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RESEARCH ARTICLE

On The Distribution of Partial Sums of Randomly Weighted Powers of Uniform Spacings

Emad-Eldin A. A. Aly[1,*]

[1]*Department of Statistics & Operations Research, Faculty of Science, Kuwait University, Safat 13060, Kuwait*

Abstract:

Objectives:

To study the asymptotic theory of the randomly wieghted partial sum process of powers of k-spacings from the uniform distribution.

Methods:

Earlier results on the distribution of the uniform incremental randomly weighted sums.

Methods:

Based on theorems on weak and strong approximations of partial sum processes.

Results and conculsions:

Our first contribution is to classify the multitude of earlier proofs in Section 3. The second contribution consists of a new class of proofs.

Keywords: Uniform spacings, Weak convergence, Gaussian process, Incremental asymptotic convergence, Random Sample, k spacings.

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1. INTRODUCTION

Let $0 = U_{(0)} \le U_{(1)} \le U_{(2)} \le \cdots \le U_{(n-1)} \le U_{(n)} = 1$ be the order statistics of a random sample of size (n-1) from the U(0,1) distribution. Let k=1,2, ... be arbitrary but fixed and assume that n=mk. The U (0,1) k-spacings are defined as

$$R_{i,k} = U_{(ik)} - U_{((i-1)k)}, i = 1,2,\cdots,m. \qquad (1)$$

Let X_1, X_2,... be $iidrv$ with $E(X_i)=\mu$, $\mathrm{Var}(X_i)=\delta^2<\infty$ and common distribution function F(.). Assume that the Xi's are independent of the Ui's. Define

$$S_m(t,k,r,F) = \begin{cases} \sum_{i=1}^{[mt]} R_{i,k}^r X_i, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases} \qquad (2)$$

where [s] is the integer part of s and r>0 is fixed.

Looking at S_m (t,k,r,F) of (2) as a weighted partial sum of the X's, Van Assche [1] obtained the exact distribution of S_2 (1, 1,1, F). Johnson and Kotz [2] studied some generalizations of Van Assche results. Soltani and Homei [3] considered the finite sample distribution of S_n ($1,1,1, F$). Soltani and Roozegar [4] considered the finite sample distribution of a case similar to S_m ($1,k,1, F$) in which the spacings (1) are not equally spaced. It is interesting to note that S_m (t,k,r, F) of (2) is also a randomly weighted partial sum of powers of k-spacings from the U(0,1) distribution.

Here, we will obtain the asymptotic distribution of the stochastic process

$$\alpha_m(t,k,r,F) = \begin{cases} m^{\frac{1}{2}}\{k^r m^{r-1} S_m(t,k,r,F) - t\mu\mu_{r,k}\}, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases} \qquad (3)$$

where

* Address correspondence to this author at Department of Statistics & Operations Research Faculty of Science, Kuwait University, Safat 13060, Kuwait;
Tel: +965 66655671; E-mail: eealy50@gmail.com

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$$\mu_{l,k} = \frac{\Gamma(k+l)}{\Gamma(k)}, k \geq 1\ l > 0 \quad \textbf{(4)}$$

and Γ(.) is the gamma function.

The motivations and justifications of this work are given next. First, as noted by Johnson and Kotz [2], S_2 *(1,1,1, F)* is a random mixture of distributions and as such it has numerous applications in Sociology and in Biology. Second, the asymptotic theory of S_m *(t,k,r, F)* is a generalization of important results of Kimball [5], Darling [6], LeCam [7], Sethuraman and Rao [8], Koziol [9], Aly [10] and Aly [11] for sums of powers of spacings from the *U*(0,1) distribution. Finally, we solve the open problem of proving the asymptotic normality of S_m *(1,k,1, F)* proposed by Soltani and Roozegar [4].

2. METHODS

2.1. The asymptotic distribution of αm (., *k,r, F*)

Let $Y_1, Y_2, \ldots$ be *iidrv* with the exponential distribution with mean 1 which are independent of the *Xi's*. By Proposition 13.15 of Breiman [12] we have for each *n*,

$$\{0 = U_{(0)}, U_{(1)}, U_{(2)}, \cdots, U_{(n-1)}, U_{(n)} = 1\} \stackrel{D}{=} \left\{0, \frac{Y_1}{\sum_{i=1}^{n} Y_i}, \frac{Y_1+Y_2}{\sum_{i=1}^{n} Y_i}, \cdots, \frac{Y_1+Y_2+\cdots+Y_{n-1}}{\sum_{i=1}^{n} Y_i}, 1\right\}.$$

Hence, for each *m*,

$$\{R_{i,k}, 1 \leq i \leq m\} \stackrel{D}{=} \left\{\frac{Z_{i,k}}{\sum_{i=1}^{m} Z_{i,k}}, 1 \leq i \leq m\right\},$$

where for $1 \leq i \leq m$,

$$Z_{i,k} = Y_{(i-1)k+1} + \cdots + Y_{ik}$$

are *iid Gamma* (*k*,1) random variables. Hence, for each *m*

$$S_m(t,k,r,F) \stackrel{D}{=} \sum_{i=1}^{[mt]} Z_{i,k}^r X_i / \left(\sum_{i=1}^{m} Z_{i,k}\right)^r, 0 \leq t \leq 1 \quad \textbf{(5)}$$

and

$$\alpha_m(\cdot,k,r,F) \stackrel{D}{=} \beta_m(\cdot,k,r,F), \quad \textbf{(6)}$$

where

$$\beta_m(t,k,r,F) = \begin{cases} 0, & 0 \leq t < \frac{1}{m}. \\ m^{\frac{1}{2}}\left\{k^r m^{r-1} \sum_{i=1}^{[mt]} Z_{i,k}^r X_i \,/ \left(\sum_{i=1}^{m} Z_{i,k}\right)^r - t\mu\mu_{r,k}\right\}, & \frac{1}{m} \leq t \leq 1 \end{cases} \quad \textbf{(7)}$$

Let $\mu_{l,k}$ be as in (4). Note that

$$E(Z_{i,k}^l) = \mu_{l,k},$$
$$E(Z_{i,k}^r X_i) = \mu\mu_{r,k}, \quad \textbf{(8)}$$
$$\sigma_{r,k}^2 = Var(Z_{i,k}^r X_i) = \sigma^2 \mu_{2r,k} + \mu^2\{\mu_{2r,k} - \mu_{r,k}^2\}$$

and

$$Cov(Z_{i,k}^r X_i, Z_{i,k}) = r\mu\mu_{r,k}.$$

The following Theorem will be needed in the sequel.

Theorem A. There exists a probability space on which a two-dimensional Wiener process $\{\underline{\mathbf{W}}^t(s) = (W_1(s), W_2(s)); s \geq 0\}$ is defined such that

$$\sup_{0 \leq s \leq 1} \left\| \left(\sum_{j=1}^{[ms]} \left(Z_{j,k}^r X_j - \mu\mu_{r,k}\right), \sum_{j=1}^{[ms]} \left(Z_{j,k} - k\right)\right)^t - \underline{\mathbf{W}}^t([ms]) \right\| \stackrel{a.s.}{=} o\left(m^{\frac{1}{4}}\right), \quad \textbf{(9)}$$

where *E* $\underline{W}$ (s)=0, *and*

$$E\underline{\mathbf{W}}(s)\underline{\mathbf{W}}^t(t) = \min(s,t) \begin{bmatrix} \sigma_{r,k}^2 & r\mu\mu_{r,k} \\ r\mu\mu_{r,k} & k \end{bmatrix}. \quad \textbf{(10)}$$

Theorem A follows from the results of Einmahl [13], Zaitsev [14] and Götze and Zaitsev [15].

The main result of this paper is the following Theorem.

Theorem 1. *On some probability space, there exists a sequence of mean zero Gaussian processes $\Gamma_m(t, k, r, F)$, $0 \leq t \leq 1$ such that*

$$\sup_{0 \leq t \leq 1} |\alpha_m(t,k,r,F) - \Gamma_m(t,k,r,F)| \stackrel{P}{=} o\left(m^{-\frac{1}{4}}\right), \quad \textbf{(11)}$$

where $\Gamma_m(t,k,r,F) \stackrel{D}{=} \Gamma(t,k,r,F)$ *for each m, and*

$$E\{\Gamma(t,k,r,F)\Gamma(s,k,r,F)\} = (t \wedge s)\sigma_{r,k}^2 - \frac{r^2\mu^2\mu_{r,k}^2}{k} ts. \quad \textbf{(12)}$$

Theorem 1 follows from (6) and the following Theorem.

Theorem 2. *On the probability space of Theorem A,*

$$\sup_{0 \leq t \leq 1} \left| \beta_m(t,k,r,F) - m^{-\frac{1}{2}}\left\{W_1(mt) - \frac{tr\mu\mu_{r,k}}{k} W_2(m)\right\} \right| \stackrel{a.s.}{=} o\left(m^{-\frac{1}{4}}\right), \quad \textbf{(13)}$$

where $\underline{W}$ *(.) is as in (9).*

Proof of Theorem 2: We will only prove here the case when $E(X)=\mu\neq 0$. The case when $\mu=0$ is straightforward and can be looked at as a special case of the case $\mu\neq 0$. Note that

$$\beta_m(t,k,r,F) = \frac{m^{\frac{1}{2}} k^r A_m(t)}{\left(\frac{1}{m}\sum_{i=1}^{m} Z_{i,k}\right)^r}, \quad \textbf{(14)}$$

where

$$A_m(t) = \frac{1}{m}\sum_{i=1}^{[mt]} Z_{j,k}^r X_j - t\mu\mu_{r,k}\frac{1}{k^r}\left(\frac{1}{m}\sum_{i=1}^{m} Z_{j,k}\right)^r$$
$$= \frac{1}{m}\sum_{i=1}^{[mt]} \left(Z_{j,k}^r X_j - \mu\mu_{r,k}\right) + \mu\mu_{r,k}\frac{([mt]-mt)}{m}$$

$$+\mu\mu_{r,k}t - t\mu\mu_{r,k}\frac{1}{k^r}\left(\frac{1}{m}\sum_{i=1}^{m}\left(Z_{j,k}-k\right)+k\right)^r. \quad \textbf{(15)}$$

It is clear that, uniformly in *t*, 0≤*t*≤1,

$$\frac{|[mt]-mt|}{m} < \frac{1}{m}. \quad \textbf{(16)}$$

By (9), (15) and (16) we have, uniformly in *t*,0≤*t*≤1,

$$A_m(t) \overset{a.s.}{=} \frac{1}{m}W_1(mt) + \frac{1}{m}(W_1([mt]) - W_1(mt)) + O\left(\frac{1}{m}\right) + \mu\mu_{r,k}t - t\mu\mu_{r,k}\left(1+\frac{1}{mk}W_2(m) + o(m^{-\frac{3}{4}})\right)^r + o(m^{-\frac{3}{4}}). \quad \textbf{(17)}$$

By Lemma 1.1.1 of Csörgö and Révész [17] we have, uniformly in *t*,0≤*t*≤1,

$$\frac{1}{m}|W_1([mt]) - W_1(mt)| \overset{a.s.}{=} O(\frac{1}{m}\sqrt{\log m}). \quad \textbf{(18)}$$

By (17) and (18) we have, uniformly in *t*,0≤*t*≤1,

$$\begin{aligned} A_m(t) &\overset{a.s.}{=} \frac{1}{m}W_1(mt) + O(\frac{1}{m}\sqrt{\log m}) + \mu\mu_{r,k}t \\ &- t\mu\mu_{r,k}\left(1+\frac{1}{mk}W_2(m)+o(m^{-\frac{3}{4}})\right)^r + o(m^{-\frac{3}{4}}) \quad \textbf{(19)} \\ &\overset{a.s.}{=} \frac{1}{m}W_1(mt) - \frac{tr\mu\mu_{r,k}}{mk}W_2(m) + o(m^{-\frac{3}{4}}). \end{aligned}$$

By the LIL

$$\begin{aligned} \left(\frac{1}{m}\sum_{i=1}^{m} Z_{i,k}\right)^r &\overset{a.s.}{=} \left(k + O\left(m^{-\frac{1}{2}}\sqrt{\log\log m}\right)\right)^r \\ &\overset{a.s.}{=} k^r + O\left(m^{-\frac{1}{2}}\sqrt{\log\log m}\right). \end{aligned} \quad \textbf{(20)}$$

By (14), (19) and (20) we have, uniformly in *t*,0≤*t*≤1,

$$\begin{aligned} \beta_m(t,k,r,F) &\overset{a.s.}{=} \frac{m^{\frac{1}{2}}k^r}{k^r + O\left(m^{-\frac{1}{2}}\sqrt{\log\log m}\right)}\left\{\frac{1}{m}W_1(mt) - \frac{tr\mu\mu_{r,k}}{mk}W_2(m) + o(m^{-\frac{3}{4}})\right\} \\ &\overset{a.s.}{=} m^{-\frac{1}{2}}\left\{W_1(mt) - \frac{tr\mu\mu_{r,k}}{k}W_2(m)\right\} + o(m^{-\frac{1}{4}}). \end{aligned}$$

This proves (13).

Corollary 1. *By* (4), (8) *and* (12),

$$\Gamma(\cdot,k,r,F) \overset{D}{=} \lambda_{r,k}W(\cdot) + \frac{r\mu\Gamma(r+k)}{\sqrt{k}\Gamma(k)}B(\cdot), \quad \textbf{(21)}$$

where

$$\lambda_{r,k}^2 = \frac{\Gamma(2r+k)}{\Gamma(k)}\sigma^2 + \mu^2\left\{\frac{\Gamma(2r+k)}{\Gamma(k)} - \frac{(r^2+k)\Gamma^2(r+k)}{k\Gamma^2(k)}\right\},$$

W(.) is a Wiener process, B(.) is a Brownian bridge and W(.) and B(.) are independent.

Corollary 2. *By* (11) *and* (21) *we have, as* m→∞,

$$\alpha_m(\cdot,k,r,F) \overset{D}{\to} \Gamma(\cdot,k,r,F) \overset{D}{=} \lambda_{r,k}W(\cdot) + \frac{r\mu\Gamma(r+k)}{\sqrt{k}\Gamma(k)}B(\cdot) \quad \textbf{(22)}$$

and, in particular,

$$\alpha_m(1,k,r,F) \overset{D}{\to} N\left(0,\lambda_{r,k}^2\right). \quad \textbf{(23)}$$

Some special cases of (22) and (23) are given . For r=1 and k≥1,

$$\Gamma(\cdot,k,1,F) = \bar{\sigma}\ \sqrt{k(k+1)}W(\cdot) + \mu\sqrt{k}B(\cdot)$$

and

$\alpha_m(1,k,1,F) \overset{D}{\to} N(0,k(k+1)\sigma^2)$. For $r > 0$ and $k = 1$,

$$\Gamma(\cdot\,,1,r,F) \overset{D}{=} \lambda_{r,1}W(\cdot) + r\mu\Gamma(r+1)B(\cdot)$$

and

$$\alpha_m(1,1,r,F) \overset{D}{\to} N(0,\lambda_{r,1}^2),$$

where

$$\lambda_{r,1}^2 = \sigma^2\Gamma(2r+1) + \mu^2\{\Gamma(2r+1) - (1+r^2)\Gamma^2(r+1)\}.$$

3. RESULTS

In this section, we will use the same notation of Section 1

3.1. The scaled sum case

Define

$$T_{m,1}(t,k,r,F) = \begin{cases} \frac{1}{\sum_{j=1}^{m} X_j}\sum_{i=1}^{[mt]} R_{i,k}^r X_i, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m} \end{cases}$$

and

$$\gamma_{m,1}(t,k,r,F) = \begin{cases} m^{\frac{1}{2}}\left\{\frac{k^r m^r}{\mu_{r,k}}T_{m,1}(t,k,r,F) - t\right\}, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}. \end{cases}$$

We can prove that

$$\gamma_{m,1}(t,k,r,F) \overset{D}{\to} \gamma_1(t,k,r,F),$$

where

$$\gamma_1(t,k,r,F) = \frac{1}{\mu\mu_{r,k}}W_1(t) - \frac{r}{k}tW_2(1) - \frac{1}{\mu}tW_3(1),$$

$(W_1(.),W_2(.),W_3(.))^t$ is a mean zero Gaussian vector with covariance $(t \wedge s)\sum_1$ and

$$\Sigma_1 = \begin{bmatrix} \sigma_{r,k}^2 & r\mu\mu_{r,k} & \sigma^2\mu_{r,k} \\ r\mu\mu_{r,k} & k & 0 \\ \sigma^2\mu_{r,k} & 0 & \sigma^2 \end{bmatrix}.$$

Let

$$\delta_{r,k}^2 = \left(\frac{\mu_{2r,k}}{\mu_{r,k}^2} - 1\right)\left(\frac{\sigma^2}{\mu^2} + 1\right) - \frac{r^2}{k}.$$

We can show that

$$\gamma_1(t,k,r,F) \overset{D}{=} \delta_{r,k}W(t) + \sqrt{\frac{r^2}{k} + \frac{\sigma^2}{\mu^2}}B(t),$$

where $W(.)$ is a Brownian Motion and $B(.)$ is a Brownian bridge and $W(.)$ and $B(.)$ are independent. Consequently,

$$\gamma_{m,1}(1,k,r,F) \overset{D}{\to} N\left(0,\delta_{r,k}^2\right).$$

When r=1,k≥1

$$\delta_{1,k}^2 = \frac{\sigma^2}{k\mu^2}.$$

When r>0,k=1

$$\delta_{r,1}^2 = \left(\frac{\Gamma(2r+1)}{\Gamma^2(r+1)} - 1\right)\left(\frac{\sigma^2}{\mu^2} + 1\right) - r^2.$$

3.2. The Centered Sum Process

Let $\overline{X} = \frac{1}{m}\sum_{j=1}^{m} X_j$ and define

$$T_{m,2}(t,k,r,F) = \begin{cases} \sum_{i=1}^{[mt]} R_{i,k}^r\left(X_i - \overline{X}\right), & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m} \end{cases}$$

and

$$\gamma_{m,2}(t,k,r,F) = \begin{cases} k^r m^{r-\frac{1}{2}} T_{m,2}(t,k,r,F), & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}. \end{cases}$$

We can prove that

$$\gamma_{m,2}(t,k,r,F) \overset{D}{\to} \gamma_2(t,k,r,F),$$

where

$$\gamma_2(t,k,r,F) = W_1(t) - \mu W_2(t) - \mu_{r,k}tW_3(1),$$

$(W_1(.), W_2(.), W_3(.))^t$ is a mean zero Gaussian vector with covariance $(t \wedge s)\sum_2$ and

$$\Sigma_2 = \begin{bmatrix} \sigma_{r,k}^2 & \mu\left(\mu_{2r,k} - \mu_{r,k}^2\right) & \sigma^2\mu_{r,k} \\ \mu\left(\mu_{2r,k} - \mu_{r,k}^2\right) & \mu_{2r,k} - \mu_{r,k}^2 & 0 \\ \sigma^2\mu_{r,k} & 0 & \sigma^2 \end{bmatrix}.$$

We can show that

$$\gamma_2(t,k,r,F) \overset{D}{=} \sigma\left\{\sqrt{\mu_{2r,k} - \mu_{r,k}^2}W(t) + \mu_{r,k}B(t)\right\},$$

where $W(.)$ is a Brownian Motion and B (.) is a Brownian bridge and $W(.)$ and $B(.)$ are independent. Consequently,

$$\gamma_{m,2}(1,k,r,F) \overset{D}{\to} N\left(0,\sigma^2\left(\mu_{2r,k} - \mu_{r,k}^2\right)\right).$$

When r=1,k≥1

$$\gamma_{m,2}(1,k,r,F) \overset{D}{\to} N(0,k\sigma^2).$$

When r> 0,k=1

$$\gamma_{m,2}(1,k,r,F) \overset{D}{\to} N\left(0,\sigma^2(\Gamma(2r+1) - \Gamma^2(r+1))\right).$$

3.3. The Renewal Process

For simplicity, we will consider the case of r=1. Define

$$S_m^*(t) = \begin{cases} \frac{1}{\mu}\sum_{i=1}^{[mt]} R_{i,k}X_i, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases}$$

$$T_m^*(t) = \begin{cases} \frac{1}{\mu}\sum_{i=1}^{[mt]} Z_{i,k}X_i/\left(\sum_{i=1}^{m} Z_{i,k}\right), & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases}$$

$$N_m(t) = \inf\{u: S_m^*(u) > t\},$$

$$M_m(t) = \inf\{u: T_m^*(u) > t\},$$

$$\alpha_m^*(t) = \begin{cases} m^{\frac{1}{2}}k\mu\{S_m^*(t) - t\}, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases}$$

$$\beta_m^*(t) = \begin{cases} m^{\frac{1}{2}}k\mu\{T_m^*(t) - t\}, & \frac{1}{m} \le t \le 1 \\ 0, & 0 \le t < \frac{1}{m}, \end{cases}$$

$$\eta_m(t) = m^{\frac{1}{2}}k\mu\{t - N_m(t)\}$$

and

$$\xi_m(t) = m^{\frac{1}{2}}k\mu\{t - M_m(t)\}.$$

By (5), for each m

$$\alpha_m^*(\cdot) \overset{D}{=} \beta_m^*(\cdot) \text{ and } \eta_m(\cdot) \overset{D}{=} \xi_m(\cdot). \quad \textbf{(24)}$$

Note that (see (3))

$$\alpha_m^*(\cdot) = \alpha_m(\cdot,k,1,F)$$

and hence, by Theorem 1

$$\sup_{0\le t\le 1}|\alpha_m^*(t) - \Gamma_m(t,k,1,F)| \overset{P}{=} o\left(m^{-\frac{1}{4}}\right),$$

where Γ_m (., k, 1, F) is as in (11).

Theorem 3. *On the probability space of Theorem A,*

$$\sup_{0\le t\le 1}|\eta_m(t) - \Gamma_m(t)| \overset{P}{=} o\left(m^{-\frac{1}{4}}(\log m \log\log m)^{\frac{1}{2}}\right),$$

where

$$\Gamma_m(t) = m^{-\frac{1}{2}}\{W_1(mt) - t\mu W_2(m)\} \quad \textbf{(25)}$$

and $\underline{W}(.)$ is as in (9).

Theorem 3 follows directly from (24) and the following Theorem.

Theorem 4. *On the probability space of Theorem A,*

$$\sup_{0\le t\le 1}|\xi_m(t)-\Gamma_m(t)| \overset{a.s.}{=} O\left(m^{-\frac{1}{4}}(\log m\log\log m)^{\frac{1}{2}}\right),$$

where $\Gamma_m(t)$ is as in (25).

Proof: By (7),

$$\beta_m^*(\cdot)=\beta_m(\cdot,k,1,F).$$

Note that

$$\xi_m(t)=\beta_m^*(M_m(t))-m^{\frac{1}{2}}k\mu\{T_m(M_m(t))-t\}.$$

Hence

$$\sup_{0\le t\le 1}|\xi_m(t)-\Gamma_m(t)|\le E_{m1}+E_{m2}+E_{m3}, \tag{26}$$

where

$$E_{m1}=\sup_{0\le t\le 1}|\beta_m^*(M_m(t))-\Gamma_m(M_m(t))|,$$

$$E_{m2}=m^{\frac{1}{2}}k\mu\sup_{0\le t\le 1}|T_m(M_m(t))-t|$$

and

$$E_{m3}=\sup_{0\le t\le 1}|\Gamma_m(M_m(t))-\Gamma_m(t)|.$$

By Theorem 2 and the LIL for Wiener processes,

$$E_{m1}\overset{a.s.}{=}o\left(m^{-\frac{1}{4}}\right). \tag{27}$$

and

$$\sup_{0\le t\le 1}|T_m(t)-t|\overset{a.s.}{=}O\left(\sqrt{m^{-1}\log\log m}\right)$$

By a Lemma of Horváth [18]

$$\sup_{0\le t\le 1}|M_m(t)-t|\le \sup_{0\le t\le 1}|T_m(t)-t|$$

and hence

$$\sup_{0\le t\le 1}|M_m(t)-t|\overset{a.s.}{=}O\left(\sqrt{m^{-1}\log\log m}\right). \tag{28}$$

By the proof of Step 5 of Horváth [18] and Theorem 2 we can show that

$$E_{m2}\overset{a.s.}{=}O\left(m^{-\frac{1}{4}}\log m\right). \tag{29}$$

As to E_{m3},

$$E_{m3}\le E_{m31}+E_{m32}, \tag{30}$$

where

$$E_{m31}=\sup|W_1(M_m(t))-W_1(t)|$$

and

$$E_{m32}=m^{-\frac{1}{2}}\mu|W_2(m)|\sup|M_m(t)-t|.$$

By (28) and Lemma 1.1.1 of Csörgö and Révész [17] we have, uniformly in $t, 0\le t\le 1$,

$$E_{m31}=\sup|W_1(t+(M_m(t)-t))-W_1(t)|$$

$$\overset{a.s.}{=}\sup_{0\le h\le m^{-\frac{1}{2}}(\log\log m)^{\frac{1}{2}}}|W_1(t+h)-W_1(t)|$$

$$\overset{a.s.}{=}O\left(m^{-\frac{1}{4}}\sqrt{\log m\log\log m}\right). \tag{31}$$

By (28) and the LIL for Wiener processes,

$$E_{m32}\overset{a.s.}{=}O\left(m^{-\frac{1}{2}}\log\log m\right). \tag{32}$$

By (30)-(32),

$$E_{m3}\overset{a.s.}{=}O\left(m^{-\frac{1}{4}}\sqrt{\log m\log\log m}\right). \tag{33}$$

By (26)-(33) we obtain Theorem 4.

4. THE RANDOM VECTOR CASE

Let $\underline{X}_1$, $\underline{X}_2$,... be *iid* random vectors with $E(\underline{X}_i)=\underline{\mu}=(\mu_1,\mu_2,\cdots,\mu_p)^t$ and $Var(\underline{X}_i)=\Sigma=[\sigma_{ij}]$. Assume that the U_is and the $R_{i,k}$s are same as in Section 1 and are independent of $\underline{X}_1$, $\underline{X}_2$,... Define

$$\underline{S}_m(t,k,r,F)=\begin{cases}\sum_{i=1}^{[mt]}R_{i,k}^r\underline{X}_i, & \frac{1}{m}\le t\le 1\\ 0, & 0\le t<\frac{1}{m}\end{cases}$$

and

$$\underline{\alpha}_m(t,k,r,F)=\begin{cases}0, & 0\le t<\frac{1}{m}.\\ m^{\frac{1}{2}}\left\{k^rm^{r-1}\underline{S}_m(t,k,r,F)\right. & \\ \left.\quad -t\mu_{r,k}\underline{\mu}\right\}, & \frac{1}{m}\le t\le 1\end{cases}$$

Theorem 5 is a generalization of Theorem 1.

Theorem 5. On some probability space, there exists a mean zero sequence of Gaussian processes $\{\underline{\Gamma}_m(t,k,r,F), 0\le t\le 1\}$ such that

$$\sup_{0\le t\le 1}\left\|\underline{\alpha}_m(t,k,r,F)-\underline{\Gamma}_m(t,k,r,F)\right\| \overset{P}{=} o\left(m^{-\frac{1}{4}}\right),$$

where, for each m,

$$\underline{\Gamma}_m(\cdot,k,r,F) \overset{D}{=} \underline{\Gamma}(\cdot,k,r,F),$$

$$E\underline{\Gamma}(s,k,r,F)\underline{\Gamma}^t(t,k,r,F) = (t\wedge s)\textstyle\sum_{r,k}^{(1)} - \frac{r^2\Gamma^2(r+k)}{k\Gamma^2(k)} ts\underline{\mu\mu}^t$$

and

$$\textstyle\sum_{r,k}^{(1)} = \frac{\Gamma(2r+k)}{\Gamma(k)}\sum + \left\{\frac{\Gamma(2r+k)}{\Gamma(k)} - \frac{\Gamma^2(r+k)}{\Gamma^2(k)}\right\}\underline{\mu\mu}^t.$$

Corollary 1 *. *By* (11) *and* (21) *we have, as* m→∞,

$$\underline{\alpha}_m(\cdot,k,r,F) \overset{D}{\to} \underline{\Gamma}(\cdot,k,r,F)$$

and, in particular,

$$\underline{\alpha}_m(1,k,r,F) \overset{D}{\to} MVN\left(\underline{0},\textstyle\sum_{r,k}^{(2)}\right),$$

where

$$\textstyle\sum_{r,k}^{(2)} = \frac{\Gamma(2r+k)}{\Gamma(k)}\sum + \left\{\frac{\Gamma(2r+k)}{\Gamma(k)} - \frac{(k+r^2)\Gamma^2(r+k)}{k\Gamma^2(k)}\right\}\underline{\mu\mu}^t.$$

Particular cases of Corollary 1* are given next.

For $r = 1$ and $k \ge 1$,

$$E\underline{\Gamma}(s,k,1,F)\underline{\Gamma}^t(t,k,1,F) = (t\wedge s)\textstyle\sum_{1,k}^{(1)} - tsk\underline{\mu\mu}^t,$$

$$\textstyle\sum_{1,k}^{(1)} = k(k+1)\sum + k\underline{\mu\mu}^t$$

and

$$\underline{\Gamma}(1,k,1,F) \overset{D}{=} MVN\left(\underline{0},k(k+1)\textstyle\sum\right).$$

For $r > 0$ and $k = 1$,

$$E\underline{\Gamma}(s,1,r,F)\underline{\Gamma}^t(t,1,r,F) = (t\wedge s)\textstyle\sum_{r,1}^{(1)} - tsr^2\Gamma^2(r+1)\underline{\mu\mu}^t,$$

$$\textstyle\sum_{r,1}^{(1)} = \Gamma(2r+1)\sum + \{\Gamma(2r+1)-\Gamma^2(r+1)\}\underline{\mu\mu}^t$$

and

$$\underline{\Gamma}(1,1,r,F) \overset{D}{=} MVN\left(\underline{0},\textstyle\sum^*\right),$$

where

$$\textstyle\sum^* = \Gamma(2r+1)\sum + \{\Gamma(2r+1) - (1+r^2)\Gamma^2(r+1)\}\underline{\mu\mu}^t.$$

CONCLUSION

We proved the weak convergence of a stochastic process defined in terms of partial sums of randomly weighted powers of uniform spacings. The asymptotic results of several important generalizations and special cases are given.

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Not applicable.

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Not applicable.

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CONFLICT OF INTEREST

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