On the order of the recursion relation of Motzkin numbers of higher rank

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Abstract. For Motzkin paths with up- and down-steps of heights 1 and 2, the minimal recursion is of order 6, not of order 4, as conjectured by Schork.

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The classical Motzkin numbers count the numbers of Motzkin paths: We consider in the Cartesian plane \( \mathbb{Z} \times \mathbb{Z} \) those lattice paths starting at \((0,0)\) that use an up-step \((1,1)\), a down-step \((1,-1)\), and a level-step \((1,0)\). Motzkin paths of length \(n\) are built of these, lead to \((n,0)\) and never go below the \(x\)-axis.

Now we consider higher rank Motzkin numbers, as suggested by Schork [2]: There are up-steps \((1,a_1)\), \((1,a_2)\), \ldots, \((1,a_r)\) with respective weights \(a_1, \ldots, a_r\), down-steps \((1,-a_1)\), \((1,-a_2)\), \ldots, \((1,-a_r)\) with respective weights \(c_1, \ldots, c_r\), and a level-step \((1,0)\) with weight \(b\).

Let us first consider the classical case \(r=1\). The generating function \(M(z)\) of these paths satisfies the equation

\[
M = 1 + bzM + azMczM,
\]

whence

\[
1 - bz - \sqrt{1 - 2bz + b^2z^2 - 4az^2c} = \frac{2az^2c}{2az^2c}.
\]

This equation is obtained by a decomposition of the Motzkin paths with respect to the first return to the \(x\)-axis.

Schork’s first problem is to find a recursion for the numbers \(m_n = [z^n]M(z)\). (The coefficient of \(z^n\) in the power series expansion of \(M(z)\), i.e., the number of (weighted) Motzkin paths of length \(n\).)

This can be automatically solved with Maple’s program \texttt{gfun} (written by Salvy et al.): The procedure \texttt{algeqtodiffeq} translates the (algebraic) equation for \(M(z)\) into an equivalent differential equation:

\[
2 + (3bz - b^2z^2 + 4az^2c - 2)M + (-z + 2bz^2 - z^3b^2 + 4z^3ac)M' = 0.
\]

The procedure \texttt{diffeqtorec} translates the differential equation into a recursion:

\[
(-b^2 + 4ac)(n+1)m_n + (5b + 2bn)M_{n+1} - (n+4)m_{n+2} = 0,
\]

which solves already this first problem.

Now let us move to the instance \(r=2\). Let us assume that the weights are all 1, so that we are just interested to count the number of (generalized) Motzkin paths. In the paper [1] we find the equation for the generating function:

\[
z^4M^4 - z^2(1 + z)M^3 + z(2 + z)M^2 - (1 + z)M + 1 = 0.
\]

Thus (again with \texttt{gfun})

\[
-4 - 100z^2 + 56z + (3750z^6 - 5000z^5 + 250z^4 + 700z^3 + 160z^2 - 92z + 4)M + (-328z^2 + 32z - 15250z^6 - 20z^3 + 4750z^4 + 11250z^7 - 650z^5)M' = 0.
\]

\[\text{After sending a draft of this note to M. Schork, he informed me that he could now also establish this recurrence together with Mansour and Sun.}\]
\[ + (5625z^8 - 7750z^7 - 1200z^6 + 3880z^5 \\ - 395z^4 - 180z^3 + 26z^2)M'' \\ + (625z^9 - 875z^8 - 250z^7 \\ + 610z^6 - 91z^5 - 23z^4 + 4z^3)M''' = 0 \]

and

\[ 625(n + 3)(n + 2)(n + 1)m_n \\ - 125(n + 3)(n + 2)(7n + 27)m_{n+1} \\ - 50(n + 3)(5n^2 + 24n + 23)m_{n+2} \\ + (41890 + 30860n + 7540n^2 + 610n^3)m_{n+3} \\ + (-6844 - 5151n - 1214n^2 - 91n^3)m_{n+4} \\ - (n + 7)(23n^2 + 301n + 976)m_{n+5} \]

\[ + 2(2n + 13)(n + 8)(n + 7)m_{n+6} = 0. \]

(This recursion also appears in [1].)

Bruno Salvy has kindly informed me that this recursion of order 6 is minimal.

Schork [2] conjectured that there should be a \((2r + 1)\)-term recursion (=order \(2r\)). Thus, the conjecture does not hold.

References
