Perfectly Deducing Bessel Mean Square Error Formula

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Abstract: In the survey teaching materials of China, deducing Bessel mean square error formula is all based on survey values with same mathematical expectation. These methods aren’t perfect. So, based on survey values without same mathematical expectation to prove Bessel mean square error formula is very necessary. Therefore, considering different mathematical expectation, it is meaningful that this paper has perfectly deduced Bessel mean square error formula.

Keywords: Bessel mean square error formula, survey values with different mathematics expectation, deducing.

1. INTRODUCTION

Bessel mean square error formula

\[ m = \pm \sqrt{\frac{vv}{n-1}} \] (1)

is a sort of basic formula about survey error. Where, \( m \) is mean square error, \( n \) is quantity of survey values; \( v_i \) (\( i=1,2,\cdots,n \)) is residual of survey values, and in survey teaching materials, \( [vv] \) generally represents \( \sum_{i=1}^{n} v_i^2 \). (Behind of this paper, the meaning of \( [vv] \) still is \( \sum_{i=1}^{n} v_i^2 \).

But, when we deduce Bessel mean square error formula in survey teaching materials of China, we often select survey values with same mathematics expectation [1-18], and generally survey values with different mathematics expectation are not selected. The literatures [4,6,8,12-15,17] are all national programming teaching materials of Chinese common higher education, and literatures [12,13,15,17] are all called as Chinese excellent surveying teaching materials. But these methods may give students viewpoint that Bessel mean square error formula is a sort of basic formula in survey values with same mathematics expectation. Where, \( \mu_1, \mu_2, \cdots, \mu_n \) their mean square error formula based on survey values with same mathematics expectation [19], their mathematics expectation are \( \mu_1, \mu_2, \cdots, \mu_n \), their variance is \( \sigma^2 \), and their density is \( \sigma_n^2 \).

2. PERFECTIVELY DEDUCING BESSEL MEAN SQUARE ERROR FORMULA

If there are equal precision independent surveying values such as \( l_1, l_2, \cdots, l_n \), their residuals are \( v_1, v_2, \cdots, v_n \), their mathematics expectation is \( \mu_1, \mu_2, \cdots, \mu_n \), their variance is \( \sigma^2 \), and their density is \( \sigma_n^2 \). As follows:

\[ f(l_i) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(l_i - \mu_i)^2}{2\sigma^2} \right) \quad (i=1, 2, \cdots, n) \] (2)

2.1. \( \mu_1 \neq \cdots \neq \mu_i \neq \cdots \neq \mu_n \quad (i=1, 2, \cdots, n) \) isn’t true

When \( \mu_1 \neq \cdots \neq \mu_i \neq \cdots \neq \mu_n \quad (i=1, 2, \cdots, n) \) isn’t true, based on Eq.2, we can receive maximum likelihood function [19].

\[ f\left( l_i \right) = \prod_{i=1}^{n} f(l_i) = \frac{1}{(2\pi)^n} \exp \left( -\frac{\sum_{i=1}^{n} (l_i - \mu_i)^2}{2\sigma^2} \right) \] (3)

Then,

\[ \ln f\left( l_i \right) = -n \ln \sqrt{2\pi} - \frac{n}{2} \ln \sigma^2 - \frac{\sum_{i=1}^{n} (l_i - \mu_i)^2}{2\sigma^2} \] (4)

2.1.1. \( \mu_i \quad (i=1, 2, \cdots, n) \) is already known

From Eq.4, we can get

\[ \frac{\partial \ln f\left( l_i \right)}{\partial \sigma^2} = \frac{n}{2\sigma^2} - \frac{\sum_{i=1}^{n} (l_i - \mu_i)^2}{2\sigma^4} = 0 \] (5)

Therefore,
\[
\sigma^2 = \frac{\sum_{i=1}^{n} (l_i - \mu_i)^2}{n}
\]

(6)

Then;

\[
m = \hat{\sigma} = \pm \sqrt{\frac{\sum_{i=1}^{n} (l_i - \mu_i)^2}{n}} = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n}} = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n}}
\]

(7)

If we want Eq.7 to have an unprejudiced character, we should change Eq.7 into Eq.8

\[
m = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-1}}
\]

(8)

2.1.2. \( \mu_i \ (i = 1, 2, \ldots, n) \) isn’t already known

From Eq.4, we can get

\[
\frac{\partial \ln f}{\partial \sigma^2} \left( \frac{1}{\mu_1 \mu_2 \cdots \mu_n \sigma^2} \right) = -\frac{n}{2\hat{\sigma}^2} + \frac{n}{2\hat{\sigma}^4} \sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2 = 0
\]

(9)

\[
\frac{\partial \ln f}{\partial \mu_i} \left( \frac{1}{\mu_1 \mu_2 \cdots \mu_n \sigma^2} \right) = \frac{l_i - \hat{\mu}_i}{\hat{\sigma}^2} = 0
\]

(10)

Therefore;

\[
\hat{\sigma}^2 = \frac{n}{\sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2}
\]

(11)

\[
\hat{\mu}_i = l_i \ (i = 1, 2, \ldots, n)
\]

(12)

From Eq.11, we can get

\[
m = \hat{\sigma} = \pm \sqrt{\frac{n}{\sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2}} = \pm \sqrt{\frac{n}{\sum_{i=1}^{n} v_i^2}} = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n}}
\]

(13)

If we want Eq.13 to have an unprejudiced character, we should change Eq.13 into Eq.14

\[
m = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-1}}
\]

(14)

So, we can say that when \( \mu_1 \neq \cdots \neq \mu_i \neq \cdots \neq \mu_n \) \((i = 1, 2, \ldots, n)\) isn’t true, and they aren’t already known, Bessel mean square error formula has been testified in this paper. But, no matter how bad or good survey values are, when we take Eq.12 into Eq.14 or Eq.13, we can only obtain in a certain case:

\[
m = 0
\]

(15)

This is a kind of special circumstance.

2.2. \( \mu_1 = \cdots = \mu_i = \cdots = \mu_n \ (i = 1, 2, \ldots, n) \) is true

Because independent surveying values \((l_1, l_2, \ldots, l_n)\) have equal precision, and \( \mu_1 = \cdots = \mu_i = \cdots = \mu_n \) \((i = 1, 2, \ldots, n)\) is true, we can assume survey values come from same mother body and assume \( \mu_i = \mu (i = 1, 2, \ldots, n) \). So, we can receive maximum likelihood function based on Eq.2 [19].

\[
f\left( \frac{1}{l_1 \mu_2 \cdots \mu_n \sigma^2} \right) = \frac{n}{\prod_{i=1}^{n} f(l_i)} = \frac{1}{(\sqrt{2\pi})^n \sigma^n} \exp \left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (l_i - \mu)^2 \right]
\]

(16)

Then;

\[
\ln f\left( \frac{1}{l_1 \mu_2 \cdots \mu_n \sigma^2} \right) = -n \ln \sqrt{2\pi} - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (l_i - \mu)^2
\]

(17)

2.2.1. \( \mu_i (i = 1, 2, \ldots, n) \) is already known

When \( \mu_i (i = 1, 2, \ldots, n) \) is already known, we can get Eq.18 from Eq.17

\[
\frac{\partial \ln f}{\partial \sigma^2} \left( \frac{1}{l_1 \mu_2 \cdots \mu_n \sigma^2} \right) = -\frac{n}{2\hat{\sigma}^2} + \frac{n}{2\hat{\sigma}^4} \sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2 = 0
\]

(18)

Therefore;

\[
\hat{\sigma}^2 = \frac{n}{\sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2}
\]

(19)

Then;

\[
m = \hat{\sigma} = \pm \sqrt{\frac{n}{\sum_{i=1}^{n} (l_i - \hat{\mu}_i)^2}} = \pm \sqrt{\frac{n}{\sum_{i=1}^{n} v_i^2}} = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n}}
\]

(20)

If we want Eq.20 to have an unprejudiced character, we should change Eq.20 into Eq.21

\[
m = \pm \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-1}}
\]

(21)

2.2.2. \( \mu_i (i = 1, 2, \ldots, n) \) isn’t already known

When \( \mu_1 = \cdots = \mu_i = \cdots = \mu_n \ (i = 1, 2, \ldots, n) \) is true, and they aren’t already known, the paper [20] has deduced Bessel mean square error formula.

5. CONCLUSIONS

In survey teaching materials of China, deducing Bessel mean square error formula is based on survey values with same mathematics expectations. Those teaching materials haven’t solved survey values with different mathematics expectation so it isn’t deemed to be perfect.
Based on survey values which don’t have the same mathematics expectation, this paper has put forward a sort of method to prove Bessel mean square error formula. The method has solved problems which haven’t been solved by survey teaching materials of China, include national programming teaching materials of Chinese common higher education and Chinese excellent surveying teaching materials.

In addition, the paper has studied all sorts of circumstances to deduce Bessel mean square error formula. So, the paper is complete.

Obviously, the paper is good for Bessel mean square error formula to be perfective, for students to understand Bessel mean square error formula, for survey teaching materials of China, include national programming teaching materials of Chinese common higher education and Chinese excellent surveying teaching materials, to be renewed.

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Author: Deng Yonghe, College of Engineering and Designing of Lishui College. The author’s major is data processing and higher education. 58 papers had been published, and among them, 55 papers had independently been finished.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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