The Co-Design Method for Robust Satisfactory $H_{\infty}/H_2$ Fault-Tolerant Event-Triggered Control of NCS with $\alpha$-Safety Degree

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Abstract: Based on discrete event-triggered communication scheme (DETCS), this paper is concerned with the satisfactory $H_{\infty}/H_2$ event-triggered fault-tolerant control problem for networked control system (NCS) with $\alpha$-safety degree and actuator saturation constraint from the perspective of improving satisfaction of fault-tolerant control and saving network resource. Firstly, the closed-loop NCS model with actuator failures and actuator saturation is built based on DETCS; Secondly, based on Lyapunov-Krasovskii function and the definition of $\alpha$-safety degree given in the paper, a sufficient condition is presented for NCS with the generalized $H_2$ and $H_{\infty}$ performance, which is the contractively invariant set of fault-tolerance with $\alpha$-safety degree, and the co-design method for event-triggered parameter and satisfactory fault-tolerant controller is also given in this paper. Moreover, the simulation example verifies the feasibility of improving system satisfaction and the effectiveness of saving network resource for the method. Finally, the compatibility analysis of the related indexes is also discussed and analyzed.

Keywords: Satisfactory $H_{\infty}/H_2$ fault-tolerant control, Co-design and compatibility analysis, Actuator saturation constraint, Discrete event-triggered communication scheme.

1. INTRODUCTION

With the continuous improvement of safety and reliability for networked control system (NCS), the study [1, 2] of fault-tolerant control of NCS is receiving more and more attention in recent decade. Moreover, there are essential differences of transmission media between the NCS and the traditional point-to-point system, and then the traditional fault-tolerant control theory and method are no longer suitable for the NCS. Thus many endeavors have been also devoted to the research of fault-tolerant control for NCS in [3, 4]. However, most of the results are based on the periodic time-triggered communication scheme (PTTCS) at present. Although the PTTCS has the universality of application and convenience of analysis, there are also some shortcomings in this scheme. For example, if all the sampled data is transmitted through a fixed sampling period, then it will lead to the high occupancy rate of network communication resource, high transmission rate of redundant data and so on. In addition, if the system selects the PTTCS, the controller is designed in an isolated way depending on the existing quality of service (QoS) for network, which can not take the compromise balance between the quality of control (QoC) for system into consideration and the QoS for network.

In order to solve the above problems, a series of event-triggered schemes are put forward by some researchers in [5-8], where the discrete event-triggered communication scheme (DETCS) is getting more and more favor of people with the advantages of simple calculation of event-triggered condition and no continuous monitoring of the system state. The DETCS can not only save the limited network resource, but also make the system design give consideration to two things of QoS for network and QoC for system. The DETCS in which the system state information is monitored only at discrete instance, was firstly proposed in [6], and the design method of controller was also given. Not long ago, a few scholars applied the DETCS to the fault-tolerant control for NCS in [9, 10].

Under the promise of ensuring the NCS in a safe and reliable work state, the satisfactory performance is also our anticipating goal for an actual fault-tolerant control system. Thus, similar to the design of traditional fault-tolerant control system, the idea of satisfactory fault-tolerant control [11, 12] was gradually introduced into the study field of NCS in [13, 14], namely, the NCS with failures can not only continue to operate in a safe and stable way, but also satisfy several constraints of performance indexes simultaneously, such as pole assignment, $\alpha$-stability, the $H_{\infty}$ and $H_2$ performance index.

Reviewing the existing fault-tolerant control results for NCS, it is not hard to get the conclusion as follows. On the
one hand, although the results achieved higher satisfaction of control performance based on PTTCs, it can not take the QoS into consideration for network in [13, 14]; On the other hand, although the QoC for system and the QoS for network were both taken into consideration, the system performance was limited to the integrity in [9, 10]. In addition, the above results didn't consider the problem of actuator saturation. However, the actuator saturation is a potential factor of system performance deterioration and system instability in [15, 16]. Because the redundant actuator shares the responsibility for the failure actuator in the fault-tolerant control system, the actuator is more liable to enter the saturated region. In such cases, it is urgently needed to consider the actuator saturation constraint in the actual design process of the fault-tolerant control system. From what has been discussed above, under the premise of considering actuator failures and actuator saturation, the co-design of network resource saving and satisfactory fault-tolerant control has been studied based on DETCS for NCS in this paper, which can implement the goal of achieving the compromise balance between QoC for system and QoS for network.

2. PROBLEM DESCRIPTION AND SOME PRELIMINARIES

2.1. The Description of Uncertain Closed-Loop Fault NCS Under DETCS

In this paper, suppose that the controlled plant model with actuator saturation constraint is given by

\[
\begin{align*}
\dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)\text{sat}(u(t)) + Ew(t) \\
z_i(t) &= C_i x(t) \\
z_2(t) &= C_2 z_1(t)
\end{align*}
\]

(1)

where \(x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, z_i(t) \in \mathbb{R}^p \) and \(z_2(t) \in \mathbb{R}^p \) denote the state vector, control input, the first controlled output and the second controlled output, respectively; The function \(\text{sat}() : \mathbb{R}^m \rightarrow \mathbb{R}^m \) denotes the standard multivariable saturation function defined as \(\text{sat}(u) = [\text{sat}(u_1), \text{sat}(u_2), \ldots, \text{sat}(u_n)]^T \), where \(\text{sat}(u_i) = \text{sign}(u_i) \min \{1, |u_i|\} \); \(w(t) \in \mathbb{R}^q \) is the external disturbance which belongs to \(L_2[0, \infty) \); \(A, B, E \) and \(C \) are parameter matrices with appropriate dimensions; \(\Delta A \) and \(\Delta B \) denote the uncertainty of system parameter, and it is assumed as norm-bounded. They are time-varying, and described as

\[
[\Delta A, \Delta B] = MF(t) [E_1, E_2]
\]

(2)

Where \(M, E_1 \) and \(E_2 \) have suitable dimensions and are constant matrices with real value; \(F(t) \) is unknown time-varying real continuous matrix function, and its elements are Lebesgue measurable; Meanwhile, \(F(t) \) satisfies \(F^T(t)F(t) \leq I \), where \(I \) is the unit matrix.

As is known to all, there are some problems in PTTCs, such as the waste of network bandwidth resource and the phenomenon of network congestion. In order to solve the above problems, it is necessary to introduce a new communication scheme with constraint into fault-tolerant control system, which can decide whether or not to send the state system information through the network. Based on [8], a kind of DETCS is presented, and its architecture is shown in Fig. (1).

Different from the traditional NCS, the event generator of Fig. (1) is located between the sensor and actuator. The function of the event generator is to judge whether or not to send out the latest sampled information to controller. We adopt the following event-triggered condition.

\[
[x(ih) - x(t_kh)]^T \Phi[x(ih) - x(t_kh)] \leq \sigma x^T(ih) \Phi x(ih)
\]

(3)

where \(\Phi \) is a symmetric positive definite matrix; \(\sigma \) is the event-triggered parameter which is a bounded positive scalar; \(h \) denotes the sampling period which drives the clock of sensor. \(x(ih) \) and \(x(t_kh) \) denote the current sampled information and the latest transmission information, respectively, where \(i_1h = t_1h = h, i = 1, 2, L \); \(\{t_ih, t_2h, t_2h, L \} \) is the release instant set of the data transmission; \(t_ih - t_2h \) denotes the release period \(h_i \) given by the condition (3) at the time \(t_ih \).
Suppose that the system state is completely measured and the system adopts the static state-feedback controller. We set the comprehensive time-delay as $\tau_q = \tau_q^m + \tau_q^a$ at the time $t_q$, where $\tau_q^m$, $\tau_q^a$ denote the transmission time-delay between the sensor and the controller, the transmission time-delay between the controller and the actuator respectively, and $\tau_q^a$ denotes the calculation time-delay. Meanwhile, considering the role of zero-order-holder, when $t \in [t_q + \tau_q^a + \tau_q^m, t_{k+1} + \tau_q^a]$, the control input can be expressed as

$$u(t) = Kx(t_q)$$

where $K$ denotes state-feedback control gain matrix.

Based on the above description, when $x(t_q)$ has reached the actuator and yet $x(t_{k+1})$ doesn’t arrive at the actuator, we define the keep interval as

$$\Omega = [t_q + \tau_q^a + \tau_q^m, t_{k+1} + \tau_q^a]$$

$$\Omega = \Delta_k^0 \cup \Delta_k^1 \cup \ldots \cup \Delta_k^{d_k}$$

where $\Delta_k^l = [t_q + \tau_q^a + \tau_q^m, t_{k+1} + \tau_q^a]$, $i_k = t_q + \tau_q^a$, $l_k = t_{k+1} - t_q - 1$; $\tau_q^a$ is the virtual network transmission delay at sample instant $i_k$.

When $t \in \Delta_k^l$, the function $\tau(t)$ is defined as

$$\tau(t) = t - i_k$$

According to (6) and (7), we describe $\tau(t)$ from two aspects of upper bound and lower bound as follows.

$$\tau_1 < \tau_q^a \leq \tau(t) \leq h + \tau_q^a + h \leq \tau_2$$

where $\tau_q^a$ is the virtual network transmission delay at sample instant $i_k$.

When $t \in \Delta_k^l$, the state-error $e(i_k)$ is defined as:

$$e(i_k) = x(t_q) - x(t_{k+1})$$

In the process of state transformation, if the following conditions are satisfied for system with any possible actuator failure in mode set $L$, the system meets $\alpha$-safety degree. It means that all closed-loop poles of the system $s_i (i = 1, 2, \ldots, n)$ satisfy $\text{Re}(s_i) < -\alpha$ and $\alpha > 0$ for the system with any possible actuator failure in mode set $L$.

Definition 1: If the $\alpha$ of $\alpha$-stability is defined as the system stability margin for a system without failure, then the $\alpha$ of $\alpha$-stability can be extended as the system safety margin for the system with any possible actuator failure in mode set $L$, and the system safety margin also can be abbreviated to $\alpha$-safety degree. It means that all closed-loop poles of the system $s_i (i = 1, 2, \ldots, n)$ satisfy $\text{Re}(s_i) < -\alpha$ and $\alpha > 0$ for the system with any possible actuator failure in mode set $L$.

Definition 2: In the process of state transformation, if the following conditions are satisfied for system with any possible actuator failure in mode set $L$,

The state trajectory that its initial state is from any point of set $R^n$ will converge to the equilibrium point, namely

$$\varphi_{\alpha} = \{x_0 \in R^n : \lim_{t \to \infty} \psi(t, x_0) = 0, \forall L\}$$

then $\varphi_{\alpha}$ is defined as fault-tolerant domain of attraction with $\alpha$-safety degree, where $\psi(t, x_0)$ is the corresponding state trajectory.

Definition 3: In the process of state transformation, if the following conditions are satisfied for system with any possible actuator failure in mode set $L$,

$$L = \text{diag}\{l_1, \ldots, l_m\}, \quad l_q \in [0, 1], \quad q = 1, 2, \ldots, m$$

Matrix $L$ denotes the mode set of system actuator failures and describes the fault extent, where $l_q = 0$ denotes that the actuator $q$ is a complete failure, $l_q \in (0, 1)$ means that the actuator $q$ is at fault state to some extent and $l_q = 1$ implies that the actuator $q$ operates in a normal state.

Combining (1), (11) with (12), the model of networked closed-loop fault system (NCFS) with actuator saturation constraint can be obtained based on DETCS as follows.

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)\text{sat}(K(x(t) - \tau(t)))$$

$$-e(i_k) + Ew(t)$$

where $t \in \Delta_k^l$; The initial state $x(t)$ is denoted by $\Psi(t)$, where $t \in [-\tau_2, 0]$. Meanwhile, set $\Psi(0)$ as $x_0$, where $\Psi(t)$ is continuous function in the interval $[-\tau_2, 0]$.

Remark 1: According to (14), many factors can be integrated into a unified framework, such as the condition of communication constraint, network time-delay, actuator saturation, actuator failure and the controller. The framework lays a solid foundation for the following co-design and compatibility analysis.
the system possesses $\alpha$-safety degree.

The state trajectory that its initial state is from any point of set $\varrho_{a_2}$ will remain inside the set $\varrho_{a_2}$, namely
\[ x_0 \in \varrho_{a_2} \Rightarrow x(t) \in \varrho_{a_2}, \forall t \geq 0, L \]

The state trajectory that its initial state is from any point of set $\varrho_{a_2} \setminus \{0\}$ will converge to the equilibrium point, namely
\[ x_0 \in \varrho_{a_2} \setminus \{0\} \Rightarrow \lim_{t \to \infty} \psi(t,x_0) = 0, \forall L \]

then $\varrho_{a_2}$ is defined as the contractively invariant set of fault-tolerant with $\alpha$-safety degree, where $\psi(t,x_0)$ is the corresponding state trajectory.

The contractively invariant set of fault-tolerant with $\alpha$-safety degree is inside the fault-tolerant domain of attraction with $\alpha$-safety degree. In general, it is difficult to get the corresponding fault-tolerant domain of attraction, thus the fault-tolerant domain of attraction with $\alpha$-safety degree can be estimated by the corresponding contractively invariant set of fault-tolerant.

If $1(F) = \{x_0 \in R^d \mid f_j x \leq 1, j = 1, 2, \ldots, m\}$, where matrix $F \in R^{m \times d}$, and $f_j$ denotes the $j$th row of matrix $F$, then $1(F)$ is defined as the area in which the feedback control $u = sat(Fx)$ is linear for $x$ in [17].

Based on ellipsoid estimation for domain of attraction, $P \in R^{m \times m}$ is one positive definite matrix. For $\rho > 0$, ellipsoid is defined as $\varepsilon(P, \rho) = \{x \in R^d, x^TPx \leq \rho\}$ where $\varepsilon(P)$ denotes $\varepsilon(P, 1)$.

**Lemma 1 [18]:** Given two feedback matrices $K \in R^{m \times n}$, $F \in R^{m \times d}$, if $x \in 1(F)$, then
\[ sat(Kx) \in co\{Y_iKx + Y_i^TFx : i = 1, 2, \ldots, 2^n\} \]

where $co\{\cdot\}$ denotes the convex hull of a group of linear feedback $Y_iKx + Y_i^TFx, Y_i \in Y$, $i = 1, \ldots, 2^n; Y$ denotes the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0; There are $2^n$ elements in $Y$. Suppose that each element of $Y$ is labeled as $Y_i$ for $i = 1, 2, \ldots, 2^n$, then $Y = \{Y_i : i \in \{1, 2, \ldots, 2^n\}\}$. Define $Y_i = I - Y_i$, clearly, if $Y_i \in Y$, $Y_i^T$ is also an element of $Y$.

**Lemma 2 [19]:** For any constant matrix $Z \in R^{m \times n}$, $Z = Z^T > 0$, scalar $\delta > 0$, vector function $x : [0, \delta] \to R^n$, such that the integrations in the following are well defined, then
\[ \int_0^\delta x^T(s)Zx(s)ds \geq \left(\int_0^\delta x^T(s)ds\right)^TZ\left(\int_0^\delta x(s)ds\right) \]

3. MAIN RESULTS

3.1. The Goal of Co-Design of Satisfactory Fault-Tolerant Event-Triggered Control and Network Communication

Based on DETCS, considering the constraint of actuator saturation and actuator failure, the goal of co-design of satisfactory $H_{\infty}/H_2$ fault-tolerant event-triggered control and network communication is achieved by seeking the control gain $K$ and the event-triggered weight matrix $\Phi$ in DETCS, which makes sure that the NCFS (14) satisfies the following conditions.

1. For allowable uncertainty of parameter, when $w(t) = 0$, the NCFS (14) possesses $\alpha$-safety degree.

2. If the system meets zero initial condition for any non-zero $w(t) \in L_2[0, \infty)$, then the system satisfies $\|z_1(t)\|_2 \leq \gamma_1\|w(t)\|_2$, where $\gamma_1$ is a given scalar, and $\|\cdot\|_2$ denotes $L_2[0, \infty]$ norm. The scalar $\gamma_1$ is the disturbance rejection level.

3. If the system meets zero initial condition for any non-zero $w(t) \in L_1[0, \infty)$, then the system satisfies $\|z_2(t)\|_2 \leq \gamma_2\|w(t)\|_2$, where $\gamma_2$ is a given scalar, and $\|\cdot\|_2$ denotes $L_1[0, \infty]$ norm. The condition can guarantee that the peak output will be smaller than a certain value. The scalar $\gamma_2$ can also be seen as the output peak rejection level.

4. Based on the premise of satisfying the above indexes and all the indexes possessing compatibility, the co-design method makes sure that the occupancy rate of network resource will be as few as possible.

3.2. The Condition of Invariant Set

**Theorem 1:** Under DETCS, considering system(14), for the given constants $r_1, r_2, r, \sigma, \alpha, \gamma_1$ and $\gamma_2$, and the given matrices $K$ and $\Phi$, existing matrices $P = P^T$, $Z = Z^T$, $Q = Q^T > 0$, $Q_1 > 0$, $Q_2 > 0$, if the above parameters satisfy the following matrix inequalities and
\(\varepsilon(P) \subset (F)\) for any possible actuator failure in mode set \(L\) and acceptable uncertainty of system parameter,

\[
\Xi_0 = \begin{bmatrix}
\Xi_{11}^{(0)} & \Xi_{12}^{(0)} & Z_1 & 0 & 0 & \Xi_{20}^{(0)} \\
* & \Xi_{22}^{(0)} & Z_2 & 2Z_0 & 0 & 0 \\
* & * & \Xi_{33}^{(0)} & 0 & 0 & 0 \\
* & * & * & \Xi_{44}^{(0)} & 0 & 0 \\
* & * & * & * & \Xi_{55}^{(0)} & 0 \\
* & * & * & * & * & -\Phi \\
\end{bmatrix} \leq 0 \quad (22)
\]

\[
\Xi_i = \begin{bmatrix}
\Xi_{11}^{(i)} & \Xi_{12}^{(i)} & Z_1 & 0 & 0 & \Xi_{20}^{(i)} \\
* & \Xi_{22}^{(i)} & Z_2 & 2Z_0 & 0 & 0 \\
* & * & \Xi_{33}^{(i)} & 0 & 0 & 0 \\
* & * & * & \Xi_{44}^{(i)} & 0 & 0 \\
* & * & * & * & \Xi_{55}^{(i)} & 0 \\
* & * & * & * & * & -\Phi \\
\end{bmatrix} \leq 0 \quad (23)
\]

\[
\begin{bmatrix}
P & C_2^T \\
* & \gamma_i^2 I
\end{bmatrix} \geq 0 \quad (24)
\]

then the state trajectory of the NCFS (14) is inside the domain of attraction \(\varepsilon(P)\), and possesses \(\alpha\) -safety degree, \(\gamma_i\) - disturbance rejection level, \(\gamma_i\) -output peak rejection level and as few as possible quantity of data communication. Namely, (11) is a satisfactory fault-tolerant control law under DETCS which can make the uncertain NCFS (14) satisfy the constraint conditions 1), 2), 3) and 4) of the design goal, where

\[
\begin{align*}
\Xi_{11}^{(i)} &= p\tilde{A} + \tilde{A}^TP + Q_i + Z_i + C_i^TC_i, \Xi_{22}^{(i)} = -(1-\mu)Q_i - 3Z_2 + \sigma \Phi, \\
\Xi_{12}^{(i)} &= p\overline{B}(\alpha K + \gamma_i^2 F)i, \Xi_{10}^{(i)} = -p\overline{B}_1(\alpha K + \gamma_i^2 F), \\
\Xi_{33}^{(i)} &= -Q_i + C_i + Z_i - Z_2, \Xi_{35}^{(i)} = -Q_i + C_i + Q_i - Z_i - 2Z_2, \\
\Xi_{44}^{(i)} &= -Q_i - 2Z_2, \Xi_{44}^{(i)} = -Q_i - 2Z_2, \Xi_{55}^{(i)} = \tau_i^2Tz_i, \tau_i = \tau_i - \tau_i.
\end{align*}
\]

**Proof:** In order to make the system (14) possess ( \(w(t) = 0\) ) \(\alpha\)-safety degree without disturbance, the state transformation \(x(t) = \exp(-\alpha t)u(t)\) is introduced into the derivation process in this paper. Based on Lemma 1, when satisfy \(\varepsilon(P) \subset (F)\), then

\[
\begin{align*}
\dot{\varepsilon}(t) &= \tilde{A}\varepsilon(t) + \overline{B}co(\gamma_i K + \gamma_i^2 F)\varepsilon(t - \tau(t)) \\
&\quad - \overline{B}co(\alpha K + \gamma_i^2 F)e_i(i_kh) \\
\varepsilon(t) &= C_iu(t) \\
\varepsilon(t) &= C_iu(t)
\end{align*}
\]

where \(\tilde{A} = A + \Delta A + \alpha I\), \(\overline{B} = \exp(\alpha T_i(t))(B + \Delta B)L\), \(\exp(\alpha T_i(t)) = \exp(\alpha T_i(t))e_i(i_kh)\), \(\varepsilon(t) = \exp(\alpha T_i(t))z_2(t) = \exp(\alpha T_i(t)).\)

If the system (25) possesses asymptotically stable performance, then the system (14) will possess \(\alpha\) -safety degree according to the Definition 1.

Construct the following Lyapunov-Krasovskii function of system (25) for \(t \in [t_1h + \tau_i h + t_{i+1}]\), we obtain

\[
\begin{align*}
\dot{V}(t) &= v^T(t)Pu(t) + \int_{t_i}^{t_{i+1}} v^T(s)Q_i\varphi(s)ds \\
&\quad + \int_{t_i}^{t_{i+1}} v^T(s)\varphi(s)\varepsilon(s)ds + \int_{t_i}^{t_{i+1}} v^T(s)Q_i\varphi(s)ds \\
&\quad + \int_{t_i}^{t_{i+1}} \tau_i v^T(s)Z_i\varphi(s)ds + \int_{t_i}^{t_{i+1}} \tau_i v^T(s)Z_i\varphi(s)ds + \int_{t_i}^{t_{i+1}} \tau_i v^T(s)Z_i\varphi(s)ds
\end{align*}
\]

where \(P^T = P > 0\), \(Q_i = Q_i^T > 0\), \(Q_i = Q_i^T > 0\), \(Q_i = Q_i^T > 0\), \(Q_i = Q_i^T > 0\), \(Z_i = Z_i^T > 0\), \(Z_i = Z_i^T > 0\).

Take the difference of Lyapunov-Krasovskii function (26) along the system (25)

\[
\begin{align*}
\dot{V}(t) &= 2v^T(t)P\dot{u}(t) + v^T(t)Q_iu(t) \\
&\quad - v^T(t - \tau_i)Q_iu(t - \tau_i) + v^T(t - \tau_i)Q_iu(t - \tau_i) \\
&\quad - v^T(t - \tau_i)Q_iu(t - \tau_i) + v^T(t - \tau_i)Q_iu(t - \tau_i) \\
&\quad - (1 - \mu)v^T(t - \tau_i)Q_iu(t - \tau_i) + \tau_i v^T(t)Z_i\dot{u}(t) \\
&\quad - \tau_i v^T(s)Z_i\dot{u}(s)ds + \tau_i v^T(t)Z_i\dot{u}(t) \\
&\quad - \tau_i v^T(s)Z_i\dot{u}(s)ds + \gamma_i^2(t)\Phi u_i(i_kh) \\
&\quad - \gamma_i^2(t)\Phi u_i(i_kh)
\end{align*}
\]

According to Lemma 1, we have

\[
2v^T(t)P\dot{u}(t) \leq \max_{i \in [1,2,3]^{2m}} \{2v^T(t)P\overline{A}u(t) \\
+ 2v^T(t)P\overline{B}(\gamma_i K + \gamma_i^2 F)u(t - \tau(t)) \\
- 2v^T(t)P\overline{B}(\gamma_i K + \gamma_i^2 F)e_i(i_kh)\}
\]

According to Lemma 2, we have

\[
\begin{align*}
-\tau_i v^T(s)Z_i\dot{u}(s)ds &\leq \left[\begin{array}{c}
u(t) \\
\dot{u}(t - \tau_i)
\end{array}\right]^{-T} \begin{bmatrix}
-Z_i & Z_i \\
* & -Z_i
\end{bmatrix} \begin{bmatrix}
u(t) \\
\dot{u}(t - \tau_i)
\end{bmatrix} \\
-\tau_i v^T(s)Z_i\dot{u}(s)ds &\leq -(u(t - \tau_i) - u(t - \tau_i))Z_i (u(t - \tau_i) - u(t - \tau_i)) \\
&\quad - u(t - \tau_i) - u(t - \tau_i))Z_i (u(t - \tau_i) - u(t - \tau_i)) - \\
&\quad (1 - \beta)(u(t - \tau_i) - u(t - \tau_i))Z_i (u(t - \tau_i) - u(t - \tau_i))
\end{align*}
\]
where $\beta = (\tau(t) - \tau_i) / \tau_i$.

When $t \in [t_i, t_i + \tau_i]$, according to (10) and $e^a(i,h) = \exp(\alpha(t - \tau(t)))e(i,h)$, we obtain

$$e^T_a(i,h)\Phi e(i,h) \leq \sigma V(t - \tau(t))\Phi V(t - \tau(t))$$  \hspace{1cm} (31)

Combining (27)-(31), we have

$$\dot{V}(u(t)) \leq \max_{i \in [1,2,\ldots,m]} \xi^T(i)(\beta \xi + 1 - \beta)\tilde{\xi}(t)$$  \hspace{1cm} (32)

where

$$\xi = [v^T(t) v^T(t - \tau_1) v^T(t - \tau_2)]^T$$

If $\xi < 0$ and $\tilde{\xi} < 0$, we can verify that the system (25) possesses asymptotic stable performance by adopting Lyapunov stable theory, namely, the system (14) possesses $\alpha$-safety degree and satisfies the condition (1) of design goal.

Then, following the similar procedure of Theorem 1 in [13], the proof can be completed without any difficulty.

### 3.3. The Co-Design Method

**Theorem 2:** Under DETCS, consider the system (14), for the given constants $\tau$, $\bar{\tau}$, $h$, $\sigma$, $\alpha$, $\gamma_1$ and $\gamma_2$, existing matrices $R > 0$, $R > 0$, $X = X^T > 0$, $V = V^T > 0$, $R = R^T > 0$, $R = R^T > 0$ and positive real number $\varepsilon > 0$. If the above parameters satisfy the following linear matrix inequalities for any possible actuator failure in mode set $L$ and acceptable uncertainty of parameter,

$$\Xi = \begin{bmatrix}
\Pi_3 & \Theta_3 & \Theta_3^* \\
* & \Lambda_3 & 0 \\
* & * & \Lambda_3^*
\end{bmatrix} < 0$$  \hspace{1cm} (33)

Then there is a feedback control law (11) which can make the state trajectory of the NCFS(14) with actuator saturation remaining inside the ellipsoid $e(P)$, and make the system possess $\alpha$-safety degree, $\gamma_1$-disturbance rejection level, $\gamma_2$-output peak rejection level and as few as possible quantity of data communication. Moreover, it is feasible to get the satisfactory $H_1$-$H_2$ fault-tolerant event-triggered controller gain $K$ and the discrete event-triggered weight matrix $\Phi$ by $K = \bar{K} = 1$ and $\Phi = \bar{\Phi}$, where

$$\Pi_3 = (\alpha + \alpha I)X + X(\alpha + \alpha I)^T - \bar{R}_1 + \bar{R}_2,$$

$$\Pi_1 = \exp(\alpha(t))BL(\gamma, \bar{R} + \gamma, \bar{F}),$$

$$\Pi_1 = 2X - \bar{R}_1, \Pi_3 = -(\alpha(t))BL(\gamma, \bar{R} + \gamma, \bar{F}),$$

$$\Pi_3 = (1\mu)(-2X + \bar{R}_1) + (-6X + 3\bar{R}_1) + \sigma(\alpha(t)) (2X - \bar{R}_1),$$

$$\Pi_2 = (X - \bar{R}_1), \Pi_3 = 4X - 2\bar{R}_1,$$

$$\Pi_3 = 4X - 2\bar{R}_1, \Pi_3 = 2X - \bar{R}_1,$$

$$\Pi_3 = -2X + \bar{R}_1 - \bar{R}_1 + \bar{R}_1 + \bar{R}_1,$$

$$\Pi_3 = 2X - \bar{R}_1 - \bar{R}_1 + \bar{R}_1 + \bar{R}_1,$$

$$\Pi_4 = 0X + \bar{R}_1 + \bar{R}_1, \Pi_3 = 4X + \bar{R}_1 + \bar{R}_1,$$

$$\Pi_5 = 2X - \bar{R}_1, \Pi_3 = -(2X - \bar{R}_1), \Pi_5 = -(2X - \bar{R}_1),$$

$$\Theta = \begin{bmatrix}
E^T & 0 & 0 & 0 & 0 & 0 \\
0 & C_1 & 0 & 0 & 0 & 0
\end{bmatrix}^T$$

$$\Theta_3 = \begin{bmatrix}
M^T & 0 & 0 & 0 & 0 & 0 \\
0 & E & \Omega_1 & 0 & 0 & \Omega_2
\end{bmatrix}^T$$

$$\Omega_3 = \exp(\alpha(t))E_2L(\gamma, \bar{R} + \gamma, \bar{F}),$$

$$\Omega_3 = -\exp(\alpha(t))E_2L(\gamma, \bar{R} + \gamma, \bar{F}),$$

$$\Lambda_3 = \text{diag}(-\gamma_1^2I, -I), \Lambda_3^* = \text{diag}(-\varepsilon^2I, -\varepsilon I)$$

**Proof:** Substituting $\bar{A} = A + \Delta A + \alpha I$, $\bar{B} = \exp(\alpha(t))$ $(B + \Delta B)L$ and (2) into $\Xi$, we have (37) according to Lemma 3,

$$\Xi = \Xi \exp(\Psi T) + \exp(\Psi T) \Xi T$$  \hspace{1cm} (37)

Limited to the space, the related expressions of $\Xi$, $\Psi$ and $\Psi$, are omitted.
Furthermore applying Schur complement, we obtain

$$
\Xi_j = \begin{bmatrix}
\Pi_j & \Theta_j & \Theta_j^* \\
* & \Lambda_j & 0 \\
* & * & \Lambda_j^*
\end{bmatrix}
$$

(38)

where the related expressions of $\Pi_j$, $\Theta_j$, $\Theta_j^*$, $\Lambda_j$ and $\Lambda_j^*$ are also omitted.

Then, similar to the proof procedure of [13, 21], we can complete the remaining proof without any difficulty.

**Remark 2:** For the given constants $\gamma_1$, $\gamma_2$, $\alpha$ and $\sigma$, the satisfactory $H_\infty / H_2$ event-triggered fault-tolerant controller given in Theorem 2 is only a $\gamma_i$-suboptimal fault-tolerant controller under DETCS, where $i = 1, 2$. For given $\alpha$, $\gamma_j$ and $\sigma$, $\gamma_j$ ($i = 1, 2; j = 1, 2$, and $i \neq j$) is optimized by (39)

$$
\min_{\gamma_j} \gamma_j \text{ s.t. } (33) \sim (36), X > 0, R_i \\
> 0, V > 0 (i = 1, \cdots, 5)
$$

(39)

Moreover, it is possible to get the optimal fault-tolerant controller and the optimal event-triggered weight matrix with the minimum rejection level $\gamma_{j_{\text{min}}}$ for NCFS(14).

**Remark 3:** If $\sigma = 0$ in (3), the model of (14) will not contain the constraint condition of communication, which makes the fault-tolerant design for the NCFS (14) under DETCS, degenerate to the case of PETCS. Thus according to Theorem 1 and Theorem 2, a series of fault-tolerant criterions and the corresponding controller design methods are obtained by setting various combinations of performance indexes under DETCS and PETCS. Due to limited space, no further details are provided here.

4. SIMULATION EXPERIMENT AND RESULT ANALYSIS

4.1. The Simulation Experiment

In order to verify the correctness of the co-design method provided in Theorem 2, we studied a classic networked control system in [13], where

$$
A = \begin{bmatrix}
-1.3 & -0.5 \\
0.7 & -1.8
\end{bmatrix},
B = \begin{bmatrix}
1 & 0.5 \\
0 & 1
\end{bmatrix},
M = \begin{bmatrix}
0.2 & 0 \\
0 & 0
\end{bmatrix},
$$

$$
F(t) = \begin{bmatrix}
\sin t & 0 \\
0 & \cos t
\end{bmatrix},
E_i = \begin{bmatrix}
0 & 0.1 \\
0 & 0.1
\end{bmatrix},
E_2 = \begin{bmatrix}
0 & 0.1 \\
0 & 0.1
\end{bmatrix},
$$

$$
C_1 = \begin{bmatrix}
0.9 & 0 \\
0 & 0.1
\end{bmatrix},
C_2 = \begin{bmatrix}
0.9 & 0 \\
0 & 0.1
\end{bmatrix},
E = \begin{bmatrix}
-0.4 & 0 \\
0.8 & 0.5
\end{bmatrix}
$$

Considering the system with disturbance input, the disturbance is described as:

$$
u(t) = \sin(2\pi t) \exp(-0.2t), 10 \leq t \leq 20
$$

For actuator normal or failures, $L$ is defined as follows:

$$
L_0 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
L_1 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix},
L_2 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix},
$$

(40)

Setting $x_0 = [1, -1]^T$, $h = 0.1s$, $\tau = 0.5s$, $\tau_1 = 0.01s$, $\tau_2 = 0.6s$; $\gamma_1 = 0.9$, $\gamma_2 = 0.5$, $\alpha = 0.1$, $\sigma = 0.6$, the simulation will be performed.

According to the Theorem 2, we can compute the satisfactory fault-tolerant controller gain matrix $K$ and the discrete event-triggered weight matrix $\Phi$ by solving the linear matrix inequalities (33) ~ (36).

$$
K = \begin{bmatrix}
0.0165 & 0.0012 \\
0.0033 & 0.0362
\end{bmatrix},
\Phi = \begin{bmatrix}
2.1709 & 0.3547 \\
0.3547 & 2.4191
\end{bmatrix}
$$

(41)

When actuator is normal or failures, the curves of states responses for NCFS (14) are shown in Fig. (2) by adopting controller $K$.

The optimal performance index of $H_\infty$ is $\gamma_{\text{min}} = 0.4450$ ($\gamma_2 = 0.5$) for NCFS (14) from (39), and the optimal satisfactory fault-tolerant event-triggered controller gain matrix and the event-triggered weight matrix are

$$
K_{\text{opt}} = \begin{bmatrix}
0.0289 & 0.0010 \\
-0.0151 & 0.0856
\end{bmatrix},
\Phi_{\text{opt}} = \begin{bmatrix}
0.9253 & -0.2709 \\
-0.2709 & 0.7717
\end{bmatrix}
$$

(42)

Similarly, the optimal performance index of $H_2$ is $\gamma_{2_{\text{min}}} = 0.4591$ ($\gamma_1 = 0.9$) for NCFS (14) from (39), and the optimal satisfactory fault-tolerant event-triggered controller gain matrix and the event-triggered weight matrix are

$$
K_{2\text{opt}} = \begin{bmatrix}
0.0349 & -0.0031 \\
-0.0083 & 0.0517
\end{bmatrix},
\Phi_{2\text{opt}} = \begin{bmatrix}
0.5191 & -0.1056 \\
-0.1056 & 0.3775
\end{bmatrix}
$$

(43)

Setting the simulation time as $t = 30s$, for the given $h$, $\alpha$, $\gamma_1$, $\gamma_2$ and $\sigma$ as above, the release time and release interval of data communication are shown in Fig. (3) for NCFS(14) based on DETCS.

4.2. The Analysis of Simulation Results

1) From the simulation curves of Fig. (2), even though the quantity of data transmission is reduced under DETCS, the NCFS (14) with constraint of actuator saturation can not only maintain the asymptotic stability, but also possess the
more satisfying safety degree, disturbance rejection level, output peak rejection level and dynamic performance. In the practical application, under the promise of $\gamma_1$ and $\gamma_2$ given in advance, the safety level and dynamic performance of NCS with failure can be also improved by appropriate increase in the safety degree $\alpha$.

2) For the given event-triggered parameter $\sigma = 0.6$ and simulation time $t_s = 30s$, in contrast to the PTTCS in which all of the 300 data need to be transmitted, there is only 83 data transmitted by the network in DETCS as shown in Fig. (3). It is shown that the DETCS driven by control demand can effectively reduce the network load and calculation resource in contrast to the PTTCS driven by physical clock.

It is seen that the method as described above can complete the co-design of satisfactory fault-tolerant controller $K$ and the event-triggered weight matrix $\Phi$ in DETCS, which can achieve the goal of taking two things of QoS and QoC for system into consideration for network.

Table 1. The comparison results of relevant variable under different triggering parameter $\sigma$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>300</td>
<td>135</td>
<td>104</td>
<td>83</td>
<td>74</td>
</tr>
<tr>
<td>$r_{v_{st}}$</td>
<td>100%</td>
<td>45%</td>
<td>34.7%</td>
<td>27.8%</td>
<td>24.7%</td>
</tr>
<tr>
<td>$\overline{h}$</td>
<td>0.100</td>
<td>0.225</td>
<td>0.292</td>
<td>0.366</td>
<td>0.407</td>
</tr>
<tr>
<td>$h_{\max}$</td>
<td>0.1</td>
<td>0.6</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. (2). The curves of state response for NCFS.

Fig. (3). Based on DETCS, the release time and release interval of transmission data in NCS.
Which indicates that there is a kind of interaction relation of the system performance and occupation of network communication. Therefore, we need to pay attention to the compromise balance will deteriorate. Therefore, when selecting a value for \( h_s \), we increase source is occupied for the given extent of state changing, the fewer \( h_s \). Furthermore, it indicates that the larger the allowable event generator.

\[
\gamma_{1 \text{min}}(\gamma_1 = 0.9) = 0.4384, \quad \gamma_{2 \text{min}}(\gamma_1 = 0.9) = 0.4485
\]

Table 2. The comparison results of \( \gamma_{1 \text{min}} \) under different safety degree \( \alpha \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \gamma_{1 \text{min}}(\gamma_1 = 0.9) )</th>
<th>( \gamma_{2 \text{min}}(\gamma_1 = 0.9) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0 )</td>
<td>0.4384</td>
<td>0.4485</td>
</tr>
<tr>
<td>( \alpha = 0.1 )</td>
<td>0.4450</td>
<td>0.4591</td>
</tr>
<tr>
<td>( \alpha = 0.2 )</td>
<td>0.4542</td>
<td>0.4728</td>
</tr>
<tr>
<td>( \alpha = 0.3 )</td>
<td>0.4684</td>
<td>0.4918</td>
</tr>
</tbody>
</table>

Table 3. A comparison of \( \gamma_{j \text{min}} \) for different \( \gamma_i \).

<table>
<thead>
<tr>
<th>( \gamma_1 )</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{2 \text{min}} )</td>
<td>0.4591</td>
<td>0.4971</td>
<td>0.5510</td>
<td>0.6320</td>
<td>0.7685</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>( \gamma_{1 \text{min}} )</td>
<td>0.4450</td>
<td>0.4842</td>
<td>0.5428</td>
<td>0.6344</td>
<td>0.7938</td>
</tr>
</tbody>
</table>

4.3. The Discussion of Compatibility Analysis

1. For given \( h = 0.1s \), \( \alpha = 0.1 \), \( \gamma_1 = 0.9 \), \( \gamma_2 = 0.5 \), when we take different value for \( \sigma \), the comparative results of relevant variable are shown in Table 1.

   It is seen from Table 1 that with the increase of \( \sigma \), namely the allowable stationary of state changing declines, the average release period \( \overline{r} \) and the maximum release period \( h_{\text{max}} \) become larger, meanwhile, the triggering times of event generator and the data transmission ratio \( r_e \) become fewer. Furthermore, it indicates that the larger the allowable extent of state changing, the fewer the communication resource is occupied for the given \( \alpha \), \( \gamma_1 \) and \( \gamma_2 \). However, if we increase \( \sigma \) too much, then the system dynamic stability will deteriorate. Therefore, when selecting a value for \( \sigma \), we need to pay attention to the compromise balance between system performance and occupation of network communication.

2. For given \( h = 0.1s \) and \( \sigma = 0.8 \), when \( \alpha \) takes different value, we can obtain the minimum disturbance rejection level \( \gamma_{1 \text{min}} \) according to (39) for \( \gamma_2 = 0.9 \). Similarly, we can obtain the minimum output peak rejection level \( \gamma_{2 \text{min}} \) according to (39) for \( \gamma_1 = 0.9 \). The comparative results of relevant variable for \( \gamma_{i \text{min}}(i = 1,2) \) are shown in Table 2.

   With the increase of safety degree \( \alpha \) seen from Table 2, the \( \gamma_{1 \text{min}} \) and \( \gamma_{2 \text{min}} \) of NCFS(14) become larger gradually, which indicates that there is a kind of interaction relation of \( \alpha \) and \( \gamma_{i \text{min}}(i = 1,2) \). Therefore, we also need to pay attention to seek the compromise balance between \( \alpha \) and \( \gamma_{i \text{min}}(i = 1,2) \) in the design of actual system, namely, we shouldn't increase \( \alpha \) blindly to pursue the better dynamic performance by sacrificing the rejection level of \( \gamma_{i \text{min}}(i = 1,2) \).

3. For given \( h = 0.1s \), \( \alpha = 0.1 \), \( \sigma = 0.8 \), we can get the minimum disturbance rejection level \( \gamma_{1 \text{min}} \) by (39), when \( \gamma_2 \) takes different value. Similarly, we also can compute the minimum output peak rejection level \( \gamma_{2 \text{min}} \) by (39) for different \( \gamma_i \). The comparative results of the relevant variable for \( \gamma_i \) and \( \gamma_{j \text{min}} \) \( (i = 1,2; j = 1,2 \), and \( i \neq j \) \) are shown in Table 3.

   It is shown that with the increase of \( \gamma_i \), the \( \gamma_{j \text{min}} \) presents a gradually decreasing trend seen from Table 3, which indicates that there is also a kind of interaction relation between \( \gamma_i \) and \( \gamma_{j \text{min}} \). Therefore, it is important to pay attention to seek the compromise balance between \( \gamma_i \) and \( \gamma_{j \text{min}} \) in the practical system design. Namely, the system is required at the cost of a good performance for another better level of performance and the reverse is also true.

**CONCLUSION**

Under the DETCS, we investigated satisfactory \( H_\infty / H_2 \) event-triggered fault tolerant control of the NCS with
CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

ACKNOWLEDGEMENTS

This work is funded and supported by the National Natural Science of China (No. 61364011 and 61463030) and the Provincial Natural Science Foundation of Gansu in China (No. 1308RJZA148).

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