The Influence of Torsional Moments on the Elastoplastic Interaction Curves of a Symmetrical I Cross-Section

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Abstract: This work deals with the influence of the pure torsional (St. Venant) moment on deriving the elastoplastic interaction curves of internal forces and moments of a symmetrical I-section. The problem is addressed employing the exact theory of torsion. The simultaneous action of normal and shear stresses due to axial forces as well as bending and torsional moments is thoroughly investigated and the results are compared to the provisions of Eurocode 3 (EC3). Is has been found that the code's provisions are partly simplified formulae leading in most cases to uneconomic design.

Keywords: Elastoplasticity, plastic torsion, interaction curves.

INTRODUCTION

Each cross-section of a structural member (beam or column) is subjected to a set of six internal forces and moments due to bending, pure torsion or axial distress and two more forces caused by warping of the cross-section (bimoment and warping torsion). Although the stress-state problem can be easily formulated for the bending moments and forces, it becomes rather difficult if one attempts to formulate it taking into account the torsion moment and apply the exact theory of torsion [1, 2]. On the other hand the usually applied simplifications for the torsional stresses distribution on a cross-section lead to formulae of questionable exactitude that sometimes overestimate the carrying capacity of the cross-section [3, 4].

The present work deals with the study of the interaction of the aforementioned internal forces and moments on a symmetrical I-section. The material is assumed elastic-perfectly plastic. The warping stresses (due to non-uniform torsion) have been neglected due to the vagueness of the way and time of their appearance during the plasticization process of the cross-section [5]. Prandtl's theory of torsion, also known as Prandtl's membrane analogy, is used for the determination of the stresses caused by pure torsion (St. Venant's torsion) in the elastic as well as in the plastic regions [6, 7]. A cross-section goes through certain stages of stress-state until its complete plasticization [8]. Decoupling of bending and torsional problems is based on the provisions of EC3.

The interaction formulae for internal forces and moments are determined for both the elastic and the plastic stage as well as for several partial plasticization stages of a cross-section using the above Prandtl's theory and the equivalence principle [9, 10].

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ELEMENTARY CONCEPTS

The Strains

We consider the infinitesimal part dx of a beam which is strained by an axial force N_x and a bending moment M_y. The beam is referred to the classic clockwise system of coordinates as it is shown in Figs. (**1** and **2**). On the plane of the neutral axis it is $\sigma_x=0$ and therefore $\varepsilon_x=0$.
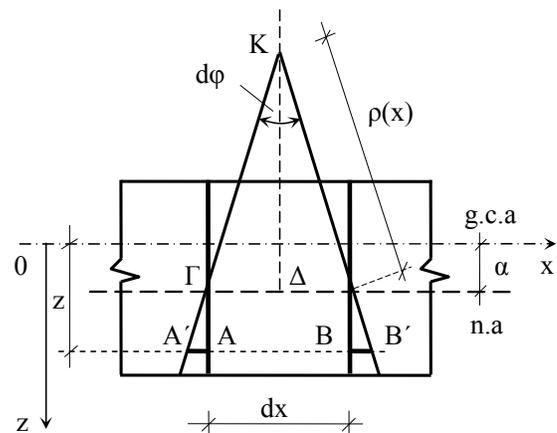

Fig. (1). Bending of an infinitesimal element dx.

Setting dx=AB, we can determine geometrically the normal strain from

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{AA' + BB'}{AB} = \frac{AA'}{AB/2} = \frac{AA'}{\Gamma\Delta}$$

and using the triangles $\Gamma A'A$ and $K\Gamma\Delta$ we have:

$$\varepsilon_x = \frac{z - \alpha}{\rho(x)} \tag{1}$$

where it is assumed

$$\cos\frac{d\varphi}{2} \cong 1.$$

The Stresses

Let us consider next the cross-section in Fig. (**2**), which is symmetric about Oz axis. It is known that, when parts of the cross-section go into the plastic region, the neutral axis moves to its final position where equilibrium of forces and moments is satisfied.
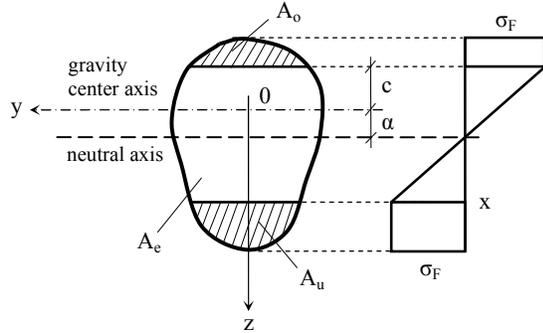

Fig. (2). Elastic-plastic stress distribution on a cross-sectional area A.

Easily, we find that:

$$\frac{\sigma_x}{\sigma_F} = \frac{z+\alpha}{\alpha+c} \tag{2}$$

If A_o and A_u are the top and bottom plastic regions of the cross-section A, and A_e is the remaining elastic part, equilibrium of forces gives:

$$\int_A \sigma_x dA = \int_{A_e} \sigma_x dA + \sigma_F (A_u - A_o) = N_x$$

and finally

$$\alpha \cdot A_e + (A_u - A_o - n_x \cdot A)(\alpha + c) = -S_e \tag{3}$$

On the other hand, equilibrium of moments gives:

$$\int_A \sigma_x z dA = \int_{A_p} \sigma_F z dA + \int_{A_e} \sigma_F \frac{z+\alpha}{\alpha+c} z dA = -M_y$$

and finally

$$\alpha \cdot S_e + (S_o - S_u - W_{yp} \cdot m_y)(\alpha + c) = -I_e \tag{4}$$

where:

$$n_x = \frac{N_x}{N_{xp}}, \ m_y = \frac{M_y}{M_{yp}}, \ S_{po} = \int_{A_o} z dA,$$
$$S_{pu} = \int_{A_u} z dA, \ S_e = \int_{A_e} z dA, \ I_e = \int_{A_e} z^2 dA \tag{5}$$

and W_{yp} is the resistant moment of the fully plastic cross-section. By solving the above system of eqs(3) and (4) we obtain the unknown quantities α and c.

CLASSICAL THEORY OF PURE TORSION

The Principle

For pure torsion, the shearing stresses τ_{xy} and τ_{xz} are independent on the normal ones $\sigma_x(y,z)$ [1]. In this case, we set:

$$u = u(x, y, z, D) \tag{6a}$$
$$D = \varphi'_x = cons \tan t \tag{6b}$$

where φ' is the constant change of the rotation angle φ of axis Ox. Using von Mises' criterion we have:

$$\sigma_x^2 + 3 \cdot (\tau_{xy}^2 + \tau_{xz}^2) = \sigma_F^2 \tag{7}$$

It is known that:

$$\tau_{xy} = \frac{\partial \Phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \Phi}{\partial y} \tag{8a} \tag{8b}$$

where Φ, is the so called stress-function.

Introducing eqs(8) into eq(7), we finally obtain:

$$\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 = \frac{1}{3}[\sigma_F^2 - \sigma_x^2(y,z)] = F^2(y,z)$$

with $\Phi = 0$ on the boundaries, or finally:

$$|grad\Phi| = F(y,z), \ \text{with} \ \Phi = 0 \ \text{on E} \tag{9a}$$

and

$$F(y,z) = \frac{1}{\sqrt{3}} \cdot [\sigma_F^2 - \sigma_x^2(y,z)]^{1/2} \tag{9b}$$

In order to solve the aformentioned boundary-value problem, we set:

$$\tau_{xy} = \frac{\partial \Phi}{\partial z} = F(y,z) \cdot \cos \varphi(y,z) \tag{10a}$$
$$\tau_{xz} = -\frac{\partial \Phi}{\partial y} = F(y,z) \cdot \sin \varphi(y,z) \tag{10b}$$

The above stress distributions satisfy eqs(9). On the other hand it is known that:

$$\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

and according to the Gauchy's principle:

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{11}$$

Because of eqs(10), eq(11) becomes:

$$\frac{\partial F}{\partial y} \cdot \cos \varphi - F \cdot \frac{\partial \varphi}{\partial y} \cdot \sin \varphi + \frac{\partial F}{\partial z} \cdot \sin \varphi + F \cdot \frac{\partial \varphi}{\partial z} \cdot \cos \varphi = 0 \tag{12}$$

The following relations are valid (see Fig. **3**):

$$\frac{\partial \varphi}{\partial \eta} = \frac{\partial \varphi}{\partial y} \cdot \frac{\partial y}{\partial \eta} + \frac{\partial \varphi}{\partial z} \cdot \frac{\partial z}{\partial \eta} = -\frac{\partial \varphi}{\partial y} \cdot \sin \varphi + \frac{\partial z}{\partial \eta} \cdot \cos \varphi \tag{13a}$$
$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial t} = \frac{\partial F}{\partial y} \cdot \cos \varphi + \frac{\partial F}{\partial z} \cdot \sin \varphi \tag{13b}$$

Introducing eqs(13) into eq(12), we obtain:

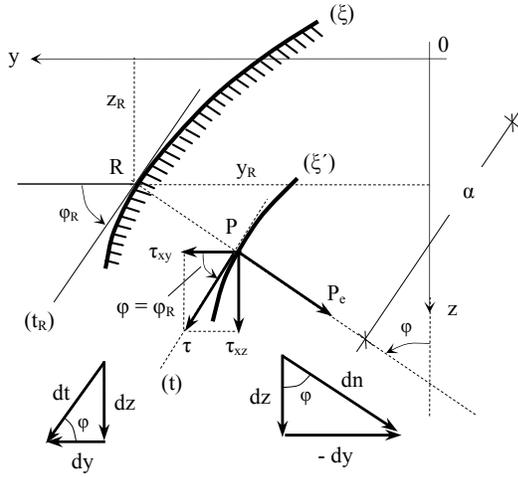$$\frac{\partial \varphi}{\partial \eta} + \frac{1}{F} \cdot \frac{\partial F}{\partial t} = 0 \tag{14}$$

Fig. (3). Geometrical conditions of the shear stresses.

where t and η are the tangential and the normal directions, respectively, at point $R(y_R, z_R)$ as shown in Fig. (**3**). Assuming that the equation of the boundary curve is known, i.e., that it has the form:

$$f(\varphi_R) = z_R + y_R \cdot \cot \varphi_R \tag{15}$$

we can integrate eq(14). Thus, the function F can be determined and, subsequently, from eq(10) the stresses τ_{xy} and τ_{xz}. Finally, the torsion moment M_x can be determined using the well-known formula:

$$M_x = 2GD \int_A (\tau_{xy} \cdot z + \tau_{xz} \cdot y) \, dA \tag{16}$$

where G is the shear modulus of steel.

The Case of a Symmetrical I-Cross-Section

Depending on the stress state, σ_x may be either constant or variable along the main axes of the cross-section. In this work, we will examine the cases of constant or linearly varying normal stress σ_x.

The Stress σₓ is Constant

The main goal is to determine the additional torsion moment M_x needed to fully plasticize a cross-section with area A subjected to a known axial force N_x. We symbolize with Φ_p the stress function of the complete plasticization of A under the action of the M_{xp} alone, and with Φ the one of the complete plasticization of A under the simultaneous action of M_x and N_x.

According to the analysis presented in the previous paragraph, we can write:

$$For \ M_{xp}: \quad \left| grad \ \Phi_p \right| = \frac{\sigma_F}{\sqrt{3}} \tag{17a}$$

$$For \ N_x + M_x: \left| grad \ \Phi \right| = \frac{1}{\sqrt{3}} \cdot [\sigma_F^2 - \sigma_x^2]^{1/2} \tag{17b}$$

But since $\sigma_x = N_x / A = const.$, it also follows that $\sigma_F^2 - \sigma_x^2 = \sigma_h^2 = const..$ According to Prandtl's membrane

theory, the membranes of eq(17), having constant inclinations, degenerate to simple roofs-in-plane with inclinations $\left| grad \ \Phi \right|$ and $\left| grad \ \Phi_p \right|$ as shown in Fig. (**4**). Therefore, the torsion moment M_x (according to the above analysis) is equal to the double of the volume that is bounded by the membrane Φ. Thus, one can write:

$$\frac{M_x}{M_{xp}} = \frac{Y}{Y_p} = \frac{\tan \vartheta}{\tan \vartheta_p} = \frac{\left| grad \ \Phi \right|}{\left| grad \ \Phi_p \right|} =$$
$$= \sqrt{1 - \left(\frac{\sigma_x}{\sigma_F}\right)^2} = \sqrt{1 - \left(\frac{N_x}{N_{xp}}\right)^2} \tag{18a}$$
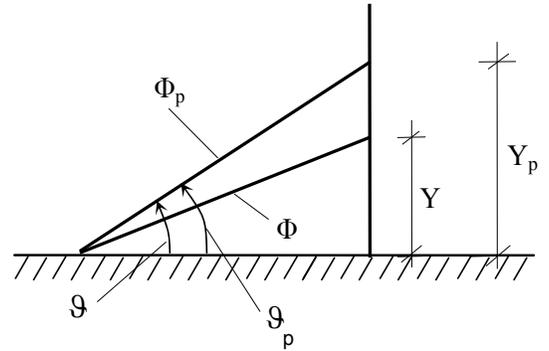
or finally:

$$n_x^2 + m_x^2 = 1 \tag{18b}$$

Fig. (4). Stress functions Φ and Φ_P.

The Stress σₓ Varies Linearly

In this case, we assume that the distribution of normal stresses σ_x along the main axes of an orthogonal cross-section is known and we proceed to determine the torsion moment M_x, which along with the given external forces N_x and M_y (or N_x and M_z) causes the complete plasticization of the cross-section. In such a case, the stress σ_x varies linearly according to the following relation:

$$\sigma_x = \frac{N_x}{A} + \frac{M_y}{W_y} \cdot \frac{z}{h/2} \tag{19}$$

Thus,

$$\frac{\sigma_x}{\sigma_F} = \frac{N_x / A}{N_{xp} / A} + \frac{M_y / W_y}{M_{yp} / W_{yp}} \cdot \frac{z}{h/2} = n_x + k_y \cdot m_y \cdot \frac{z}{h/2} \tag{20}$$

where

$$n_x = \frac{N_x}{N_{xp}}, \quad m_y = \frac{M_y}{M_{yp}}, \quad k_y = \frac{W_{yp}}{W_y},$$

and N_{xp} is the axial force which acting alone on the cross-section causes its complete plasticization, and M_{yp} is the bending moment which acting alone causes the complete plasticization of the cross-section. Hence, eq(20) can be written as:

$$\frac{\sigma_x}{\sigma_F} = n_x + \frac{z}{\lambda}, \quad where: \quad \lambda = \frac{h}{2 \cdot k_y \cdot m_y} \tag{21}$$

Equations (9a) and (9b) give:

$$\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 = \frac{\sigma_F}{3} \cdot \left[1 - \left(n_x + \frac{z}{\lambda}\right)^2\right] = F^2(z) \tag{22}$$

with $\Phi = 0$ on (L_b).

In order to be able to determine the moment $M_x = 2V_\Phi$ through the calculation of the volume V_Φ that is bounded below the membrane $\Phi(y,z)$, we must first solve the above boundary-value problem, as it is described by eq(22). For the solution of this problem, we apply eq(13b):

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial y} \cdot \cos\varphi + \frac{\partial F}{\partial z} \cdot \sin\varphi = -\frac{\sigma_F^2}{3} \cdot \frac{\lambda \cdot n_x + z}{\lambda^2 \cdot F} \cdot \sin\varphi$$

and eq(13a) becomes:

$$\frac{\partial \varphi}{\partial \eta} = -\frac{1}{F} \cdot \frac{\partial F}{\partial t} = \frac{1}{F^2} \cdot \left(\frac{\sigma_F^2}{3} \cdot \frac{\lambda \cdot n_x + z}{\lambda^2}\right) \cdot \sin\varphi =$$

$$= \frac{\frac{\lambda \cdot n_x + z}{\lambda^2}}{1 - \left(\frac{\lambda \cdot n_x + z}{\lambda^2}\right)^2} \cdot \sin\varphi$$

Since $dz = d\eta \cdot \cos\varphi$, the above equation can be written as follows:

$$\frac{\partial \varphi}{\partial z} \cdot \cos\varphi = \frac{\lambda \cdot n_x + z}{\lambda^2 - (\lambda \cdot n_x + z)^2} \cdot \sin\varphi$$

or

$$\cot\varphi \cdot d\varphi = \frac{\lambda \cdot n_x + z}{\lambda^2 - (\lambda \cdot n_x + z)^2} \cdot dz = -\frac{d[\lambda^2 - (\lambda \cdot n_x + z)^2]}{2 \cdot [\lambda^2 - (\lambda \cdot n_x + z)^2]}$$

or, in a more concise form

$$\int \cot\varphi \cdot d\varphi = -\frac{1}{2} \cdot \int \frac{d[\lambda^2 - (\lambda \cdot n_x + z)^2]}{\lambda^2 - (\lambda \cdot n_x + z)^2}$$

and

$$\ln(\sin\varphi) = \ln\frac{c}{\sqrt{\lambda^2 - (\lambda \cdot n_x + z)^2}}$$

or, finally

$$\sin\varphi = \frac{c}{\sqrt{\lambda^2 - (\lambda \cdot n_x + z)^2}} \tag{23}$$

For a random point R at the boundary αβ (see Fig. **5**), a value $\sin\varphi_R$ corresponds to each value of z_R, and hence, from eqs(23) we obtain:

$$c = \sin\varphi_R \cdot \sqrt{\lambda^2 - (\lambda \cdot n_x + z_R)^2}$$
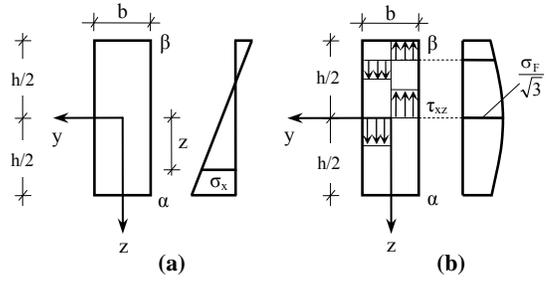
Thus, eq(23) becomes:

Fig. (5). (a) Distribution of normal stresses and (b) shear stresses.

$$\sin\varphi = \sqrt{\frac{\lambda^2 - (\lambda \cdot n_x + z_R)^2}{\lambda^2 - (\lambda \cdot n_x + z)^2}} \cdot \sin\varphi_R \tag{24}$$

Along the boundary αβ (as shown in Fig. **5**), it is $\sin\varphi_R = 1$ and eq.(10) gives $\tau_{xz} = F(z) \cdot \sin\varphi$, or:

$$\tau_{xz} = \frac{\sigma_F}{\sqrt{3}} \cdot \sqrt{1 - \left(\frac{\lambda \cdot n_x + z}{\lambda}\right)^2} \cdot \sqrt{\frac{\lambda^2 - (\lambda \cdot n_x + z_R)^2}{\lambda^2 - (\lambda \cdot n_x + z)^2}}$$

and

$$\tau_{xz} = \frac{\sigma_F}{\sqrt{3}} \cdot \sqrt{\frac{\lambda^2 - (\lambda \cdot n_x + z_R)^2}{\lambda^2}} \tag{25}$$

where z_R is now equal to z, because it expresses the ordinate of the boundary αβ. So, eqs(25) give for $z_R = z$ the diagram of Fig. (**5b**), while the additional torsion moment M_x will be:

$$M_x = \int_{-h/2}^{h/2} \tau_{xz} \left(\frac{b}{2}\right)^2 dz = \frac{\sigma_F \cdot b^2}{4 \cdot \sqrt{3}} \cdot \int_{-h/2}^{h/2} \sqrt{1 - \left(\frac{\lambda \cdot n_x + z}{\lambda}\right)^2} \cdot dz =$$

$$= \frac{\sigma_F \cdot b^2}{4 \cdot \sqrt{3}} \cdot \lambda \int_{-h/2}^{h/2} \sqrt{1 - \left(\frac{\lambda \cdot n_x + z}{\lambda}\right)^2} \cdot d\left(\frac{\lambda \cdot n_x + z}{\lambda}\right) =$$

$$= \frac{\sigma_F \cdot b^2}{4 \cdot \sqrt{3}} \cdot \frac{\lambda}{2} \cdot \left[\frac{\lambda \cdot n_x + z}{\lambda} \sqrt{1 - \left(\frac{\lambda \cdot n_x + z}{\lambda}\right)^2} + \arcsin\left(\frac{\lambda \cdot n_x + z}{\lambda}\right)\right]_{-h/2}^{h/2}$$

or finally:

$$M_x = \frac{\sigma_F b^2 \lambda}{8 \cdot \sqrt{3}} \left[\left(n_x + \frac{h}{2\lambda}\right)\sqrt{1 - \left(n_x + \frac{h}{2\lambda}\right)^2} - \right.$$

$$- \left(n_x - \frac{h}{2\lambda}\right)\sqrt{1 - \left(n_x - \frac{h}{2\lambda}\right)^2} + \tag{26a}$$

$$\left. + \arcsin\left(n_x + \frac{h}{2\lambda}\right) - \arcsin\left(n_x - \frac{h}{2\lambda}\right)\right]$$

Introducing the expression of λ from eq(21b) into the above expression, we obtain:

$$\frac{M_x}{\sigma_F} = \frac{M_x}{M_{xp}/J_{dp}} = \frac{b^2 \lambda h}{16 k_y m_y \sqrt{3}} \left[(n_x + k_y m_y)\sqrt{1 - (n_x + k_y m_y)^2}\right.$$

$$- (n_x - k_y m_y)\sqrt{1 - (n_x - k_y m_y)^2} +$$

$$\left. + \arcsin(n_x + k_y m_y) - \arcsin(n_x - k_y m_y)\right]$$

and finally:

$$m_x = \frac{1}{J_{dp}} \cdot \frac{b^2 \lambda h}{16 k_y m_y \sqrt{3}} \Big[(n_x + k_y m_y)\sqrt{(n_x + k_y m_y)} -$$

$$- (n_x - k_y m_y)\sqrt{(n_x - k_y m_y)} \qquad (26b)$$

$$+ \arcsin(n_x + k_y m_y) - \arcsin(n_x - k_y m_y) \Big]$$

where J_{dp} for an orthogonal cross-section is given by

$$J_{dp} = \frac{h b^2}{4\sqrt{3}} .$$

INTERACTION CURVES

Since the methodology presented in the previous paragraph is too extensive, we will examine in detail only a limited number of cases that are the most characteristic ones. This technique can be applied to all other cases.

The Case $N_x + M_x$

This case is already presented above. If N_x is the acting axial force and M_x the searched additional torsion moment, the following relation is valid:

$$n_x^2 + m_x^2 = 1 \qquad (27)$$

where $n_x = N_x/N_{xp}$ and $m_x = M_x/M_{xp}$.

The Case $M_x + M_y$

We assume that the distribution of the stresses, caused by a bending moment M_y, along the Oz axis is known (see Fig. **6**). If A is the area of a doubly symmetric I-cross-section, we search for the additional torsion moment M_x which acting simultaneously with the given bending moment M_y causes the complete plasticization of the cross-section A. The cross-section goes to its complete plasticization through the following steps:

1st Step

This step starts from $M_y=0$ up to the plasticization of the first external fiber under the moment M_{y1}. It is $M_{y1}=\sigma_F \cdot W_y$, or finally:

$$m_{y1} = \frac{M_{y1}}{M_{yp}} = \frac{W_y}{W_{yp}} \qquad (28)$$

2nd Step

This step starts from M_{y1} up to the plasticization of both flanges under the moment M_{y2}. It is

$$M_{y2} = M_{yp} - 2 \cdot \frac{1}{2} \cdot \sigma_F t_w \frac{h_o^2}{3} ,$$

or finally:

$$m_{y2} = \frac{M_{y2}}{M_{yp}} = 1 - \frac{h_o A_w}{6 W_{yp}} \qquad (29)$$

3rd Step

This step starts from M_{y2} up to the complete plasticization of the entire cross-section under the moment M_{yp}.

The 1st Step: Torsion on the Entire Cross-Section

The Part I (Flanges)

In this case, since it is $t_f \ll h$, and the direction of shear stresses τ is parallel to the flanges' large dimension, one can consider the mean stress of each flange (see also Fig. **6**). It is:

$$\frac{\sigma_o}{\sigma_F} = \frac{M_y / W_y}{M_{yp} / W_{yp}} = k_y \cdot m_y$$

where k_y can be obtained from eq(20). Thus:

$$\frac{\sigma_m}{\sigma_F} = k_y \cdot m_y \cdot \frac{h + 2 h_o}{2 h} \qquad (30)$$

and from eq(18a) we have:

$$\frac{M_{xI}}{M_{xIp}} = \frac{|grad\,\Phi|}{|grad\,\Phi_p|} = \frac{\frac{1}{\sqrt{3}}\sqrt{\sigma_F^2 + \sigma_m^2}}{\sigma_F / \sqrt{3}} = \sqrt{1 - \left(\frac{\sigma_m}{\sigma_F}\right)^2}$$

or

$$M_{xI} = \sigma_F \cdot J_{dpI} \sqrt{1 - k_y^2 \left(\frac{h + 2 h_o}{2 h}\right)^2 m_y^2} \qquad (31)$$
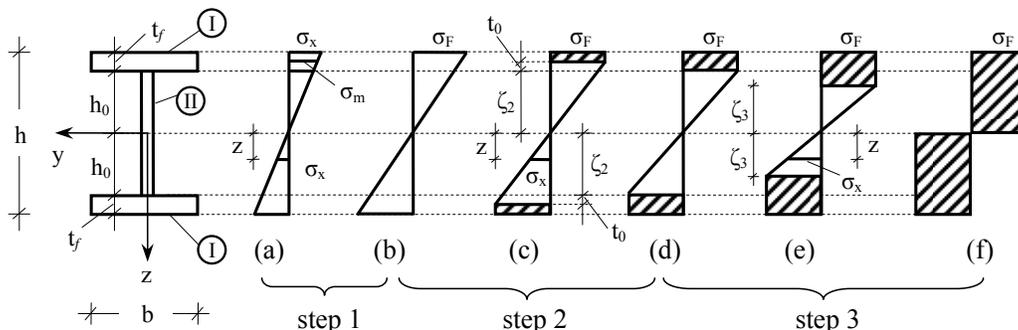
where

$$J_{dpI} = \frac{b \cdot t_f^2}{2 \cdot \sqrt{3}}$$

for both flanges.

The Part II (Web)

It is (see Fig. **6**):

Fig. (6). Normal stresses on an I-section during the plasticization progress.

$$\frac{\sigma_x}{\sigma_F} = \frac{\sigma_o}{\sigma_F} \cdot \frac{z}{h/2} = \frac{z}{\lambda} \qquad (32)$$

where

$$\lambda = \frac{h}{2 \cdot k_y \cdot m_y}$$

With $n_x = 0$ and λ computed from eq(32), eq(26a) gives:

$$M_{xII} = \frac{\sigma_F t_w^2}{4\sqrt{3}} \left[h_o \sqrt{1 - \left(\frac{2 h_o k_y}{h} m_y \right)^2} + \right.$$
$$\left. + \frac{h}{2 k_y m_y} \arcsin\left(\frac{2 h_o k_y}{h} m_y \right) \right] \qquad (33)$$

Thus, the total torsion moment is the summation of the above two parts M_{xI} and M_{xII}:

$$M_x = \sigma_F J_{dpI} \sqrt{1 - k_y^2 \left(\frac{h + \rho_y}{2} \right)^2 m_y^2} +$$
$$+ \frac{\sigma_F t_w^2}{4\sqrt{3}} \left[h_o \sqrt{1 - \rho_y^2 m_y^2} + \frac{h}{2 k_y m_y} \arcsin\left(\rho_y m_y \right) \right]$$

and finally:

$$m_{x1} = \frac{1}{J_{dp}} \left\{ J_{dpI} \sqrt{1 - k_y^2 \left(\frac{h + \rho_y}{2} \right)^2 m_y^2} + \right.$$
$$\left. + \frac{t_w^2}{4\sqrt{3}} \left[h_o \sqrt{1 - \rho_y^2 m_y^2} + \frac{h}{2 k_y m_y} \arcsin\left(\rho_y m_y \right) \right] \right\} \qquad (34)$$

with $k_y = W_{yp} / W_y$ and $\rho_y = 2 h_o k_y / h$.

The 2nd Step: Torsion on the Web and a Part of Flanges ($h_o < \zeta < h/2$)

Since an exact calculation of distance ζ_2 (Fig. **6c**) leads to the solution of a 3rd order algebraic equation and observing that ζ_2 varies almost linearly within thickness t_f, one can determine ζ_2 through the following approximate relation:

$$\zeta_2 = \frac{h}{2} - \frac{6 \cdot t_f \cdot (W_y - W_{yp} \cdot m_y)}{6 \cdot (W_y - W_{yp}) + h_o \cdot A_w} \qquad (35)$$

We divide the elastic region of the cross-section in two parts: the elastic part of the flanges and elastic part of the web (as shown in Fig. **6c**).

The Part I (Flanges)

Since the thickness t_f of the flanges is small compared to the height h, one can consider that the normal stresses in Fig. (**6c**) have a uniform distribution along the thickness t_o. So, one can use the mean stress:

$$\sigma_m = \frac{\sigma_F + \sigma_x(h_o)}{2} = \sigma_F \cdot \frac{h_o + \zeta_2}{2 \cdot \zeta_2} \qquad (36)$$

Using eq(19a), we obtain the following relation:

$$M_{xI} = \sigma_F \cdot J_{dpI} \cdot \sqrt{1 - \left(\frac{h_o + \zeta_2}{2\zeta_2} \right)^2} \qquad (37)$$

where

$$J_{dpI} = \frac{b \cdot t_o^2}{2\sqrt{3}} = \frac{b \cdot (\zeta_2 - h_o)^2}{2\sqrt{3}}$$

The Part II (Web)

The normal stress σ_x at a distance z (Fig. **6c**) is given by:

$$\sigma_x(z) = \sigma_F \cdot \frac{z}{\zeta_2} \qquad (38)$$

that has the same form with the expression of eq(32). Following the same procedure as the one presented above, one can obtain the relation:

$$\tau_{xz} = \frac{\sigma_F}{\sqrt{3}} \sqrt{\frac{\zeta_2^2 - z^2}{\zeta_2^2}} \quad for \quad -h_o \leq z \leq h_o \qquad (39)$$

The torsion moment undertaken by part II will be:

$$M_{xII} = \int_{-h_o}^{h_o} \tau_{xz} \left(\frac{t_w}{2} \right)^2 dz = \frac{\sigma_F}{4\sqrt{3}} \cdot t_w^2 \cdot \zeta_2 \cdot$$
$$\cdot \frac{1}{2} \cdot \left[\frac{z}{\zeta_2} \sqrt{1 - \left(\frac{z}{\zeta_2} \right)^2} + \arcsin \frac{z}{\zeta_2} \right]_{-h_o}^{h_o}$$

or finally

$$M_{xII} = \frac{\sigma_F \cdot t_w^2}{4\sqrt{3}} \cdot \zeta_2 \cdot \left[\frac{h_o}{\zeta_2} \cdot \sqrt{1 - \left(\frac{h_o}{\zeta_2} \right)^2} + \arcsin \frac{h_o}{\zeta_2} \right] \qquad (40)$$

Thus, the total torsion moment is the sum of the above two parts M_{xI} and M_{xII} :

$$M_x = \sigma_F \cdot \frac{b \cdot (\zeta_2 - h_o)^2}{2\sqrt{3}} \cdot \sqrt{1 - \left(\frac{h_o + \zeta_2}{2\zeta_2} \right)^2} +$$
$$+ \frac{\sigma_F \cdot t_w^2}{4\sqrt{3}} \cdot \left[h_o \cdot \sqrt{1 - \left(\frac{h_o}{\zeta_2} \right)^2} + \zeta_2 \cdot \arcsin \frac{h_o}{\zeta_2} \right]$$

and finally:

$$m_x = \frac{1}{J_{dp}} \cdot \left\{ \frac{b \cdot (\zeta_2 - h_o)^2}{2\sqrt{3}} \cdot \sqrt{1 - \left(\frac{h_o + \zeta_2}{2\zeta_2} \right)^2} + \right.$$
$$\left. + \frac{t_w^2}{4\sqrt{3}} \cdot \left[h_o \cdot \sqrt{1 - \left(\frac{h_o}{\zeta_2} \right)^2} + \zeta_2 \cdot \arcsin \frac{h_o}{\zeta_2} \right] \right\} \qquad (41)$$

where ζ_2 can be computed from eq(35).

The 3rd Step: Torsion Only on the Web ($\zeta_3 < h_o$)

In this case (see Fig. **6e**), we have:

$$M_y = M_{yp} - t_w \cdot \sigma_F \cdot \zeta_3^2 / 3$$

or finally:

$$\zeta_3 = \sqrt{\frac{3 W_{yp}}{t_w} \cdot (1 - m_y)} \tag{42}$$

On the other hand, we know that (see Fig. **6e**):

$$\sigma_x(z) = \sigma_F \cdot \frac{z}{\zeta_3} \quad for \quad |z| \le \zeta_3 \tag{43}$$

The above expression has the same form with eq(38). Following the same procedure as the one presented above, we obtain the expression:

$$\tau_{xz} = \frac{\sigma_F}{\sqrt{3}} \sqrt{\frac{\zeta_3^2 - z^2}{\zeta_3^2}} \quad for \quad -\zeta_3 \le z \le \zeta_3$$

The torsion moment will be:

$$M_x = \frac{\sigma_F}{4\sqrt{3}} \cdot t_w^2 \frac{\zeta_3}{2} \left[\frac{z}{\zeta_3} \cdot \sqrt{1 - \left(\frac{z}{\zeta_3}\right)^2} + \arcsin\frac{z}{\zeta_3} \right]_{-\zeta_3}^{\zeta_3} =$$

$$= \frac{\sigma_F \cdot t_w^2 \cdot \pi}{8\sqrt{3}} \cdot \zeta_3$$

or finally

$$m_x = \frac{\pi \cdot t_w^2}{8\sqrt{3} \cdot J_{dp}} \cdot \sqrt{\frac{3 W_{yp}}{t_w} \cdot (1 - m_y)} \tag{44}$$

The Case $M_x + M_z$

Following the procedure given in the previous paragraph, one can distinguish between the following two cases (see Fig. **7**):
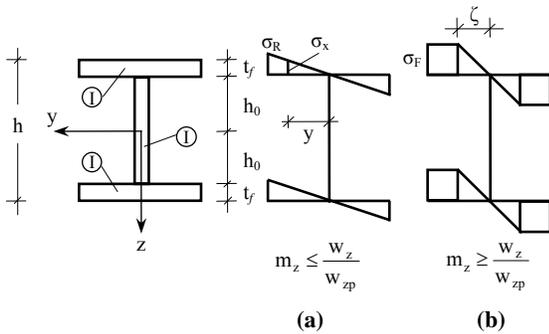

(a)   **(b)**

Fig. (7). Normal stress σ_x for bending moment M_z.

Torsion on the Entire Cross-Section

Neglecting the very small part of bending moment M_z that can be undertaken by the web and assuming that the flanges undertake the whole bending moment M_z, we have:

$$\frac{\sigma_R}{\sigma_F} = \frac{M_z / W_z}{M_{zp} / W_{zp}} = k_z \cdot m_z$$

and

$$\frac{\sigma_x}{\sigma_R} = \frac{y}{b/2}$$

thus $\sigma_x = \sigma_R \cdot 2 y / b = \sigma_F \cdot k_z \cdot m_z \cdot 2 y / b$, or finally:

$$\frac{\sigma_x}{\sigma_F} = \frac{y}{\lambda}, \quad where: \quad \lambda = \frac{b}{2 \cdot k_z \cdot m_z} \tag{45}$$

For the flanges, it is valid that

$$\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 = \frac{\sigma_F}{\sqrt{3}} \cdot \left[1 - \left(\frac{y}{\lambda}\right)^2\right]$$

and, therefore,

$$M_{xI} = 2 \int_{-b/2}^{b/2} \frac{\sigma_F \cdot t_f^2}{4\sqrt{3}} \cdot \sqrt{1 - \left(\frac{y}{\lambda}\right)^2} \, dy =$$

$$= 2 \cdot \frac{\sigma_F \cdot t_f^2}{4\sqrt{3}} \cdot \frac{\lambda}{2} \left[\frac{y}{\lambda} \sqrt{1 - \left(\frac{y}{\lambda}\right)^2} + \arcsin\frac{y}{\lambda} \right]_{-b/2}^{b/2} \tag{46}$$

$$= 2 \cdot \frac{\sigma_F \cdot t_f^2}{4\sqrt{3}} \cdot \left[\frac{b}{2} \cdot \sqrt{1 - (k_z \cdot m_z)^2} + \frac{b}{2 \cdot k_z \cdot m_z} \cdot \arcsin(k_z \cdot m_z) \right]$$

On the other hand, normal stresses do not develop within the web due to the axial force but only shear stresses due to torsion. Thus,

$$M_{xII} = \sigma_F \cdot J_{dpII} \tag{47}$$

The total torsion moment is the summation of the above two parts M_{xI} and M_{xII} :

$$m_x = \frac{1}{J_{dp}} \cdot \{ J_{dpII} + \frac{t_f^2}{2\sqrt{3}} \cdot [\frac{b}{2} \cdot \sqrt{1 - (k_z \cdot m_z)^2} + \frac{b}{2 \cdot k_z \cdot m_z} \cdot \arcsin(k_z \cdot m_z)]\} \tag{48}$$

where

$$J_{dpII} = \frac{h_o \cdot t_w^2}{2\sqrt{3}} \cdot$$

Torsion on the Web and a Part of Flanges

This case exists only if

$$m_z \ge W_z / W_{zp} \tag{49}$$

Then, the following relation is valid:

$$M_z = M_{zp} - \sigma_F \cdot t_f \cdot 2 \cdot \zeta^2 / 3$$

or finally:

$$\zeta = \sqrt{\frac{3 W_{zp}}{2 t_f} \cdot (1 - m_z)} \tag{50}$$

and through an analysis analogous to the one presented above, we obtain:

$$M_{xI} = 2 \cdot \frac{\sigma_F \cdot t_f^2 \cdot \pi}{8 \sqrt{3}} \cdot \zeta \qquad (51)$$

Thus, the total torsion moment is the summation of the above two parts M_{xI} and M_{xII}, with M_{xII} computed from eq(47)

$$m_x = \frac{1}{J_{dp}} \cdot \left[J_{dpII} + \frac{\pi \cdot t_f^2}{4 \sqrt{3}} \cdot \sqrt{\frac{3 \cdot W_{zp}}{2 \cdot t_f} \cdot (1 - m_z)} \right] \qquad (52)$$

The Case $M_x + M_y + V_z$

In the present case (existence of M_y and V_z) and since the flanges undertake an almost negligible part of the shear force (~2%), we can assume that the shear force is undertaken solely by the web. The shear force V_{zp} (i.e., the shear force needed for complete plasticization of the cross-section) will be:

$$V_{zp} = A_w \cdot \sigma_F / \sqrt{3} = 2 \cdot h_o \cdot t_w \cdot \sigma_F / \sqrt{3} \qquad (53)$$

Torsion on the Entire Cross-Section

The stresses σ_x and τ are given by the following expressions:

$$\sigma_x = \frac{M_y}{J_y} \cdot z, \qquad \tau = \frac{V_z}{J_y \cdot t_w} \left[\frac{A_f}{4}(h - t_f) + t_w \frac{h_o^2 - z^2}{2} \right] \qquad (54)$$

The most unfavorable stress combination occurs at $z=(h-2t_f)/2$, and since

$$\sigma_h = \sigma_F = \sqrt{\sigma_x^2 + 3\tau^2}$$

the following condition must be also valid in order for the cross-section to be able to undertake an additional torsion moment:

$$V_z^2 \leq \frac{16 \cdot t_w^2}{A_f^2 (h - t_f)^2} \cdot \left[\frac{\sigma_F^2 \cdot J^2}{3} - \left(\frac{h - 2 \cdot t_f}{2} \right)^2 \cdot M_y^2 \right] \qquad (55)$$

The distribution of shear stresses in a rectangular cross-section due to simultaneous action of torsion moment M_x and shear force V_z may be regarded similarly to the distribution of normal stresses caused by axial force N_x and bending moment M_y. Thus, we consider the model shown in Fig. (**8**), where t_V is the width of the web that can be plasticized by the shear force V_z.
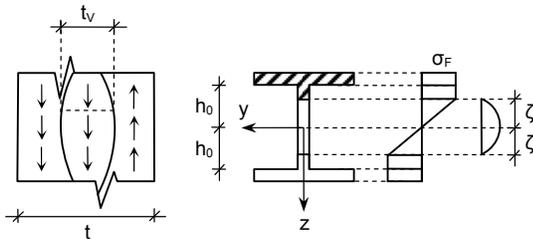

Fig. (8). Shear stress distribution in the web.

In this case, the following relation is valid

$$t_V \cdot \frac{\sigma_F}{\sqrt{3}} \cdot dz = t_w \cdot \tau \cdot dz$$

and taking into account the second expression of eq(54), the above relation results $t_V = \mu(z) \cdot v$, where $v = V_z/V_{zp}$ and

$$\mu(z) = \frac{A}{J_y} \cdot \left[\frac{A_f}{4} \cdot (h - t_f) + t_w \cdot \frac{h_o^2 - z^2}{2} \right] \qquad (56)$$

Thus, the searched torsion moment of the web will be

$$M_{xII} = \int_{-h_o}^{h_o} \tau_{xz} \cdot \frac{t_w^2 - t_v^2}{4} dz$$

and

$$m_x = \frac{1}{J_{dp}} \cdot \left\{ J_{dpI} \cdot \sqrt{1 - \left(\frac{k_y + \rho_y}{2} \right)^2 \cdot m_y^2} + \right.$$
$$+ \frac{t_w^2}{4\sqrt{3}} \left[h_o \sqrt{1 - \rho_y^2 \cdot m_y^2} + \frac{h}{2 k_y m_y} \cdot \arcsin(\rho_y m_y) \right] -$$
$$\left. - v^2 \cdot \int_{-h_o}^{h_o} \frac{(\rho_1 - \rho_2 \cdot z^2)^2}{4 \cdot \sqrt{3}} \sqrt{1 - \left(\frac{z}{\lambda} \right)^2} \cdot dz \right\} \qquad (57)$$

$with: \quad k_y = W_{yp} / W_y, \quad \rho_y = 2 h_o \cdot k_y / h$

Torsion on the Web and a Part of Flanges

The distribution of shear stresses τ is the shown in Fig. (**8**). Under the same assumptions eq(36) is valid, and thus we finally obtain:

$$m_x = \frac{1}{J_{dp}} \cdot \left\{ \frac{b \cdot (\zeta - h_o)^2}{2 \sqrt{3}} \cdot \sqrt{1 - \left(\frac{h_o + \zeta}{2\zeta} \right)^2} + \right.$$
$$+ \frac{t_w^2}{4\sqrt{3}} \cdot \left[h_o \cdot \sqrt{1 - \left(\frac{h_o}{\zeta} \right)^2} + \zeta \cdot \arcsin \frac{h_o}{\zeta} \right] -$$
$$\left. - v^2 \cdot \int_{-h_o}^{h_o} \frac{(\rho_1 - \rho_2 \cdot z^2)^2}{4 \cdot \sqrt{3}} \sqrt{1 - \left(\frac{z}{\lambda} \right)^2} \cdot dz \right\} \qquad (58)$$

Torsion Only on the Web

It can be easily proved that the distribution of shear stresses starts from the boundaries of the plastic regions of the cross-section, as shown in Fig. (**7b**). Thus the part of the web that receives the shear stresses due to V_z, may as well behave like a rectangular cross-section with dimensions $2 \cdot \zeta \cdot t_w$. Evidently, we may write:

$$\sigma_x = \sigma_F \cdot \frac{z}{\zeta}, \qquad \tau = \frac{3 \cdot V_z}{4 \cdot t_w} \cdot \frac{1}{\zeta} \cdot \left[1 - \left(\frac{z}{\zeta} \right)^2 \right] \qquad (59)$$

and

$$\tau_{max} = \frac{3 \cdot V_z}{4 \cdot \zeta \cdot t_w}$$

When τ_{max} reaches the limit value $\tau_F = \sigma_F / \sqrt{3}$, eq(8c) gives:

$$\zeta = \frac{3 \cdot \sqrt{3} \cdot V_z}{4 \cdot t_w \cdot \sigma_F} \qquad (60)$$

According to the diagram in Fig. (**7a**), one can write $M_y = M_{yp} - \sigma_F \cdot t_w \cdot \zeta^2 / 3$, or because of eq(40) and taking into account eq(33), we find that in order for the cross-section to be able to undertake a torsion moment M_x the following condition must apply:

$$V_z^2 \leq \frac{16}{9} \cdot \frac{t_w \cdot W_p}{A_w^2} \cdot \left(1 - \frac{M_y}{M_{yp}}\right) \cdot V_{zp}^2 \qquad (61)$$

The width t_v, using eq(39b), is:

$$t_v = \frac{3 \cdot \sqrt{3}}{4 \cdot t_w \cdot \zeta \cdot \sigma_F} \left[1 - \left(\frac{z}{\zeta}\right)^2\right] =$$

$$= \frac{3 \cdot h_o}{2 \cdot \zeta} \cdot \left[1 - \left(\frac{z}{\zeta}\right)^2\right] \cdot \frac{V_z}{V_{zp}} \qquad (62)$$

and, the searched torsion moment will be:

$$m_x = \frac{\pi \cdot t_w^2}{8\sqrt{3} \cdot J_{dp}} \cdot \sqrt{\frac{3 \cdot W_{yp}}{t_w} \cdot (1 - m_y)} -$$

$$- \frac{1}{J_{dp}} \cdot v^2 \cdot \int_{-\zeta}^{\zeta} \frac{\sqrt{3} \cdot h_o}{2 \cdot \zeta} \cdot \left[1 - \left(\frac{z}{\zeta}\right)^2\right]^2 \sqrt{1 - \left(\frac{z}{\zeta}\right)^2} \cdot dz \qquad (63)$$

The Special Case $M_x + V_z$

Following the same procedure as in the previous section, we obtain:

$$m_x = 1 - \frac{1}{4 J_{dp} \sqrt{3}} \cdot v^2 \cdot \int_{-h_o}^{h_o} \mu(z) \cdot \sqrt{1 - \left(\frac{z}{h_o}\right)^2} \, dz$$

for $V_z \leq A_w/A$, and

$$m_x = 1 - \frac{(A \cdot v - A_w)^2}{8 \cdot b \cdot J_{dp} \sqrt{3}} -$$

$$- \frac{1}{4 J_{dp} \sqrt{3}} \cdot \left(\frac{A_w}{A}\right)^2 \cdot \int_{-h_o}^{h_o} \mu(z) \cdot \sqrt{1 - \left(\frac{z}{h_o}\right)^2} \, dz \qquad (64)$$

where

$$\mu(z) = \frac{A_w}{J_y} \left[\frac{A_f}{4}(h - t_f) + t_w \cdot \frac{h_o^2 - z^2}{2}\right]$$

The Case $N_x + M_x + M_y$.

In this case, the analysis is very complicated since it depends on the values of N_x (and, therefore, on the plastic part of the cross-section) and also on the values of M_y. The paths
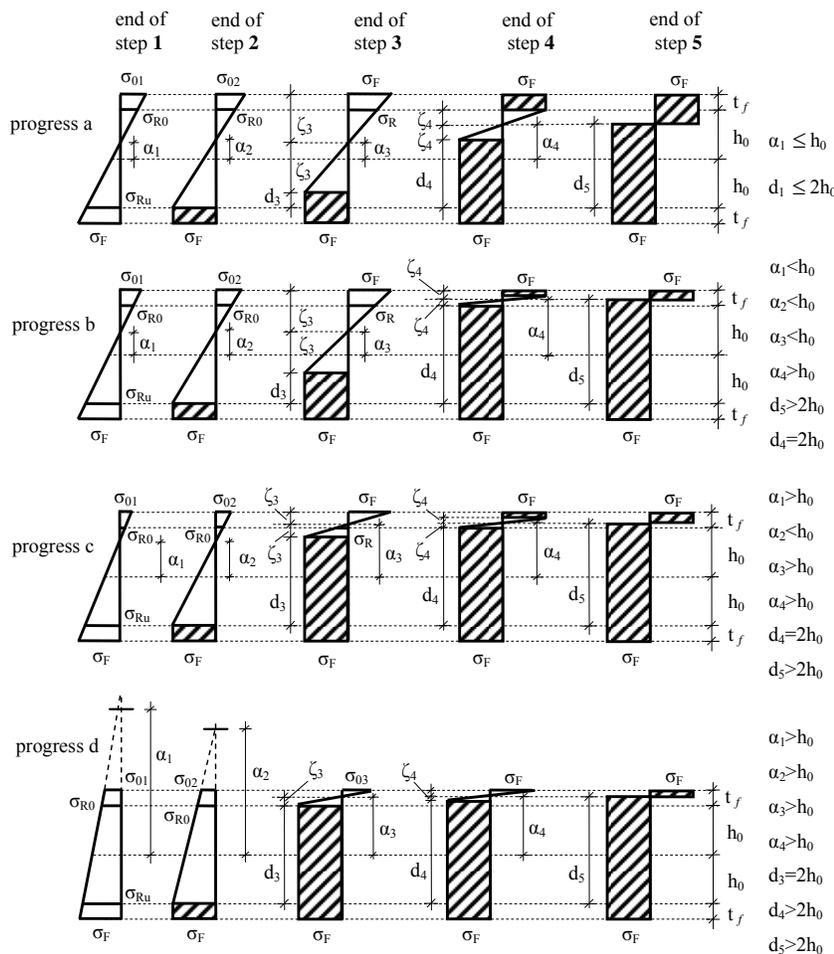

Fig. (9). Plasticization processes for a symmetrical I-section.

leading to the complete plasticization of the cross-section and the corresponding steps are shown in Fig. (**9**). The analysis is carried out following the same steps as in the preceding paragraphs and a detailed description is out of the scope of this paper.

NUMERICAL RESULTS & CONCLUSIONS

Considering first the simultaneous action of axial force N_x and torsion moment M_x, we obtain the diagram of Fig. (**10**) showing the interaction curve m_x - n_x in non-dimensional form. It is observed that for intermediate values of the axial force and the torsion moment the interaction curve is smooth and represents the limiting combinations N_x - M_x corresponding to the fully plastic cross-section.
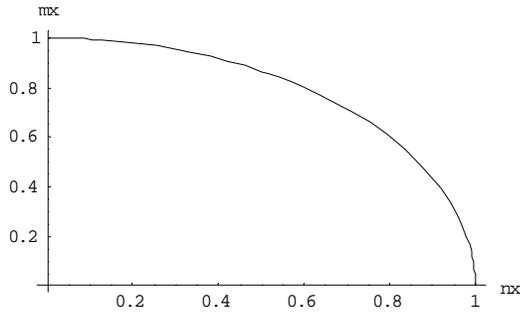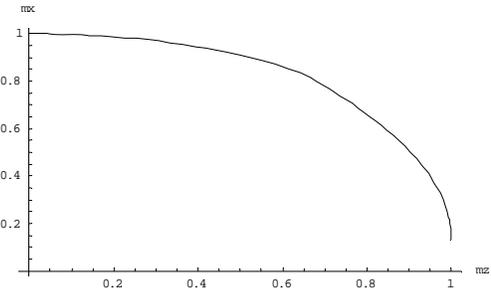

Fig. (10). Interaction curve $m_x - n_x$.

In Figs (**11a**) and (**11b**), one can see the interaction curves m_x - m_y for HEB and IPE cross-sections, respectively.

(a)

———— HE-B 300   - - - - - - - HE-B 1000

(b)

———— IPE 300   - - - - - - - IPE 1000
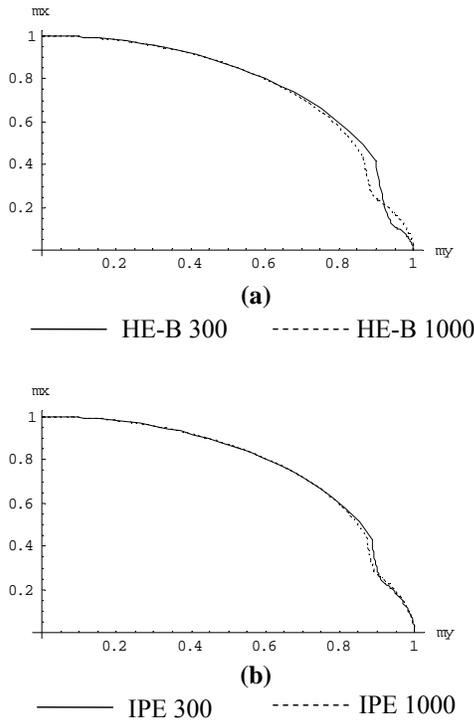
Fig. (11). Interaction curves m_x-m_y for (**a**) wide-flange and (**b**) short-flange I-sections.

It is observed that for small or intermediate values of the bending moment M_y, both interaction curves are smooth and almost coincident, while for higher values of M_y near the plastic resistance value, a sudden drop is observed on both curves due to phase-chance in the plasticization process accompanied by a 15% -20% difference between the short- and wide-flange I-cross-sections.

The diagram of Fig. (**12**) shows the interaction curve m_x – m_z. One can observe the limited carrying capacity of the I-section in bending moment M_z and torsion moment M_x when comparing to the corresponding diagrams m_x – m_y.

Fig. (12). Interaction curve m_x - m_z.

Fig. (**13**) shows the interaction curves m_x-m_y for different values of the ratio $v=V_z/V_{zp}$. The presence of a shear force V_z with simultaneously acting moments M_y and M_x causes a gradual reduction of the carrying capacity of the I-section with a maximum 10% drop.

The interaction curves v-m_x are shown in Fig. (**14**) according to the proposed approach (continuous line) and according to the provisions of EC3-Annex G (doted line). One can observe that EC3 always underestimates the carrying capacity of I-sections leading to uneconomic design. It should be noted at this point that the provisions of Eurocode 3 refer to only this loading case for which, a comparison between the results shows that the code's provisions are partly simplified but certainly uneconomic.

———— v = 0.2   - - - - - - v = 0.6   ·········· v = 1
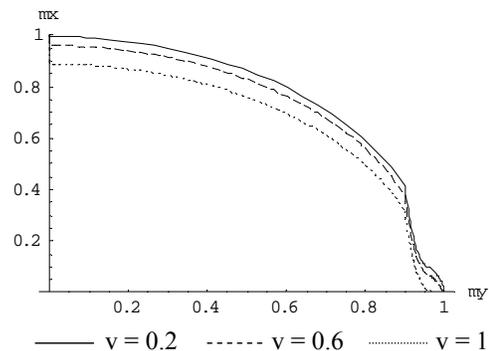
Fig. (13). Interaction curves m_x – m_y for different values of the ratio $v=V_z/V_{zp}$.

In Fig. (**15**), the interaction curves m_x-m_y are shown for (a) short-flange IPE sections and (b) for wide-flange HEB sections and for different values of the axial force n_x. In both cases, a high reduction of the carrying capacity of the cross-section is observed for high values of the bending moment
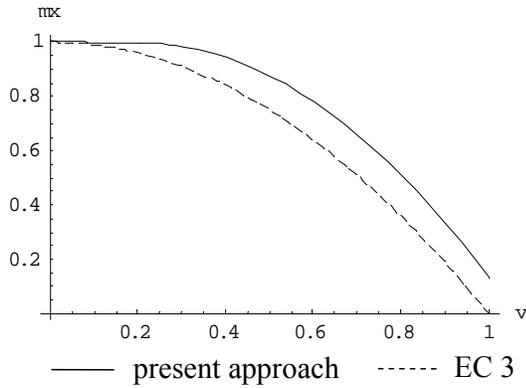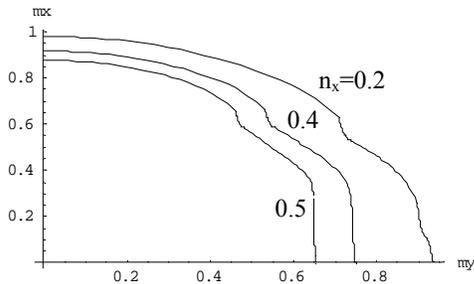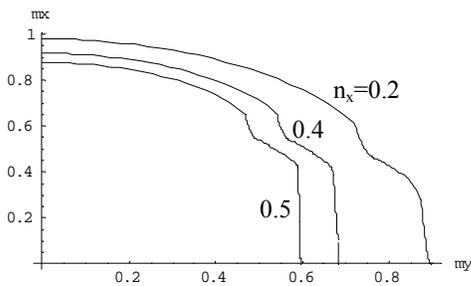
Fig. (14). Interaction curves m_x -v.

M_y. Also, an additional reduction of the carrying capacity of the I-section with a maximum 15% drop is observed when a significant axial force N_x is present.

(a) IPE – profiles (short flange)

(b) HE-B profiles (wide flange)

Fig. (15). Interaction curves m_x – m_y for various values of $n_x=N_x/N_{x,pl}$.

Finally, we can conclude that following the present approach one can achieve the fulfillments of a very economic and safe design.

LIST OF SYMBOLS

σ	=	normal stresses
ε	=	normal strains
τ_{xy}	=	shear stresses
σ_F	=	yield stress
Φ	=	stress function
F	=	failure criterion
N_x	=	axial force
M_x	=	torsional moment
M_y	=	bending moment about y
M_z	=	bending moment about z
e	=	index for sizes in the elastic region
p	=	index for sizes in the plastic region
f	=	index for flange sizes
w	=	index for web sizes
W	=	first moment of inertia
J	=	second moment of inertia
J_d	=	polar moment of inertia
n_x	=	non-dimensional axial force
m_x	=	non-dimensional torsional moment
m_y	=	non-dimensional bending moment about y
m_z	=	non-dimensional bending moment about z
v_z	=	non-dimensional shear force along z

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