\gamma X_2$, $x_2=X_2$, $x_3=X_3$, where γ is the shear strain. From equation (8), we have the following stress matrix:

$$[\mathbf{T}]=\begin{bmatrix} g(\mathbf{E}) & b\gamma/2 & 0 \\ b\gamma/2 & g(\mathbf{E}) & 0 \\ 0 & 0 & g(\mathbf{E}) \end{bmatrix}. \tag{13}$$

Here, $g_X(\mathbf{E})=c|\gamma|/\sqrt{2}$ and $g_Y(\mathbf{E})=0$. For both materials X and Y the relation shear stress-shear strain is the same as in linear elasticity:

$$T_{12}=b\gamma/2. \tag{14}$$

By the same physical argument used in linear elasticity, we require that $b>0$.

On the other hand, the behaviour of materials X and Y are quite different in what regards the normal stresses T_{11}, T_{22} and T_{33}. For material Y, they are all zero, as in the classic linear theory. For material X, however,

$$T_{11}=T_{22}=T_{33}=c|\gamma|/\sqrt{2}; \tag{15}$$

they are all equal but non-zero; they vary linearly with the absolute value of the shear strain. The sign of the normal stresses depends on the sign of c, which, as we will see below, may in principle be positive or negative. Here the so-called Poynting effect (difference between T_{11} and T_{22}) of finite elasticity does not appear.

b) Laterally constrained extension and compression

Here a body is extended ($\varepsilon>0$) or compressed ($\varepsilon<0$) axially (in the direction of $\mathbf{e}_2$), while no strain occurs latterally: $x_1=X_1$, $x_2=X_2+\varepsilon X_2$, $x_3=X_3$. Therefore, from equation (8):

$$[\mathbf{T}]=\begin{bmatrix} a\varepsilon+g(\mathbf{E}) & 0 & 0 \\ 0 & (a+b)\varepsilon+g(\mathbf{E}) & 0 \\ 0 & 0 & a\varepsilon+g(\mathbf{E}) \end{bmatrix} \tag{16}$$

In this case, $g_X(\mathbf{E})=g_Y(\mathbf{E})=c|\varepsilon|$.

For either material the response is the same. In extension, the lateral stress resulting from the impossibility of lateral deformation is $T_{11}=T_{33}=(a+c)\varepsilon$, which should be positive on physical grounds; hence, we impose $(a+c)>0$. In compression, the lateral stress is $T_{11}=T_{33}=(a-c)\varepsilon$; as we expect it to be negative, we impose $(a-c)>0$. Therefore, we must have $a>|c|$ (in other words, $a>0$ and $-a<c<a$).

The axial stress is related to the axial strain ε by $T_{22}=(a+b)\varepsilon+c|\varepsilon|$, which shows clearly that the stiffness is different in extension and compression.

c) hydrostatic expansion and compression

Now we examine the hydrostatic expansion ($\varepsilon>0$) and compression ($\varepsilon<0$) given by $\mathbf{E}=\varepsilon\mathbf{I}$. From equation (8):

$$\mathbf{T}=[(3a+b)\varepsilon+g(\mathbf{E})]\mathbf{I}. \tag{17}$$

In this case we have $g_X(\mathbf{E})=\sqrt{3}c|\varepsilon|$ and $g_Y(\mathbf{E})=3c|\varepsilon|$.

For material X, in extension, $\mathbf{T}=(3a+b+\sqrt{3}c)\varepsilon\mathbf{I}$, whereas, in compression, $\mathbf{T}=(3a+b-\sqrt{3}c)\varepsilon\mathbf{I}$. It is thus clear that the relation tr$\mathbf{T}$/3tr$\mathbf{E}$, which we can interpret as a hydrostatic bulk modulus, is different in expansion (K_e) and compression (K_p):

$$K_e=a+b/3+c/\sqrt{3}, \tag{18}$$

$$K_c=a+b/3-c/\sqrt{3}. \tag{19}$$

These bulk moduli are positive, since $a>|c|$.

We pass now to material Y. For the hydrostatic expansion and compression, we obtain

$$K_e=a+b/3+c, \tag{20}$$

$$K_c=a+b/3-c, \tag{21}$$

which are also positive, since $a>|c|$.

For both materials, $K_e>K_c$ or $K_e<K_c$ according to whether $c>0$ or $c<0$.

To end this section, it is convenient to repeat the restrictions on the material constants obtained above: a and b are positive; c may be positive or negative but must satisfy $|c|<a$.

6. THE INVERSE STRESS-STRAIN RELATION

It is often useful for the solution of many problems to express $\mathbf{E}$ as a function of $\mathbf{T}$. This will be done in this section for materials X and Y.

a) Material X

From equation (11) we get

$$\mathbf{E}=-\frac{a}{b(3a+b)}tr(\mathbf{T})\mathbf{I}+\frac{1}{b}\mathbf{T}-\frac{c}{(3a+b)}\|\mathbf{E}\|\mathbf{I}. \tag{22}$$

Now, in order to find the norm of $\mathbf{E}$, we first calculate the norm of both members of equation (22). A quadratic equation is then obtained: $A\|\mathbf{E}\|^2+B\|\mathbf{E}\|+C=0$, with $A=1-3H^2$, $B=-2H(3F+G))tr\mathbf{T}$, $C=-\|Ftr(\mathbf{T})\mathbf{I}+G\mathbf{T}\|^2$, where $F=-a/[b(3a+b)]$, $G=1/b$, $H=-c/(3a+b)$. As $H^2<1/3$ (since $|c|<a$), then $A>0$; and, since $C<0$, there are two non-zero real roots with different signs. The positive root, denoted here $R(\mathbf{T})$, is the norm of **E**. Returning to equation (22) we finally get the inverse relation for material X:

$$\mathbf{E} = [Ftr(\mathbf{T}) + HR(\mathbf{T})]\mathbf{I} + G\mathbf{T}\,. \tag{23}$$

b) Material Y

For material Y the situation is simpler: there is one linear inverse relation for $tr\mathbf{E}\leq 0$ and another one for $tr\mathbf{E}\geq 0$, as it will be shown below. From equation (12) we obtain

$$\mathbf{E} = Ltr\mathbf{TI} + N\mathbf{T} + M\left|tr\mathbf{E}\right|\mathbf{I}\,, \tag{24}$$

where L=-a/[b(3a+b)], M=-c/(3a+b), N=1/b. In what follows, to get more compact equations we introduce S=1/(3a+b+3c) and V=1/(3a+b-3c).

For $tr\mathbf{E}\leq 0$ we have $tr\mathbf{E}=Vtr\mathbf{T}$ and, from equation (24), we get

$$\mathbf{E}=(L+MV)tr(\mathbf{T})\mathbf{I}+N\mathbf{T}. \tag{25}$$

On the other hand, for $tr\mathbf{E}\geq 0$ we have $tr\mathbf{E}=Str\mathbf{T}$; so, from equation (24):

$$\mathbf{E}=(L+MS)tr(\mathbf{T})\mathbf{I}+N\mathbf{T}. \tag{26}$$

7. FINAL REMARKS

We have briefly studied the response of small–strain hypoplastic equations, and have seen that they can represent, in principle, solids that have different linear elastic behaviour in compression and in extension. However, no application to experimental data has been done yet; this will be a next step.

We should bear in mind that equations (11) and (12) are special cases of equation (8), which defines a class of materials. Other expressions for the function g can be chosen, when the objective is the application to a particular material.

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Received: April 04, 2008 Revised: July 07, 2008 Accepted: July 21, 2008