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Static Analysis of Gradient Elastic Bars, Beams, Plates and Shells

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Abstract: A review on the response of gradient elastic structural components, such as bars, beams, plates and shells, to static loading is provided. The simplified form II gradient elastic theory of Mindlin with just one elastic constant (the gradient elastic modulus) in addition to the two classical elastic moduli is employed to derive the governing equations of equilibrium and buckling of the aforementioned structural components. All possible boundary conditions (classical and non-classical) are obtained with the aid of variational formulations of the problems associated with these components. Thus, well posed boundary value problems are solved analytically and the response of gradient elastic bars, beams, plates and shells to static loading is determined. In all cases, the effect of the microstructure consists of stiffening the structure, which results in decreasing deflections and increasing buckling loads for increasing values of the gradient elastic modulus.

Keywords: Gradient elasticity, static analysis, stability analysis, bars, beams, plates, shells.

1. INTRODUCTION

Classical theory of elasticity does not take into account the effect of the microstructure of the material and as a result of that this theory is characterized by the local character of stress and the absence of an internal length scale. However, for structural components or structures, such as bars, beams, plates or shells having extremely small overall dimensions comparable to the internal length scale of their material, microstructural effects are important and have to be taken into account when studying their mechanical behavior and response to loading. Structures of this extremely small size find applications in modern nanoelectronic and nanomechanical devices.

For the above type of structures, use of generalized or higher-order theories of linear elasticity is necessary for the study of their mechanical behavior. These theories are characterized by the microstructural effects of non-locality of stress and the existence of internal length scales, i.e., additional elastic moduli with dimensions of length. Among these theories, one can mention here the general elasticity with microstructure due to Mindlin [1], the micropolar elasticity due to Eringen [2], which is similar to that of the Cosserat brothers [3], the couple stress elasticity due to Toupin [4] and Koiter [5] and the nonlocal theory of elasticity due to Eringen [6]. A review of higher-order theories of elasticity can be found, e.g., in the book of Vardoulakis and Sulem [7] and the review article of Lakes [8].

The most general and widely used of all these theories, especially during the last 15 years or so, is that version of Mindlin's [1] theory associated with the second gradient of strain, i.e., the simplified form II theory with a strain energy density depending on strain gradients. The simplified gradient elastic theory of form II due to Mindlin [1], is difficult to be used in practical applications as it contains five constants in addition to the two classical Lamé constants. For this reason, only one or two constants in addition to the two Lamé constants are retained in the theory when applied to practical engineering problems. These constants represent material lengths related to volumetric (most widely used) and surface strain energy [7].

In this review paper, the above gradient elasticity theory with just one constant (the gradient elastic modulus with dimensions of length) in addition to the two classical Lamé constants as applied to structural components, such as, bars, beams, plates and shells under static loading is considered. Only analytical works on the subject are considered with emphasis on the works of the authors and their co-workers. One can mention here the works of Altan *et al.* [9], Tsepoura *et al.* [10] and Papargyri-Beskou and Beskos [11] on bars under tension, Vardoulakis *et al.* [12], Aifantis [13], Papargyri-Beskou *et al.* [14], Vardoulakis and Giannakopoulos [15] and Giannakopoulos and Stamoulis [16] on beams under bending, buckling or torsion, Lazopoulos [17], Papargyri-Beskou and Beskos [18] and Papargyri-Beskou *et al.* [19] on plates under bending including buckling and Papargyri-Beskou and Beskos [20] on buckling of circular cylindrical shells.

For reasons of completeness, one can also mention in this introduction analytic works on static analysis of bars, beams and plates with material behavior based on generalized or higher-order linear theories of elasticity other than the simple gradient elastic theory of Mindlin [1]. Thus, one can mention the works of Ariman [21], Gauthier and Jahsman [22], Krishna Reddy and Venkatasubramanian [23], Yang and Lakes [24], Park and Lakes [25], Lakes [8, 26] and McFarland and Colton [27] using Cosserat / micropolar theories, Ellis and Smith [28], Yang and Lakes [24], Yang *et al.* [29],

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Lam *et al.* [30], Park and Gao [31], Ma *et al.* [32], Kong *et al.* [33] and Tsiatas [34] using couple-stress and modified couple-stress theories and Pisano and Fuschi [35], Peddieson *et al.* [36], Sudak [37], Duan and Wang [38], Reddy [39], Reddy and Pang [40], Wang *et al.* [41] and Zhang *et al.* [42] using nonlocal elasticity theories.

In the following sections, after a brief presentation of the aforementioned simple gradient elasticity theory with just one constant in addition to the two classical elastic constants, the governing equations of equilibrium (including buckling) for gradient elastic bars, beams, plates and cylindrical shells are presented together with their classical and non-classical boundary conditions as derived with the aid of variational principles. Representative boundary value problems associated with the above structures under static loading are considered and the effect of microstructure on their response is assessed. It is found that in all cases, the effect of the microstructure consists of stiffening the structure and consequently on leading to smaller deflections and higher buckling loads.

2. SIMPLE STRAIN GRADIENT ELASTICITY

The simplest possible version of the simplified form II theory of strain gradient linear elasticity due to Mindlin [1] with five constants besides the two Lamé constants is the one with just one constant in addition to the two Lamé constants. The constitutive equations for this simple theory have the form

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - \nabla \cdot \boldsymbol{\mu}$$
$$\boldsymbol{\tau} = 2\,\mu\,\boldsymbol{\varepsilon} + \lambda\, tr\,\boldsymbol{\varepsilon}\mathbf{I} \qquad (1)$$
$$\boldsymbol{\mu} = g^2 \nabla \boldsymbol{\tau} = g^2\left[2\mu\nabla\,\boldsymbol{\varepsilon} + \lambda\nabla\left(\text{tr}\,\boldsymbol{\varepsilon}\right)\mathbf{I}\right]$$

In the above, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the total and the classical Cauchy stress tensors, respectively, $\mathbf{I}$ is the unit tensor, $\boldsymbol{\mu}$ is the double stress tensor and $\boldsymbol{\varepsilon}$ and $\text{tr}\boldsymbol{\varepsilon}$ are the strain tensor and its trace, which are expressed in terms of the displacement vector **u** as

$$\boldsymbol{\varepsilon} = \left(1/2\right)\left(\nabla\mathbf{u} + \mathbf{u}\nabla\right)$$
$$\text{tr}\boldsymbol{\varepsilon} = \nabla \cdot \mathbf{u} \qquad (2)$$

Furthermore, g is the volumetric strain energy gradient modulus or simply gradient elastic modulus, representing the internal or characteristic length of the material microstructure and λ and μ are the two classical Lamé moduli. One should note that, while $\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\varepsilon}$ and $\mathbf{I}$ are second-order tensors, $\boldsymbol{\mu}$ is a third order tensor associated with the microstructure of the material.

When dealing with bars and beams, one has only one normal stress component σ along the longitudinal direction, which, in view of Eqs (1), reads as

$$\sigma = E\,\varepsilon - g^2 E\left(d^2\varepsilon / dx^2\right) \qquad (3)$$

where E is the modulus of elasticity. When dealing with flexural plates or shells, one has a state of plane stress and Eqs (1) can be written in terms of Cartesian plane coordinates x and y as

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) - g^2\frac{E}{1-\nu^2}\nabla^2\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \qquad (4)$$
$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy} - g^2\frac{E}{1+\nu}\nabla^2\varepsilon_{xy}$$

where ν is the Poisson's ratio and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Lamé constants λ and μ are given in terms of E and ν as

$$\lambda = E\nu / \left(1+\nu\right)\left(1-2\nu\right)$$
$$\mu = E / 2\left(1+\nu\right) \qquad (5)$$

3. GOVERNING EQUATION OF AXIAL BAR

The following developments are based on the work of Tsepoura *et al.* [10]. Consider a straight prismatic bar of constant cross section A and length L under a static distributed axial load p creating a tensile stress σ and a displacement u along its longitudinal axis x. On the basis of Eq. (2) one can write the strain-displacement relation

$$\varepsilon = du / dx \qquad (6)$$

The equilibrium equation along the x direction has the simple form

$$A\frac{d\sigma}{dx} + p = 0 \qquad (7)$$

Combining Eqs (3) and (6) one receives

$$\sigma = Eu' - g^2 Eu''' \qquad (8)$$

where primes indicate differentiation with respect to x. Substitution of Eq. (8) in Eq. (7) results in the governing equilibrium equation

$$AE\left(u'' - g^2 u^{IV}\right) + p = 0 \qquad (9)$$

The above equation for g = 0 reduces to the classical case.

The boundary conditions of the axial bar problem, as obtained through a variational principle [10], satisfy the equations

$$\left[P\left(L\right) - AE\left[u'\left(L\right) - g^2u'''\left(L\right)\right]\right]\delta u\left(L\right) -$$
$$\left[P\left(0\right) - AE\left[u'\left(0\right) - g^2u'''\left(0\right)\right]\right]\delta u\left(0\right) = 0 \qquad (10)$$
$$\left[R\left(L\right) - AEg^2u''\left(L\right)\right]\delta u'\left(L\right) -$$
$$\left[R\left(0\right) - AEg^2u''\left(0\right)\right]\delta u'\left(0\right) = 0$$

where P and R are the classical axial force and the double axial force, respectively.

For example, if one assumes the two classical boundary conditions to be u(0) and u(L) prescribed and the corresponding non-classical ones to be $u'\left(0\right)$ and $u'\left(L\right)$ prescribed, then $\delta u(0)=\delta u(L)=0$ and $\delta u'(0)=\delta u'(L)=0$ and Eqs (10) are all satisfied. In view of Eqs (10), one can observe that when dealing with the classical boundary conditions either u or P have to be prescribed at x=0 and x=L, while for the case of non-classical ones either u' or R have to be prescribed at x=0 and x=L.

4. GOVERNING EQUATION OF FLEXURAL BEAM

The following developments are based on the work of Papargyri-Beskou *et al.* [14]. Consider a Bernoulli-Euler prismatic flexural beam of constant cross-section A, moment of inertia I and length L under a distributed lateral load q(x) along its longitudinal axis x, which undergoes a lateral deflection $\upsilon(x)$. The strain-displacement relation reads

$$\varepsilon=-z d^2\upsilon / dx^2 \tag{11}$$

where ε is the normal to the cross-section strain and z is measured along the height of the cross-section of the beam.

The equilibrium equations require that the axial force resultant be zero and the bending moment due to the normal to the cross-section stress σ to be equal to the external moment M, i.e.,

$$\int_A \sigma dA=0 \qquad \int_A \sigma z dA=-M \tag{12}$$

The first of Eqs (12) indicates that the axis x is a centroidal one, while the second one with the aid of Eqs (3) and (11) and the fact that the cross-sectional moment of inertia $I=\int_A z^2 dA$ yields

$$EI\left(-\upsilon''+g^2\upsilon^{IV}\right)=-M \tag{13}$$

Taking into account that $dM/dx=V$ and $dV/dx=-q$, where V is the shear force, one finally obtains from (13), the governing equation of equilibrium in the form

$$EI\left(\upsilon^{IV}-g^2\upsilon^{VI}\right)+q=0 \tag{14}$$

The above equation for g=0 reduces to the classical case of beam bending.

In case there is only a constant axial compressive force P acting on the beam, one has q=0 and can easily obtain with the aid of Eq. (14) the beam buckling governing equation in the form [14]

$$EI\left(\upsilon^{IV}-g^2\upsilon^{VI}\right)+P\upsilon''=0 \tag{15}$$

The boundary conditions of the flexural beam problem, as obtained through a variational principle [14], satisfy the equations

$$\begin{gathered}
\left[V(L)-EI\left[\upsilon'''(L)-g^2\upsilon^{V}(L)\right]\right]\delta\upsilon(L)-\\
\left[V(0)-EI\left[\upsilon'''(0)-g^2\upsilon^{V}(0)\right]\right]\delta\upsilon(0)=0\\
\left[M(L)-EI\left[\upsilon''(L)-g^2\upsilon^{IV}(L)\right]\right]\delta\upsilon'(L)-\\
\left[M(0)-EI\left[\upsilon''(0)-g^2\upsilon^{IV}(0)\right]\right]\delta\upsilon'(0)=0\\
\left[m(L)-EIg^2\upsilon'''(L)\right]\delta\upsilon''(L)-\\
\left[m(0)-EIg^2\upsilon'''(0)\right]\delta\upsilon''(0)=0
\end{gathered} \tag{16}$$

where m is the double bending moment. For example, if one assumes the four classical boundary conditions to be $\upsilon(0)$, $\upsilon(L)$, $\upsilon'(0)$ and $\upsilon'(L)$ prescribed and the corresponding non-classical ones to be $\upsilon''(0)$ and $\upsilon''(L)$ prescribed, then $\delta\upsilon(0)=\delta\upsilon(L)=0$, $\delta\upsilon'(0)=\delta\upsilon'(L)=0$, $\delta\upsilon''(0)=\delta\upsilon''(L)=0$ and Eqs (16) are all satisfied. In view of Eqs (16) one can observe that, when dealing with the classical boundary conditions, either the deflection υ or the shear forces $V=EI\left(\upsilon'''-g^2\upsilon^{V}\right)$ and the strain υ' or the bending moments $M=EI\left(\upsilon''-g^2\upsilon^{IV}\right)$ at the boundary of the beam have to be specified. For the case of the non-classical boundary conditions, one has to specify either the boundary strain gradient υ'' or the boundary double moments $m=EIg^2\upsilon'''$.

When there is an axial compressive force P acting on the beam, the above Eqs (16) are modified by adding the following terms in the left hand side of the first of these equations: $-P\upsilon'(L)\delta\upsilon(L)+P\upsilon'(0)\delta\upsilon(0)$ [10].

5. GOVERNING EQUATION OF FLEXURAL PLATE

The following developments are based on the work of Papargyri-Beskou and Beskos [18]. Consider a Kirchhoff flat, thin, flexural plate of constant thickness h geometrically described by its middle surface (x,y). The plate experiences a lateral deflection w = w(x,y) along the normal axis z to the plane of the plate due to a lateral distributed static load q = q(x,y). For this plate one can write the strain-displacement relations

$$\varepsilon_{xx}=-z\frac{\partial^2 w}{\partial x^2},\quad \varepsilon_{yy}=-z\frac{\partial^2 w}{\partial y^2},\quad \varepsilon_{xy}=-z\frac{\partial^2 w}{\partial x\,\partial y} \tag{17}$$

the equilibrium equations

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}+q=0 \tag{18}$$

and the moment-stress relations

$$\begin{gathered}
M_x=\int_{-h/2}^{h/2}\sigma_{xx} z\,dz,\quad M_y=\int_{-h/2}^{h/2}\sigma_{yy} z\,dz,\\
M_{xy}=-\int_{-h/2}^{h/2}\sigma_{xy} z\,dz
\end{gathered} \tag{19}$$

where σ_{xx}, σ_{yy} are normal and σ_{xy} shear stresses and M_x, M_y are bending and M_{xy} twisting moments.

Introducing Eqs (17) into Eqs (4) and the resulting expressions into Eqs (19), one can obtain the moment-displacement relations in the form

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + g^2 D\left[\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial y^4} + (1+\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2}\right]$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) + g^2 D\left[\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2} + (1+\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2}\right] \tag{20}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} - g^2 D\,(1-\nu)\left(\frac{\partial^4 w}{\partial x^3 \partial y} + \frac{\partial^4 w}{\partial y^3 \partial x}\right),$$

where the flexural rigidity $D = Eh^3/12(1-\nu^2)$. Thus, finally, introducing Eqs (20) into Eq. (18), one can obtain the governing equation of equilibrium in the form

$$D\nabla^4 w - g^2 D \nabla^6 w = q \tag{21}$$

where $\nabla^4 w = \nabla^2(\nabla^2 w)$ and $\nabla^6 w = \nabla^2(\nabla^4 w)$.

The above equation for g=0 reduces to the classical case of plate bending.

In case q=0 and the only loading consists of the in-plane compressive forces P_x, P_y and shear force P_{xy}, one can easily obtain with the aid of Eq. (21) the plate buckling governing equation in the form [18]

$$D\nabla^4 w - g^2 D\nabla^6 w + P_x \frac{\partial^2 w}{\partial x^2} + P_y \frac{\partial^2 w}{\partial y^2} + 2P_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{22}$$

The boundary conditions of the plate bending problem can be obtained through a variational principle [19]. However, they are not repeated here due to their complexity. They basically involve prescribed $w, \partial w/\partial n$, total shear forces V and normal bending moments M_n (classical boundary conditions involving though terms with g) as well as $\partial^2 w/\partial n^2$ and higher order (double) moments M_{nn} (non-classical boundary conditions), with the n being the normal to the boundary unit vector. These conditions can be all of deformation type, all of action type or appropriate combinations of the two types. For example, for a clamped all around circular plate one has the classical w = 0 and $\partial w/\partial n = 0$ and the non-classical $\partial^2 w/\partial n^2 = 0$ boundary conditions. For a simply supported all around circular plate one has the classical w = 0 and $M_n = 0$ and the non-classical $\partial^2 w/\partial n^2 = 0$ or $M_{nn} = 0$ boundary conditions.

6. GOVERNING EQUATION OF CYLINDRICAL SHELL

The following developments are based on the work of Papargyri-Beskou and Beskos [20]. Consider a circular cylindrical thin shell element of constant radius R and thickness h, which is geometrically described by its middle surface (x,y) and experiences displacements u, υ and w along the axes x, y and z, respectively due to a compressive axial force P_x. The x axis is along the axis of the cylinder, the y axis along the circumferential direction, while the z axis along the radial direction.

On the basis of Donnell's theory of circular cylindrical thin shells one has the strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial \upsilon}{\partial y} - \frac{w}{R}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} \tag{23}$$

the equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{24}$$

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

and the force-stress and moment-stress relations

$$N'_x = \sigma_{xx} h, \quad N_y = \sigma_{yy} h, \quad N_{xy} = \sigma_{xy} h \tag{25}$$

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} z\,dz, \quad M_y = \int_{-h/2}^{h/2} \sigma_{yy} z\,dz,$$

$$M_{xy} = -\int_{-h/2}^{h/2} \sigma_{xy} z\,dz$$

where N_x, N_y are the in-plane normal and N_{xy} the in-plane shearing total forces, M_x, M_y are the bending moments and M_{xy} the twisting moments and $N_x = N'_x + P_x$.

Introducing Eqs (23) into Eqs (4) and the resulting equations into Eqs (25) one can obtain the force-displacement relations

$$N'_x = \frac{Eh}{1-v^2}\left(\frac{\partial u}{\partial x} + v\frac{\partial \upsilon}{\partial y} - v\frac{w}{R}\right) - g^2 \frac{Eh}{1-v^2}\nabla^2\left(\frac{\partial u}{\partial x} + v\frac{\partial \upsilon}{\partial y} - v\frac{w}{R}\right)$$

$$N_y = \frac{Eh}{1-v^2}\left(\frac{\partial \upsilon}{\partial y} - \frac{w}{R} + v\frac{\partial u}{\partial x}\right) - g^2 \frac{Eh}{1-v^2}\nabla^2\left(\frac{\partial \upsilon}{\partial y} - \frac{w}{R} + v\frac{\partial u}{\partial x}\right) \tag{26}$$

$$N_{xy} = \frac{Eh}{2(1+v)}\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) - g^2 \frac{Eh}{2(1+v)}\nabla^2\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)$$

and the moment-displacement relations (20). Introducing Eqs (26) and (20) into the equilibrium equations (24) and neglecting higher order terms, one can obtain three coupled equations of equilibrium in terms of u, υ and w. Finally, elimination of u and υ between the above three equations results in the governing equation of equilibrium in terms of w of the form

$$D\nabla^8 w - P_x \nabla^4 \frac{\partial^2 w}{\partial x^2} + \frac{Eh}{R^2}\frac{\partial^4 w}{\partial x^4} - 2g^2 D\nabla w^{10} + g^2 P_x \nabla^6 \frac{\partial^2 w}{\partial x^2} - g^2 \frac{2Eh}{R^2}\left(\frac{\partial^6 w}{\partial x^6} + \frac{\partial^6 w}{\partial x^4 \partial y^2}\right) = 0 \tag{27}$$

where $\nabla^8 w = \nabla^2(\nabla^6 w)$ and $\nabla^{10} w = \nabla^2(\nabla^8 w)$. The above equation for g = 0 reduces to the classical case of buckling of circular cylindrical thin shells.

Concerning the boundary conditions of the static circular cylindrical shell problem, no variational principle has been established as yet so as to have all possible of these conditions. However, for simple problems one can make reasonable assumptions about them and proceed with the solution as shown in the examples section.

7. APPLICATION EXAMPLES

In this section some representative examples are presented taken from previous works of the authors and their co-workers in order to assess the effect of the microstructure on the response of gradient elastic components to static loading.

7.1. Axial Bar in Tension [10]

Consider a bar of length L, built-in at one end (x=0) and under an axial static load P at the other end (x=L). Thus, the classical boundary conditions are

$$u(0)=0,\quad AE\left[u'(L)-g^2u'''(L)\right]=P \tag{28}$$

while the non-classical ones are assumed with the aid of Eqs (10) to be

$$R(0)=AEg^2u''(0)=0,\quad u'(L)=\varepsilon_0 \tag{29}$$

The solution of Eq. (9) with q=0 is

$$u(x)=c_1 e^{x/g}+c_2e^{-x/g}+c_3x+c_4 \tag{30}$$

where the constants c_1-c_4 are obtained with the aid of the boundary conditions (28) and (29) and have the form

$$c_1=-c_2=(g/L)\left[\varepsilon_o-(P/AE)\right]/2\cosh(L/g),\quad c_3=P/AE,\ c_4=0 \tag{31}$$

Fig. (**1**) shows the variation of $\bar{u}=u/(PL/AE)$ versus the dimensionless distance $\xi=x/L$ for various values of g/L for the case of $\varepsilon_o=0.6$. One can observe that the response $\bar{u}$ decreases for increasing values of g/L and this implies that the microstructural effect consists of stiffening the bar.

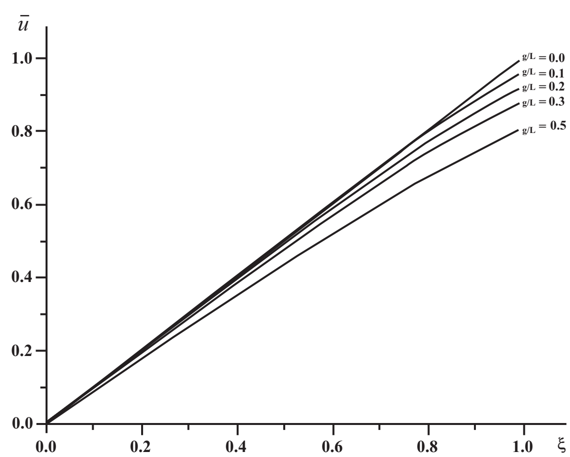

Fig. (1). Axial displacement of the bar along its length for various values of g/L.

7.2. Bending and Buckling of a Beam [14]

Consider a cantilever beam of length L with its built-in end at x=0 under a static uniformly distributed vertical load q. Thus, the classical boundary conditions are

$$\upsilon(0)=\upsilon'(0)=0,\quad M(L)=V(L)=0 \tag{32}$$

while the non-classical ones are assumed with the aid of Eqs (16) to be

$$\upsilon''(0)=\upsilon'''(L)=0 \tag{33}$$

The solution of Eq. (14) is

$$\upsilon(x)=c_1x^3+c_2x^2+c_3x+c_4+c_5g^4 \sinh(x/g)+c_6g^4\cosh(x/g)-(q/24EI)x^4 \tag{34}$$

where the constants c_1-c_6 are obtained with the aid of boundary conditions (32) and (33) and have the form

$$\begin{aligned}
&c_1=qL/6EI,\quad c_2=-(qL^2/4EI)\left[2(g/L)^2+1\right]\\
&c_3=(qL^3/2EI)(g/L)\left[2(g/L)^2+1\right]\tanh(L/g)\\
&c_4=-(qL^4/2EI)(g/L)^2\left[2(g/L)^2+1\right]\\
&c_5=-(q/2EI)\left[2+(L/g)^2\right]\tanh(L/g)\\
&c_6=(q/2EI)\left[2+(L/g)^2\right]
\end{aligned} \tag{35}$$

Fig. (**2**) shows the variation of the lateral displacement υ versus the non-dimensional distance $\xi=x/L$ for various values of g/L. It is observed that the response υ decreases for increasing values of g/L, and this implies that the microsructural effect consists of stiffening the beam.

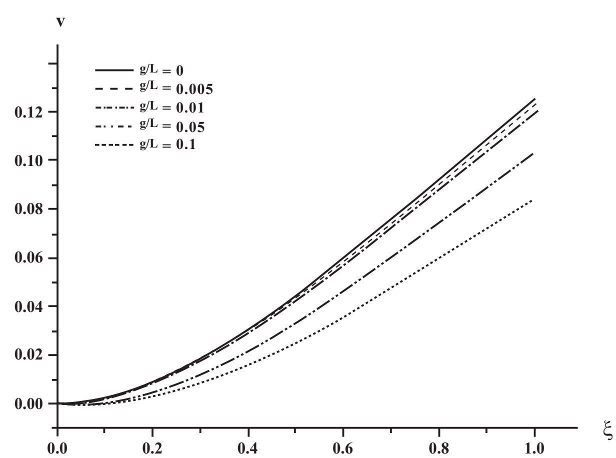

Fig. (2). Lateral deflection of the cantilever beam along its length for various values of g/L.

Consider now a simply supported beam under the action of an axial compressive force P. The solution of the governing equation (15) of this buckling problem is

$$\upsilon(x) = c_1 x + c_2 x^2 + c_3 \sin\xi x + c_4 \cos\xi x + c_5 \sinh\theta x + c_6 \cosh\theta x \quad (36)$$

where

$$\xi = \left(1/\sqrt{2}g\right)\sqrt{-1+\sqrt{1+4g^2k^2}}$$
$$\theta = \left(1/\sqrt{2}g\right)\sqrt{1+\sqrt{1+4g^2k^2}} \quad (37)$$
$$k^2 = P/EI$$

and $c_1 - c_6$ are constants to be determined with the aid of the boundary conditions. The classical boundary conditions are

$$\upsilon(0) = \upsilon(L) = 0, \quad M(0) = M(L) = 0 \quad (38)$$

while the non-classical ones are assumed with the aid of the modified Eqs (16) to be

$$\upsilon''(0) = \upsilon''(L) = 0 \quad (39)$$

Thus, use of Eqs (38) and (39) in (36) results in

$$c_1 = c_2 = c_4 = c_5 = c_6 = 0 \quad (40)$$

$$\upsilon(x) = c_3 \sin\xi x \quad (41)$$

$$\sin\xi L = 0 \quad (42)$$

Equation (42) is used to obtain the buckling load in the form

$$P_{cr}/P_{cr}^{o} = 1 + \pi^2 (g/L)^2 \quad (43)$$

where $P_{cr}^{o} = \pi^2 EI/L^2$ is the classical buckling load. Fig. (**3**) depicts the variation of P_{cr}/P_{cr}^{o} versus $\pi^2 (g/L)^2$ and shows that the buckling load increases for increasing values of g/L. This is in agreement with the previously observed stiffening effect of the microstructure.

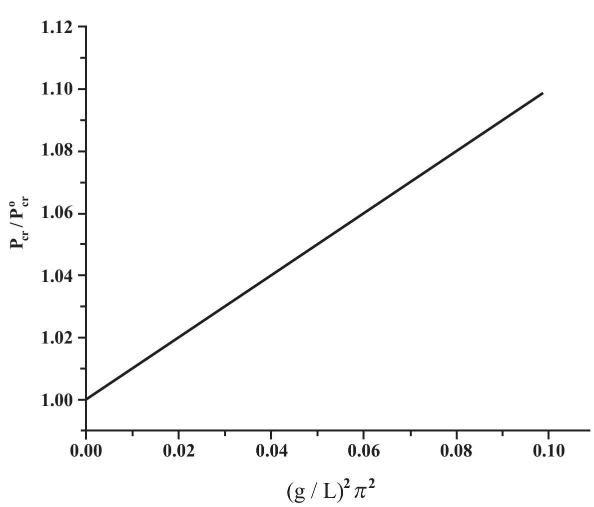

Fig. (3). Buckling (critical) load of the simply supported beam versus $\pi^2 (g/L)^2$.

7.3. Bending and Buckling of a Plate [18, 19]

Consider a simply supported all around rectangular plate with sides a and b along the x and y directions respectively, subjected to a lateral uniformly distributed load q. The classical boundary conditions of the problem are

$$w = 0, \quad M_x = 0 \quad \text{at } x = 0, a \quad (44)$$
$$w = 0, \quad M_y = 0 \quad \text{at } y = 0, b$$

where the moments M_x and M_y are given in terms of the derivatives of w by Eqs (20). The non-classical boundary conditions are assumed with the aid of [19] to be of the type $\partial^2 w/\partial n^2 = 0$ at all sides of the plate, i.e., as

$$w_{xx} = 0, \text{ at } x = 0, a \quad (45)$$
$$w_{yy} = 0, \text{ at } y = 0, b$$

where subscripts indicate differentiation. In addition, on physical grounds, one has that

$$w_{yy} = 0 \quad \text{at } x = 0, a \quad (46)$$
$$w_{xx} = 0 \quad \text{at } y = 0, b$$

meaning that the curvature along any side is zero.

A solution of the governing equation (21) is assumed of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (47)$$

Because of the above form of w, the conditions of w=0 at all sides are automatically satisfied. Furthermore, one can observe that even order derivatives of w with respect to x or y have the same form as w in (47) and this implies that

$$w_{xxxx} = w_{yyyy} = \quad w_{xxyy} = 0 \qquad \text{at } x = 0, a \quad (48)$$
$$w_{yyyy} = w_{xxxx} = w_{yyxx} \qquad \text{at } y = 0, b$$

In view of Eqs (45), (46) and (48), one can see from Eq. (20) that conditions $M_x = 0$ and $M_y = 0$ in (44) are identically satisfied by the chosen form of w. Thus, in conclusion, the expression (47) for w satisfies automatically all boundary conditions.

Assuming that load q can be expressed as

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (49)$$

one can easily find by substituting Eqs (47) and (49) in (21) that

$$W_{mn} = q_{mn}/D\left[\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 + g^2\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^3\right] \quad (50)$$

For the special case of a square plate with b = a and m = n = 1 one can obtain from Eqs (47) and (50) the normalized central deflection W_{11} as

$$w_{11}/w_{11}^{\ c} = 1/\left[1 + 2\pi^2 (g/a)^2\right] \quad (51)$$

where w_{11}^{c} corresponds to the classical case (g = 0). Fig. (**4**) depicts w_{11}/w_{11}^{c} versus g/a and shows that the deflection decreases for increasing values of g/a, again demonstrating the stiffening effect of the microstructure.

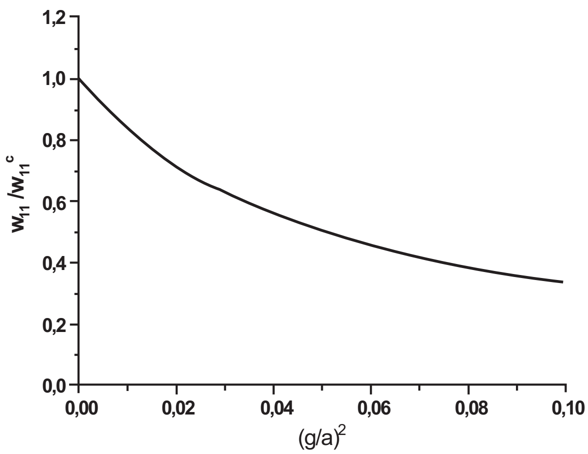

Fig. (4). Central deflection of the simply supported square plate versus $(g/a)^2$.

Consider now the buckling problem of the previous square plate with b=a and q=0 as described by the governing equation (22) with $P_y = P_{xy} = 0$. Assuming again a solution of the form (47) one can easily find out from (22) that the buckling load P_{cr} is

$$P_{cr}/P_{cr}^{o} = 1 + 2\pi\left(g/a\right)^2 \tag{52}$$

where P_{cr}^{o} corresponds to the classical case (g=0). Fig. (**5**) depicts Eq. (52) and shows that P_{cr} increases for increasing values of g/a, exactly as in the case of beams, again indicating that the microstructural effect results in stiffening of the plate.

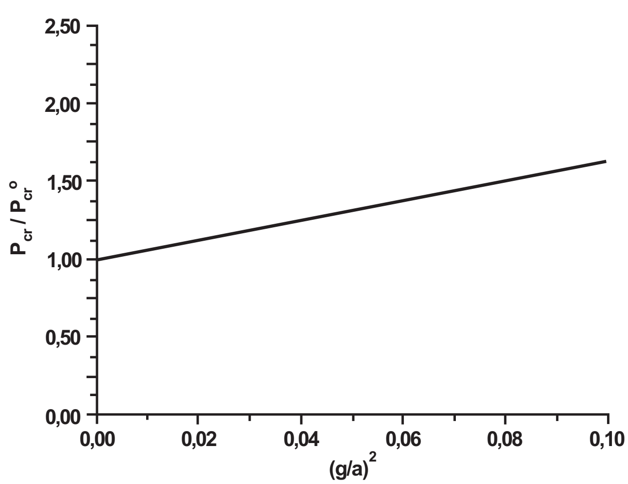

Fig. (5). Buckling (critical) load of the simply supported square plate versus $(g/a)^2$.

7.4. Buckling of a Circular Cylindrical Shell [20]

Consider the buckling problem of the circular cylindrical shell of section 6. It is assumed that the shell has a length L and is simply supported at both ends x=0 and x=L. The classical boundary conditions of the problem are

$$w = 0, \quad M_x = 0 \quad \text{at } x = 0, L \tag{53}$$

where M_x is given by Eq. $(20)_1$. On physical grounds one also has

$$w_{yy} = 0 \quad \text{at } x = 0, L \tag{54}$$

Non-classical boundary conditions would normally involve prescribed values for higher order derivatives of w and/or higher order moments at the two ends x = 0, L. Since higher order moments are expressible in terms of higher order derivatives of w, all boundary conditions can be expressed in terms of higher order derivatives of w. Here these conditions are assumed to be

$$w_{xx} = 0 \quad \text{at } x = 0, L \tag{55}$$

In accordance with the classical case, a solution of Eq. (27) is assumed of the form

$$w = w_o \sin\frac{m\pi x}{L}\sin\frac{\beta\pi y}{L} \tag{56}$$

where w_o denotes the displacement amplitude and

$$\beta = nL/\pi R \tag{57}$$

with m and n positive integers. For the above expression (56) for w one has that w and all second and fourth order derivatives of w become zero at x = 0, L meaning that the assumed solution w satisfies all boundary conditions (classical and non-classical) automatically. Furthermore, the assumed w works with the governing equation (27) which consists of only even order derivatives of w. Indeed, substitution of w in Eq. (27) results in the expression

$$\begin{aligned} K_x = {} & \frac{\left(m^2+\beta^2\right)^2}{m^2\left[1+\bar{g}^2\pi^2\left(m^2+\beta^2\right)\right]} + \\ & \frac{Z^2 m^2}{\left(m^2+\beta^2\right)^2\left[1+\bar{g}^2\pi^2\left(m^2+\beta^2\right)\right]} + \\ & +2\bar{g}^2\pi^2\frac{\left(m^2+\beta^2\right)^3}{m^2\left[1+\bar{g}^2\pi^2\left(m^2+\beta^2\right)\right]} + 2\bar{g}^2\pi^2 \\ & \frac{Z^2\left(m^4+\beta^4\right)}{\left(m^2+\beta^2\right)^2\left[1+\bar{g}^2\pi^2\left(m^2+\beta^2\right)\right]} \end{aligned} \tag{58}$$

where

$$Z = \frac{2\sqrt{3}}{\pi^2}\frac{L^2}{Rh}\sqrt{1-v^2}, \quad K_x = \frac{\sigma_x h L^2}{D\pi^2}, \quad \bar{g} = g/L \tag{59}$$

with $P_x = -\sigma_x h$. For $\bar{g} = 0$ one recovers the classical K_x^c. In order to find the critical value of P_x or equivalently K_x

one has to determine the minimum of the function $K_x = K_x(m, \beta \text{ or } n)$. This is accomplished numerically for the representative case of a shell with h = R/200, L = 5R and v = 0.3. Fig. (**6**) depicts the ratio K_x^m / K_x^{mc} versus $\bar{g} = g / L$, where the superscript m means minimum and c classical. It is observed that the buckling load increases for increasing values of the microstructural parameter $\bar{g}$ as in the cases of beams and plates. However, whereas in beams and plates this ratio increases without bound for large values of g/L, in circular cylindrical shells this ratio attains a finite value. This indicates that the stiffening effect due to the microstructure is bounded.

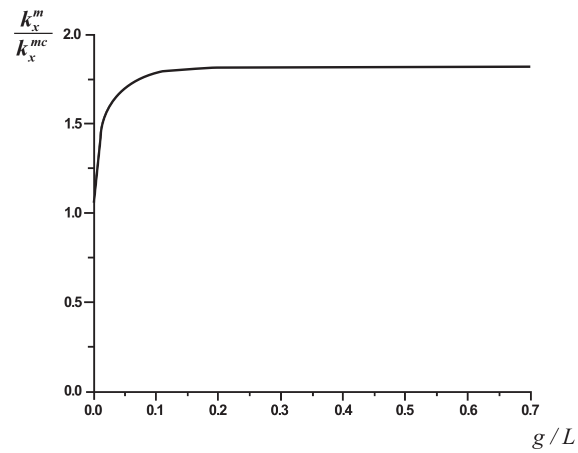

Fig. (6). Buckling (critical) load of the simply supported circular cylindrical shell versus g / L.

8. QUALITATIVE COMPARISONS WITH OTHER HIGHER-ORDER THEORIES

On the basis of all the aforementioned examples and others reported in other works dealing with gradient elasticity([8, 9, 12, 13, 15-17]), one can conclude that for the simple gradient elasticity theories, increasing values of the gradient coefficient, which manifests macroscopically the microstructural effects, lead to the stiffening of the structure and hence to lower deflections and higher buckling loads with respect to the classical case.

It is now interesting to observe what is the effect of the microstructure on the static response of bars, beams and plates on the basis of other higher-order theories. Thus, it is easy to see by looking, e.g., at references [21-27] and [28-34], that the microstructural effect on the basis of Cosserat / micropolar and couple-stress theories, respectively, consists of stiffening the structure and hence lowering deflections and increasing buckling loads, exactly as in the case of the simple gradient elasticity theory. On the other hand, it is also easy to see by looking at references [35-42], that the microstructural effect on the basis of the nonlocal theory of elasticity consists of making the structure more flexible and hence increasing deflections and lowering buckling loads.

The above observations have also been made in particular cases by, e.g., Papargyri-Beskou *et al.* [19] when comparing the response of gradient elastic plates with that of couple-stress and micropolar elastic plates [28, 21] and Papargyri-Beskou and Beskos [20] when comparing buckling of gradient elastic shells and beams with that of nonlocal beams [37].

9. CONCLUSIONS

On the basis of the results of the previous sections the following conclusions can be stated:

1) When the internal length of the material microstructure of a linear elastic bar, beam, plate or circular cylindrical shell is comparable to the overall geometry of that structure, use of higher order elasticity theories is necessary. In this work, the simple gradient elasticity theory with just one constant (gradient coefficient or internal length) besides the two classical elastic constants is successfully employed.

2) The governing equations of equilibrium for gradient elastic bars, beams, plates and circular cylindrical shells are derived and found to be ordinary (for bars and beams) or partial (for plates and shells) differential equations of an order which is higher by two than in the corresponding classical cases. As a result of that, one expects additional non-classical boundary conditions to the classical ones for a well posed boundary value problem. All possible boundary conditions (classical and non-classical) can be rationally obtained with the aid of variational principles.

3) The solution of representative boundary value problems involving the static response and buckling load of gradient elastic bars, beam, plates and circular cylindrical shells demonstrates that the effect of the microstructure consists of stiffening these structures, which thus, exhibit lower deflections and higher buckling loads.

4) Qualitative comparisons of the results of the simple gradient elastic theory used here against those of other higher-order elastic theories on the basis of the static response of bars, beams and plates, reveal that the simple gradient elastic theory produces results in agreement with those of the couple-stress and micropolar elastic theories and exactly opposite with those of the nonlocal elasticity theory.

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