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Determination of the Thickness of a Power-Law Fluid Driven by the Penetration of a Long Gas Bubble in a Rectangular Channel Using a Singular Perturbation Method

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Abstract: In the present work an analytical and numerical study is presented in order to determine the residual fluid film thickness of a power-law fluid on the walls of a rectangular horizontal channel when it is displaced by another immiscible fluid of negligible viscosity. The mathematical model describes the motion of the displaced fluid and the interface between both fluids. In order to obtain the residual film thickness, m, we used a singular perturbation technique: the matching asymptotic method; in the limit of small capillary number, Ca. The main results indicated that the residual film thickness of the non-Newtonian fluid decreases for decreasing values of the power-law index, which is in qualitative agreement with experimental results.

Keywords: Gas-assisted power-law fluid displacement, gas bubble dynamics, matched asymptotic, lubrication theory.

INTRODUCTION

The steady displacement of Newtonian and non-Newtonian fluids by long bubbles, confined in vertical and horizontal cylindrical ducts or parallel plates has received considerable attention during the past decades, due to the fundamental and practical importance of this process in many industrial applications. Typical examples appear in film coating, bubble columns, gas-assisted injection molding, lubrication theory, oil recovery in naturally fractured reservoirs, etc. The theoretical and experimental studies of this fluid dynamic problem are, in general, very complex because several physical difficulties associated with the formation and deposition of a liquid film in the wall of the container are presented. Usually in these problems the important step is to determine the thickness of the film, considering two basic situations: a) when the plate is moved steadily out of a bath of the liquid with uniform velocity and b) the slow displacement of one fluid by another, taking into account that the walls are fixed. For Newtonian fluids this problem has been widely studied for different cases and parting from the pioneer analytical works of Bretherton [1] and Cox [2] and the experimental work of Taylor [3], the basic mechanics of deposition films are now well understood. In this direction, we can emphasize the following works: Wilson [4] studied analytically the film coating in a plate when it is drawn steadily out of a bath of the liquid: the drag-out problem. Using the method of matched asymptotic expansions to predict the thickness of the film for small values of the capillary number Ca, this author showed that the analysis of Landau [5] for this same problem, represents the leading order of an asymptotic solution, where the higher order correction terms were included to improve the Landau's solution. Park and Homsy [6] studied in a Hele Shaw cell the two-phase displacement in the gap between closely spaced planes. These authors also used matched asymptotic expansions to estimate the thickness of the film for small values of the capillary number Ca, confirming the previous results obtained by Bretherton [1]. Schwartz *et al.* [7] in an effort to validate the theoretical prediction of Bretherton, developed an experimental procedure to measure the average thickness of the wetting film left behind during the slow passage of an air bubble in a water-filled capillary tube. He confirmed that for bubbles of length less than 20 tube radii, the agreement is very good. The above works have a common characteristic: the order of magnitude of the experimental and theoretical predicted thicknesses is sufficient to avoid the presence of intermolecular forces. In this direction, Teletzke *et al.* [8] distinguished the frontier between *thick* and *thin* films, recognizing that as liquid film thickness approaches molecular dimensions (thicknesses $\leq 100\ nm$), the intermolecular forces are of the same order of magnitude of viscous and capillary forces. A presentation of these new topics can be found in the excellent book of Middleman [9], where the presence of other effects as surface-active impurities, slip at apparent contact lines, air entrainment in coating fluids, Marangoni shear stresses, etc can complicate enormously the treatment of the problem. For instance, Münch [10] using numerical and asymptotic schemes derived the thickness of a film, which forms at the tip of a capillary meniscus due to the presence of thermally induced Marangoni shear stress effects. Therefore, for Newtonian fluids the analysis of thin and ultrafilms is an active area of fundamental development that offers new theoretical perspectives and controversial aspects.

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However, the specialized literature to treat these problems for non-Newtonian fluids is scarce in spite of that there are abundant evidence and interesting applications. The pioneering works of Poslinski *et al.* [11] and Poslinski and Coyle [12] found that the fraction of the non-Newtonian fluid deposited on the walls is less than that of a corresponding Newtonian fluid at low capillary number. In particular their numerical results showed that fluids with shear thinning viscosity exhibit a smaller fractional coverage than Newtonian fluid at the same capillary number. Fractional coverage was also found to be smaller for a fluid with a smaller shear thinning index.

Kamisli and Ryan [13] studied the two dimensional flow of a power-law fluid analytically using a singular perturbation method and experimentally in order to determine the residual liquid film thickness for a circular tube and for a rectangular channel. The analytical results of this work indicated that the residual liquid film thickness of non-Newtonian fluids increases with decreasing power-law index. Later these authors, Kamisli and Ryan [14], developed a mathematical analysis to study the motion of long bubbles into Newtonian and non-Newtonian fluid confined in a horizontal circular tube, rectangular channels, and square cross-sectional channels. The model results are in qualitative agreement with previous experimental observations and give also a good agreement with a previous numerical solution obtained by Poslinski *et al.* [11].

In order to identify the effects of fluid elasticity on the fractional coverage, Huzyak and Koelling [15] studied the penetration of a long gas bubble through a tube filled with a viscoelastic fluid. The authors developed experiments with two Newtonian fluids and two highly elastic constant shear viscosity fluids. The fractional coverage was characterized in terms of the capillary and Deborah numbers; and found that the fractional coverage for viscoelastic fluids is a strong function of the tube diameter.

In the present work, we study the motion of an inviscid fluid into a power-law fluid confined in a horizontal rectangular channel. We develop a mathematical model using a singular perturbation method in order to determine the residual fluid film thickness of pseudoplastic fluid (characterized by a power-law index less than unity, $n<1$) on the walls.

METHODOLOGY

Consider that a fluid of negligible viscosity is injected displacing with constant velocity, U, a power-law fluid confined in a rectangular channel, as shown schematically in Fig. (**1**). The walls of the channel are infinite long in the direction perpendicular to the plane $x-y$ and are separated a distance $2R$ small enough to neglect the gravitational effects, so that the solution is symmetric about the midplane. We assume that the displaced fluid totally wets the wall, leaving a film on the wall as displacement proceeds. The surface tension between the fluids is known and uniform; therefore the tangential force balance at the interface is equal to zero. On the other hand, we consider that the flow is very slow and the capillary number, which will be defined later, is small. Bretherton [1] found that for sufficiently small capillary number the viscous stresses appreciably modify the static profile of the interface for regions only very near to the wall; therefore the displacement process can be considered as a singular perturbation problem.

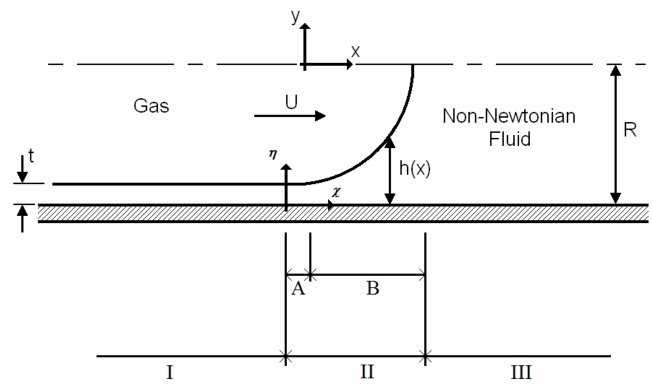

Fig. (1). Simplified physical model.

In the present work, we propose that the fluid which is found confined is described by the power-law fluid model that is expressed by $\tau_{ij}=-2K\varepsilon_{ij}^{n}$, where τ_{ij} is the stress tensor; ε_{ij} is the strain-rate tensor, K and n are material parameters; the first is the index of consistency and the later is the power-law index. Using the power-law model in the mass and conservation equations in Cartesian coordinates and taking into account that the problem is stationary with uniform properties, we obtain that the governing equations for the displaced fluid are

$$\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\partial \tilde{v}}{\partial \tilde{y}}=0, \tag{1}$$

$$\rho\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)=-\frac{\partial \tilde{P}}{\partial \tilde{x}}+ 2K\left[\frac{\partial}{\partial \tilde{x}}\left(\left[\frac{\partial \tilde{u}}{\partial \tilde{x}}\right]^{n}\right)+\frac{\partial}{\partial \tilde{y}}\left(\left[\frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{x}}\right)\right]^{n}\right)\right], \tag{2}$$

$$\rho\left(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right)=-\frac{\partial \tilde{P}}{\partial \tilde{y}}+ 2K\left[\frac{\partial}{\partial \tilde{x}}\left(\left[\frac{1}{2}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{\partial \tilde{v}}{\partial \tilde{x}}\right)\right]^{n}\right)+\frac{\partial}{\partial \tilde{y}}\left(\left[\frac{\partial \tilde{v}}{\partial \tilde{y}}\right]^{n}\right)\right]. \tag{3}$$

In Eq. (1)–(3), $\tilde{u}$ and $\tilde{v}$ represent the axial and radial velocities, respectively; P is the pressure in the displaced fluid and $\tilde{x}$ and $\tilde{y}$ are the Cartesian coordinates. These equations must be solved with the appropriate boundary conditions. At the inner surface of the plates, we impose the well known no-slip condition. At the interface we have three conditions. The first is the kinematic boundary condition, which describes that the interface is impermeable for both fluids and is given by

$$\tilde{v}=\tilde{u}\tilde{h}', \tag{4}$$

where $\tilde{h}$ is the interface position and is only function of the axial coordinate, $\tilde{h} = \tilde{h}(\tilde{x})$; and the prime refers to differentiation with respect to $\tilde{x}$; therefore $\tilde{h}' = \frac{d\tilde{h}}{d\tilde{x}}$.

The other conditions are the dynamic conditions: the tangential and normal stress boundary conditions. The first one expresses that the tangential force balance at the interface is zero because the surface tension is uniform. The second condition establishes that the normal stresses on the two sides of the interface are balanced by surface tension. Using the power-law model for the displaced fluid the dynamic conditions are given by the following expressions

$$2K\left(\frac{1}{2}\left[\frac{\partial\tilde{u}}{\partial\tilde{y}}+\frac{\partial\tilde{v}}{\partial\tilde{x}}\right]\right)^{n}\left(1-\tilde{h}'^{2}\right)- 2K\left[\left(\frac{\partial\tilde{u}}{\partial\tilde{x}}\right)^{n}-\left(\frac{\partial\tilde{v}}{\partial\tilde{y}}\right)^{n}\right]\tilde{h}'=0 \quad (5)$$

$$\frac{2K}{\left(1+\tilde{h}'^{2}\right)}\left[\left(\frac{2^{n}}{2}\left[\frac{\partial\tilde{u}}{\partial\tilde{y}}+\frac{\partial\tilde{v}}{\partial\tilde{x}}\right]\right)^{n}\tilde{h}'-\left(\frac{\partial\tilde{u}}{\partial\tilde{x}}\right)^{n}\tilde{h}'^{2}\right]- \frac{2K}{\left(1+\tilde{h}'^{2}\right)}\left(\frac{\partial\tilde{v}}{\partial\tilde{y}}\right)^{n}+\tilde{P}=-\sigma\left(\frac{1}{R_x}+\frac{1}{R_y}\right) \quad (6)$$

Eq. (5) and (6) represent the tangential and normal stress boundary condition, respectively; where σ is the interface surface tension. In the normal stress condition, Eq. (6), we have used the planar approximation for the curvature in the y direction, equivalent to setting $R_y^{-1}=0$. The principal radius of curvature R_x can be expressed in terms of the interface position as: $R_x^{-1}=\tilde{h}''\left(1+\tilde{h}'^{2}\right)^{-3/2}$.

To obtain the fraction of the liquid deposited on the walls of the tube, m, we use the technique of matched asymptotic expansions. In order to apply the mentioned technique we divided the domain into three regions shown in Fig. (**1**): the constant film thickness region (region I), the front region (region II) and the static region (region III). In region I the problem can be solved using the classical lubrication theory if the film thickness is known. In region III the flow is parabolic in y direction and according with the reference frame in the midplane, this region is static. A detailed analysis of region II is required to obtain the solution.

In the limit of small capillary number the solution in region I does not match smoothly with the solution in the constant film thickness region. Therefore, in order to obtain the complete solution the front region must be subdivided in two. The capillary static region (region B) in which the shape of the interface is nearly circular and pressure forces and interfacial tension are important. On the other hand, the transition region (region A) where the shape of the interface is deformed by viscous traction and thus viscous forces also become important.

It is necessary to examine region II more carefully, therefore in the following we only studied this region. In order to obtain representative dimensionless parameters and to explore the most appropriate scaling for the governing equations presented lines below; we apply an order of magnitude analysis using the scales shown in Fig. (**1**). We consider that the interface tension, σ, is constant and that the fluid of negligible viscosity is injected with a small constant velocity, U. In the capillary static region, we use the following scales to apply the order of magnitude analysis

$$\tilde{x}\sim R,\ \tilde{y}\sim R,\ \tilde{h}\sim R,\ \tilde{u}\sim U,\ \tilde{v}\sim U,\ \tilde{P}\sim\frac{\sigma}{R}.$$

Using the characteristic scales in the governing equations and in the boundary conditions, we define the Reynolds and capillary numbers for a power-law fluid, respectively, as follow

$$\mathrm{Re}=\frac{2^{n-1}\rho U^{2-n}R^{n}}{K},\quad Ca=\frac{KU^{n}R^{1-n}}{\sigma 2^{n-1}}.$$

According with the characteristic values associated to the problem the Reynolds number is very small, $\mathrm{Re}<<1$; therefore in Eqs. (2) and (3) the convective terms can be neglected. From the order of magnitude analysis and in the limit of small capillary number, we obtain that the pressure in the capillary static region is constant and confirms that the interfacial tension and the pressure forces are the dominant effects in the mentioned region.

To apply the order of magnitude analysis to the transition region we must consider than the thickness, δ, and the length, L_x, of this region are unknown. The scales for the transition region are given by,

$$\tilde{x}\sim L_x,\ \tilde{y}\sim\delta,\ \tilde{h}\sim\delta,\ \tilde{u}\sim U,\ \tilde{v}\sim V,\ \tilde{P}\sim\frac{\sigma}{R}.$$

Using the scales given above in the governing equations and the boundary conditions, it is easily to obtain the unknown variables for this region; which are given by the following expressions:

$$\delta\sim Ca^{\frac{2}{2n+1}}R,\quad L_x\sim Ca^{\frac{1}{2n+1}}R,\quad V\sim UCa^{\frac{1}{2n+1}}. \quad (7)$$

Using the results of the order of magnitude analysis, the equations for the capillary static region can be non-dimensionalized using the following scales:

$$x=\frac{\tilde{x}\text{-}Ut}{R},\ y=\frac{\tilde{y}}{R},\ h=\frac{\tilde{h}}{R},\ u=\frac{\tilde{u}}{U},\ v=\frac{\tilde{v}}{U},\ P=\frac{\tilde{P}}{ó/R}.$$

The dimensionless governing equations can be written for the capillary static region as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (8)$$

$$\mathrm{Re}\,Ca\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial P}{\partial x}+ 2^n Ca\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^n\right]+Ca\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^n\right] \quad (9)$$

$$\mathrm{Re}\,Ca\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial P}{\partial y}+ Ca\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^n\right]+2^n Ca\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial y}\right)^n\right] \quad (10)$$

Eq. (8) is the dimensionless mass conservation equation, and Eqs (9) and (10) are the dimensionless motion equations. The boundary conditions in dimensionless variables are given by the following expressions:

$$u=-1,\ v=0 \quad \text{at} \quad y=-1\ , \quad (11)$$

$$v=uh' \quad \text{at} \quad y=-h\ , \quad (12)$$

$$\frac{1}{2^n}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^n\left(1-h'^2\right)-\left[\left(\frac{\partial u}{\partial x}\right)^n-\left(\frac{\partial v}{\partial y}\right)^n\right]h'=0 \quad \text{at} \quad y=-h\ , \quad (13)$$

$$\frac{2Ca}{\left(1+h'^2\right)}\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^n h'-2^{n-1}\left(\frac{\partial u}{\partial x}\right)^n h'^2\right]- \frac{2^n Ca}{\left(1+h'^2\right)}\left(\frac{\partial v}{\partial y}\right)^n+P=\frac{h''}{\left(1+h'^2\right)^{3/2}} \quad \text{at} \quad y=-h\ . \quad (14)$$

As we mentioned previously the problem is suitable to apply the matched asymptotic expansion technique. For this problem the small parameter is the capillary number; then we proposed the following expansions for the capillary static region:

$$\phi(x)=\sum_{j=0}^{\infty}Ca^{\frac{j}{2n+1}}\phi_j(x,y)\ , \quad (15)$$

in which ϕ represent indistinctly dimensionless pressure, velocity or interface position and the subscript j refers to the order of the variable. Substituting the above expansion into Eq. (8)–(14) and collecting terms of the same power of Ca, we obtain a set of equations in the capillary static region. The leading order of Eqs. (9) and (10) are given by,

$$\frac{\partial P_0}{\partial x}=\frac{\partial P_0}{\partial y}=0\ , \quad (16)$$

where the subscript 0 denotes the zeroth order approximation. Eq. (16) implies that the leading order to the pressure is constant, as it was mentioned in the order of magnitude analysis. In order to determine the value of the constant pressure, the interfacial boundary conditions have to be examined. The zeroth order interface position can be obtained from the normal stress boundary condition subjected to the conditions at the front tip of the interface: $h'\to\infty$ when $x\to L$ and $h=0$ at $x=L$. We obtain the zero order solution as:

$$h_0=-\frac{1}{P_0}\left(1-\left[P_0\left(x_0-L\right)-1\right]^2\right)^{1/2}\ , \quad (17)$$

where P_0 is the pressure jump across the interface and L is the distance from the tip of the front to the origin of the reference frame.

In the transition region, the problem has to be rescaled. In this region, the pressure, viscous force and interfacial tension are all important. Using the order of magnitude results, Eq. (7), we proposed the following change of variables to obtain the dimensionless governing equations.

$$\chi=\frac{x+l}{Ca^{\frac{1}{2n+1}}},\ \eta=\frac{y+1}{Ca^{\frac{2}{2n+1}}},\ \delta=\frac{1-h}{Ca^{\frac{2}{2n+1}}},\ u=u,$$

$$V=\frac{v}{Ca^{\frac{1}{2n+1}}},\ P=P.$$

In the change of variables l is a shift of coordinates that is determined with the matching condition. Using an equivalent expansion as in the capillary static region, the zeroth order equations for the transition region are given by,

$$\frac{\partial u_0}{\partial\chi}+\frac{\partial V_0}{\partial\eta}=0, \quad (18)$$

$$-\frac{\partial P_0}{\partial\chi}+\frac{\partial}{\partial\eta}\left[\left(\frac{\partial u_0}{\partial\eta}\right)^n\right]=0, \quad (19)$$

$$\frac{\partial P_0}{\partial\eta}=0\ . \quad (20)$$

$$u_0=-1,\ V_0=0 \quad \text{at} \quad \eta=0\ , \quad (21)$$

$$V_0=u_0\frac{\partial\delta_0}{\partial\chi} \quad \text{at} \quad \eta=\delta\ , \quad (22)$$

$$\left(\frac{\partial u_0}{\partial\eta}\right)^n=0 \quad \text{at} \quad \eta=\delta\ , \quad (23)$$

$$P_0=-\frac{\partial^2\delta_0}{\partial\chi^2} \quad \text{at} \quad \eta=\delta\ . \quad (24)$$

From the above Eq. (18)–(24), it is possible to obtain an ordinary differential equation for the interface given by,

$$\frac{d^3\delta_0}{d\chi^3}=\frac{\left(\delta_0-\theta_0\right)^n}{\left(\delta_0\right)^{2n+1}}\left(\frac{2n+1}{n}\right)^n\ . \quad (25)$$

Eq. (25) is a non-linear and third-order differential equation, in which θ_0 means the leading order for the constant film thickness in the transition region that is determined by a matching condition. For the numerical integration, Eq. (25) is transformed by using a change of variable originally proposed by Park and Homsy [6]. The transformation is given by,

$$H_0 = \frac{\delta_0}{\theta_0}, \quad X = \frac{\chi + s}{(t_0)^{(n+2)/3}} \left(\frac{2n+1}{n} \right)^{n/3} \quad . \tag{26}$$

In the above equation s is an arbitrary constant that shifts the coordinates and is determined by the matching condition. Using Eq. (26) into Eq. (25) we obtain the canonical form for a power-law fluid expressed as:

$$\frac{d^3 H_0}{dX^3} = \frac{(H_0 - 1)^n}{(H_0)^{2n+1}} \quad . \tag{27}$$

Eq. (27) must be solved with appropriate boundary conditions which are provided by the matching with the uniform film thickness region (I) and with the near region (III) to the front tip. Therefore, we proposed for the first region that for,

$$X \to -\infty; \; H_0 \to 1 \; . \tag{28}$$

For simplicity, we propose a new change of variable for Eq. (27) of the following form: $\varphi = H_0 - 1$. Therefore the matching condition is $X \to -\infty; \; \varphi \to 0$. Using the change of variable in Eq. (28) and linearizing it about $H_0 = 1$, according with the matching condition, we obtain the following equation:

$$\frac{d^3 \varphi}{dX_0^3} = \varphi^n . \tag{29}$$

Eq. (29) presents two different cases. The first one is the Newtonian limit with $n = 1$. Under this condition Eq. (29) is an ordinary linear equation and equi-dimensional in φ; therefore the solution for the Newtonian case, valid for the matching condition can be given by

$$\varphi = F_1 e^X \quad , \tag{30}$$

where F_1 is a constant and may take any value due to the arbitrary shift of coordinates s. For $n < 1$ (pseudoplastic fluid) Eq. (27) is a non-linear equation, therefore it does not accept an exponential solution [16].

The second boundary conditions for Eq. (27) is given when the interface is closed to the tip. This matching condition with region III is $H_0 >> 1$ when $X \to \infty$. Therefore Eq. (27) in this limit can be rewritten as follow,

$$\frac{d^3 H_0}{dX^3} \approx 0 \; . \tag{31}$$

The solution for Eq. (31) is

$$H_0 \sim \frac{1}{2} AX^2 + BX + C \; . \tag{32}$$

In the above equation A, B and C are constants that will be determine by numerical integration of Eq. (27), using a fourth order Runge-Kutta method. The results for different values of the power-law index are shown in Table **1** and in Fig. (**2**). The initial values to integrate Eq. (27) are given by Eq. (30) for any value of the power index, n. For non-Newtonian fluids we use an iterative form.

Table 1. Values for Constants of Integration A, B and C for Different Values of the Power-Law Index

n	A	B	C
1	0.643	-0.535	3.015
0.8	0.588	3.794	14.147
0.6	0.567	5.877	31.963
0.5	0.567	6.582	39.521
0.4	0.573	7.186	46.353
0.3	0.582	7.726	52.448
0.2	0.588	8.146	57.468

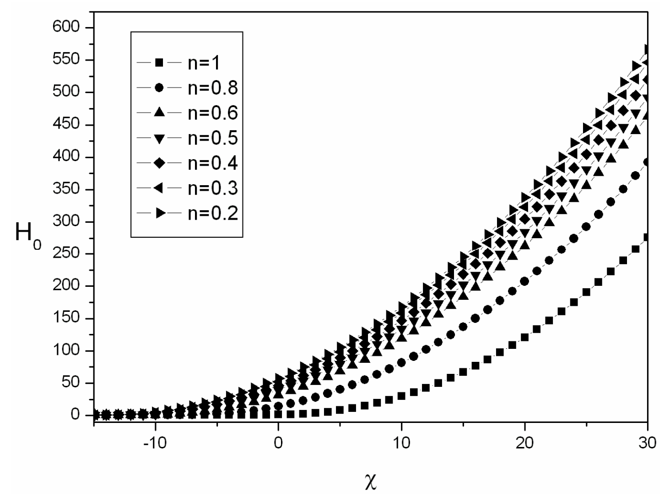

Fig. (2). Results of the numerical integration of Eq. (27) for different power-law indexes.

Once the constants of integration are known, the Eq. (32) can be expressed in dimensionless variables for the transition region is

$$\delta_0(\chi) = \frac{A}{2} \frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}} \chi^2 + \left[As \frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}} + B \frac{\left(\frac{2n+1}{n}\right)^{n/3}}{\theta_0^{(n-1)/3}} \right] \chi + \left[\frac{As^2}{2} \frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}} + Bs \frac{\left(\frac{2n+1}{n}\right)^{n/3}}{\theta_0^{(n-1)/3}} + C\theta_0 \right] \; . \tag{33}$$

The solution is incomplete until the thickness t_0 is determined. This variable and all other quantities are determined by matching with the solution in the capillary static region. The matching condition is given by the following equation:

$$\lim_{x \to -l} h(x) = \lim_{\chi \to \infty} \left(1 - \delta(\chi) Ca^{2/2n+1}\right) \; . \tag{34}$$

By expanding $h(x)$ about $x = -l$ using Taylor series expansion, rewriting the expansion in transition region variables and comparing it with the left-hand side term by term, matching conditions for each order can easily be determined.

We obtain the following set of equations up to terms of second order. The zero order are

$$h_0(-l)=1\ , \tag{35}$$

$$h'_0(-l)=0\ , \tag{36}$$

$$h''_0(-l)=-A\frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}}\ . \tag{37}$$

The first order equations are given by

$$h_1(-l)=0\ , \tag{38}$$

$$h'_1(-l)=-As\frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}}-B\frac{\left(\frac{2n+1}{n}\right)^{n/3}}{\theta_0^{(n-1)/3}}\ , \tag{39}$$

$$h''_1(-l)=0\ . \tag{40}$$

Finally the second order equations are

$$h_2(-l)=-\frac{As^2}{2}\frac{\left(\frac{2n+1}{n}\right)^{2n/3}}{\theta_0^{(2n+1)/3}}-Bs\frac{\left(\frac{2n+1}{n}\right)^{n/3}}{\theta_0^{(n-1)/3}}-C\theta_0\ , \tag{41}$$

$$h'_2(-l)=0\ , \tag{42}$$

$$h''_2(-l)=0\ . \tag{43}$$

Eqs. (35) and (36) give the zeroth order for pressure and to define the position of the reference frame for the transition region. The pressure jump across the interface and the shift of coordinates can be determined from the previous equations combined with Eq. (17). Then the leading order for the pressure jump and the shift of coordinates are given by the following expressions

$$P_0=-1\ , \tag{44}$$

$$l=1-L\ . \tag{45}$$

Knowing P_0 and l, Eq. (37) can be solved and the residual film thickness is determined. Then, the film thickness expressed in dimensionless variables of the capillary static region is given by

$$t_0=Ca^{\frac{2}{2n+1}}\left[A\left(\frac{2n+1}{n}\right)^{2n/3}\right]^{\frac{3}{2n+1}}\ . \tag{46}$$

The fraction of liquid deposited on the walls is defined as: $m=\frac{A_T-A_B}{A_T}$; where A_T is the transverse area and A_B is transverse area of the injected fluid. For this particular case, we have a rectangular transverse area; therefore $m=2t$. Then the liquid fraction is

$$m=2Ca^{\frac{2}{2n+1}}\left[A\left(\frac{2n+1}{n}\right)^{2n/3}\right]^{\frac{3}{2n+1}} \tag{47}$$

The higher order corrections for both regions can be determined in a similar way. However, the details are not presented in this work.

RESULTS

In order to validate the model developed in this work, we compare our theoretical predictions with the results obtained by Park and Homsy [6]. These authors studied the two-phase displacement in three dimensions in a Hele Shaw cell and presented the following expressions for the thickness of the film and the pressure jump across the interface. The results are given by

$$t=1.337Ca^{2/3}\left[1+\varepsilon^2\left(\frac{1}{4}\pi f''+[f']^2\right)\right]+O\left(\varepsilon^2 Ca, Ca^{4/3}\right)\ , \tag{48}$$

$$P=1+3.80Ca^{2/3}-\frac{1}{4}\pi f''\varepsilon^2+O\left(Ca,\varepsilon^2 Ca^{2/3}\right)\ . \tag{49}$$

In the above equations $\varepsilon=\frac{R}{L_T}$, where L_T is the width of the Hele Shaw cell; f represents the tip projection of the interface onto the $x-z$ plane and is function of the z coordinate. In the case that we are studying $L_T\to\infty$, $\varepsilon=0$ and $f=L$. Therefore, the results from Eqs. (44) and (46), taking into account higher order corrections, are

$$t=1.33751Ca^{2/3} \tag{50}$$

$$P=1+3.7346Ca^{2/3} \tag{51}$$

For Newtonian fluid case Eqs. (50) and (51) are in good agreement with Eqs. (48) and (49).

Eq. (20) expresses the fraction m of fluid remaining after passage a fluid of negligible viscosity for any power-law

Table 2. Fluid Fraction Expressions for Different Values of the Power-Law Index

Power-law index n	Fluid Fraction m
1	$2.675\,Ca^{2/3}$
0.8	$2.237\,Ca^{10/13}$
0.6	$1.874\,Ca^{10/11}$
0.5	$1.708\,Ca$
0.4	$1.541\,Ca^{10/9}$
0.3	$1.357\,Ca^{5/4}$
0.2	$1.118\,Ca^{10/7}$

index. From this equation is clearly shown that the fraction m depends on two parameters: the capillary number, Ca, and the fluid power index, n. The influence of both parameters on the fluid fraction is clearly shown in Table **2**.

In Figs. (**3**) and (**4**) we show the fluid fraction for different values of the power index. It is clear that for smallest values of the capillary number we obtain the minimum fluid fraction independently of the fluid properties. On the other hand for any given capillary number with decreasing the power-law index the fluid fraction decreases. This result is in qualitative agreement with the experimental results obtained by Kamisli and Ryan [13].

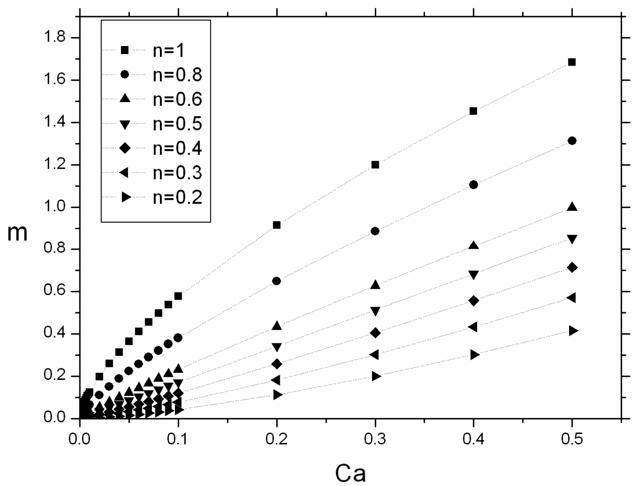

Fig. (3). Fluid fraction versus capillary number.

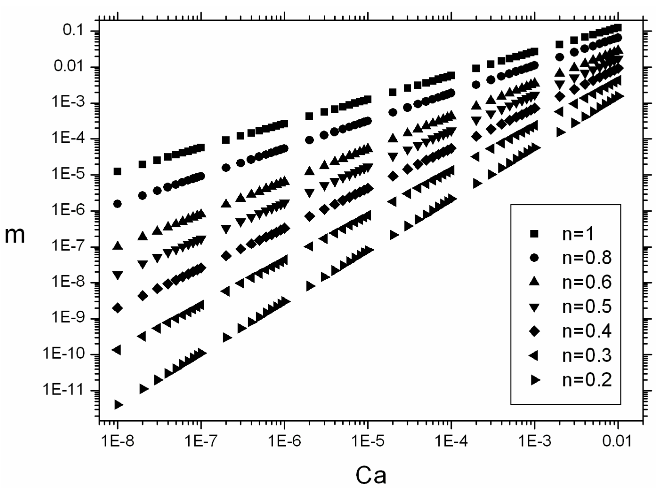

Fig. (4). Fluid fraction versus capillary number (log-log).

The results show in the present study contrast with the theoretical results obtained by Kamisli and Ryan [13] using perturbation methods to determine the residual film thickness in gas-assisted power-law displacement. These authors concluded that a perturbation analysis using a power-law constitutive equation does not correctly predict the variation of the residual liquid film thickness as a function of the power law index. In the cited study above the expansions for the capillary static and the transition regions are developed in terms of a small parameter $Ca^{1/3}$, where Ca is the modified capillary number and is independent of the power-law index. For any power-law index, the dimensionless variables for the transition region suggested by Kamisli and Ryan [13], are given by

$$\chi = \frac{x+l}{Ca^{1/3}},\ \eta = \frac{y+1}{Ca^{2/3}},\ \delta = \frac{1-h}{Ca^{2/3}},\ u = u,$$
$$V = \frac{v}{Ca^{1/3}},\ P = P.$$

We can appreciate that the expansions and the scaling of the dimensionless variables proposed by Kamisli and Ryan [13] produces a divergence between experimental and numerical results. However, our scaling is in accordance with in the determination of the residual film thickness. These can be seen in Eq. (15) and in the dimensionless variables for the transition region. For the Newtonian fluid, $n = 1$, we recover the expansions and dimensionless variables proposed by the authors mentioned above. Nevertheless for $n \neq 1$ the expansions proposed in [13] are not valid, because the residual film thickness depends on the displaced fluid properties.

In order to analyze this behavior, we present Fig. (**5**), in which we have plotted the ratio of the non-Newtonian fluid fraction, m_{n-n}, with the Newtonian fluid fraction, m_n, for three values of the capillary number. Fig. (**5**) shows that the ratio of fluid fractions decreases for decreasing values of the power-law index; and confirms that the minimum fluid fraction is obtained with smaller capillary numbers.

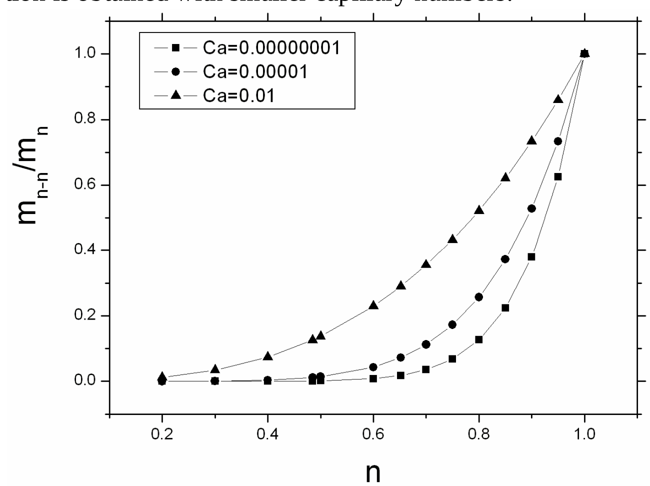

Fig. (5). Fluid fraction ratio versus power-law index.

DISCUSSION AND CONCLUSION

The residual liquid film thickness is determined applying a perturbation analysis using a power-law constitutive equation. The theory predicted that the residual fraction increases with increasing values of the power-law index which is in qualitative agreement with the experimental observations of previous investigators. Furthermore, it is possible to recover the results obtained previously by many authors for the Newtonian case. It is important to note the relevance of the order of magnitude analysis in this kind of problems, because the appropriate scales of the transition region and the series expansions for the perturbation technique are fundamental to obtain the residual film thickness and the pressure jump across the interface.

According to the results for any power-law index, the minimum residual liquid film thickness is obtained for the smaller capillary numbers. This result is important to determine the best conditions for oil recovery in naturally fractured reservoirs.

ACKNOWLEDGEMENTS

This work has been supported by the research grants IN103206 of the DGAPA, UNAM and 43010-Y of the Consejo Nacional de Ciencia y Tecnología at Mexico.

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Received: April 01, 2008 Revised: June 06, 2008 Accepted: July 01, 2008