Exponential Stability for Switched Systems with Mixed Time Delays

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Abstract: The global exponential stability problem for a class of switched systems with mixed time delays is investigated in this paper. LMI-based delay-dependent and delay-independent criteria are proposed to guarantee the exponential stability for our considered systems. Razumikhin-like approach and Leibniz-Newton formula are used to find the stability results. Finally, a numerical example is illustrated to show the improved results from using this method.

Keywords: Global exponential stability, switched neutral system, LMI-based approach.

INTRODUCTION

The time-delay phenomenon is encountered in many practical systems, such as aircraft stabilization, neural networks, nuclear reactors, ship stabilization, and systems with lossless transmission lines. It is well known that the existence of time delay in a system may cause instability or bad system performance in open and closed-loop systems. Hence stability analysis for time-delay systems has received considerable attention in recent years [1-3]. In many physical systems, the system models can be described by functional differential equation of neutral type, in which the models depend on the state of delay but also depend on state derivatives. Neutral system examples include distributed networks, heat exchanges, and processes including steam. Stability conditions have been proposed to guarantee the stability for neutral systems [4-5]. On the other hand, stability for switched time-delay systems has been an attractive research topic which is composed of subsystems with their own parameterizations [6-7]. Hence we will consider the stability problem for switched neutral systems in this paper. Based on Razumikhin-like approach and Leibniz-Newton formula [8], delay-dependent and delay-independent results are provided. The LMI approach [9] is an efficient and powerful tool in solving some control problems; such as H\textsubscript{\infty} control, state feedback control, and observer-based control. Hence, we will recommend the LMI-based results. A numerical example is provided to demonstrate the proposed results.

The notation used throughout this paper is as follows. For a matrix $A$, we denote the transpose by $A^T$, spectral norm by $\|A\|$, symmetric positive (negative) definite by $A > 0$ ($A < 0$). $A \preceq B$ means that matrix $B - A$ is symmetric positive semi-definite. For a vector $x$, we denote the Euclidean norm by $\|x\|$ and $\|x\|_2 = \sup_{\theta \in [0,T]} \|x(t + \theta)\| + \|x(t)\|^\pi$. $I$ denotes the identity matrix.

PROBLEM FORMULATION AND MAIN RESULTS

Consider the following switched neutral system with mixed time delays:

$$\dot{x}(t) - D\dot{x}(t - \tau) = A_0x(t) + A_\sigma x(t - h(t)), \quad t \geq 0,$$ (1a)

$$x(t) = \phi(t), \quad t \in [-H, 0],$$ (1b)

where $x \in \mathbb{R}^n$, $x_i$ is state at time $t$ defined by $x_i(\theta) = x(t + \theta)$, $\forall \theta \in [-H, 0]$, $\sigma$ is a switching signal which is a piecewise constant function and may depend on $t$ or $x$, $\sigma$ takes its values in the finite set $\{1, 2, \ldots, N\}$, time-varying delay $0 \leq h(t) \leq H$, $\dot{h}(t) \leq h_\sigma$, $h_\sigma > 0$, $\tau > 0$, $H = \max \{h_\sigma, \tau\}$. Matrices $D$, $A_0$, and $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, N$, are constant, and the initial vector $\phi \in C_1$, where $C_1$ is the set of differentiable functions from $[-H, 0]$ to $\mathbb{R}^n$.

Define the functions $\lambda_i(t)$, $i = 1, 2, \ldots, N$, in the following:

$$\lambda_i(t) = \begin{cases} 1, & \sigma = i, \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, 2, \ldots, N. \quad (2)$$

The switched system in (1) can be rewritten as follows:

$$\dot{x}(t) - D\dot{x}(t - \tau) = \sum_{i=1}^{N} \lambda_i(t) [A_i x(t) + A_{i\sigma} x(t - h(t))], \quad t \geq 0,$$ (3a)

$$x(t) = \phi(t), \quad t \in [-H, 0],$$ (3b)

where $\lambda_i(t)$ is defined in (2) and $\sum_{i=1}^{N} \lambda_i(t) = 1, \quad \forall \ t \geq 0$. 

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Definition 1: The system (1) is said to be the globally exponentially stable with convergence rate $\alpha > 0$, if there are two positive constants $\alpha$ and $\Psi$ such that

$$\|x(t)\| \leq \Psi \cdot e^{-\alpha t}, \quad t \geq 0.$$

Now we present a delay-dependent condition for the global exponential stability of system (1) with (2).

Theorem 1: System (1) is global exponentially stable with convergence rate $0 < \alpha < \min\{\|P\|/\tau, \|D\|/\tau\}$, if $\tau > 0$ and there exist some $n \times n$ matrices $P_i, Q, R, S, T > 0$, a matrix $U$, such that the following LMI conditions hold for all $i = 1, 2, \cdots, N$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} \\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} & \Sigma_{45} \\ \Sigma_{51} & \Sigma_{52} & \Sigma_{53} & \Sigma_{54} & \Sigma_{55} \end{bmatrix} < 0,$$

(4)

where * represents the symmetric form in the matrix and

$$\Sigma_{11} = 2\alpha \cdot P + PA_{h0} + A_{h0}^T P + Q + S - e^{-2\alpha h_0} \cdot R,$$

$$\Sigma_{12} = -2\alpha \cdot PD - A_{h0}^T PD, \quad \Sigma_{13} = PA_{h0} + e^{-2\alpha h_0} \cdot R,$$

$$\Sigma_{14} = A_{h0}^T U^T, \quad \Sigma_{15} = -A_{h0}^T U^T D,$$

$$\Sigma_{22} = 2\alpha \cdot D^T PD - e^{-2\alpha h_0} \cdot S,$$

$$\Sigma_{33} = -e^{-2\alpha h_0} \cdot \left[(1-h_0) \cdot Q + R^T\right], \quad \Sigma_{34} = A_{h0}^T U^T,$$

$$\Sigma_{35} = -A_{h0}^T U^T D, \quad \Sigma_{44} = T + h_0^2 \cdot R - U^T,$$

$$\Sigma_{45} = (U + U^T) D,$$

$$\Sigma_{55} = -D^T \left[(U + U^T) \cdot D - e^{-2\alpha t} \cdot T\right].$$

Proof: Define the Lyapunov functional

$$V(x_i) = e^{2at} \cdot \left(x(t) - Dx(t - \tau)\right)^T P \left(x(t) - Dx(t - \tau)\right)$$

$$+ \int_{t-h_0}^t e^{2a s} \cdot x^T(s) Q x(s) ds$$

$$+ h_0 \cdot \left(\int_{t-h_0}^t e^{2a s} \cdot (s - (t - h_0)) \cdot x^T(s) R x(s) ds\right)$$

$$+ \int_{t-h_0}^t e^{2a s} \cdot x^T(s) S x(s) ds + \int_{t-h_0}^t e^{2a s} \cdot \dot{x}^T(s) T \dot{x}(s) ds,$$

(5)

where $P, Q, R, S, T > 0$. The time derivatives of $V(x_i)$, along the trajectories of system (3) satisfy

$$\dot{V}(x_i) = e^{2at} \cdot \left[2\alpha \cdot (x(t) - Dx(t - \tau))^T P \left(x(t) - Dx(t - \tau)\right)\right]$$

$$+ e^{2at} \cdot \sum_{i=1}^N \lambda_i(t) \cdot \left[A_{h0} x(t) + A_{h0} x(t - h(t))\right]^T P \left[A_{h0} x(t) + A_{h0} x(t - h(t))\right]$$

$$+ e^{2at} \cdot \sum_{i=1}^N \lambda_i(t) \cdot \left[(x(t) - Dx(t - \tau))^T P \left(A_{h0} x(t) + A_{h0} x(t - h(t))\right)\right]$$

$$+ \int_{t-h_0}^t e^{2a s} \cdot x^T(s) Q x(s) ds$$

$$+ h_0 \cdot \left(\int_{t-h_0}^t e^{2a s} \cdot (s - (t - h_0)) \cdot x^T(s) R x(s) ds\right)$$

$$+ \int_{t-h_0}^t e^{2a s} \cdot x^T(s) S x(s) ds + \int_{t-h_0}^t e^{2a s} \cdot \dot{x}^T(s) T \dot{x}(s) ds,$$

(6a)

where $X^T = [x^T \quad x^T \quad x^T \quad x^T \quad \dot{x}^T \quad \dot{x}^T \quad \dot{x}^T(t - \tau)],$ matrices $\Sigma_i$, $i = 1, 2, \cdots, N$, are defined in (4). From the conditions $\Sigma_i < 0$, $i = 1, 2, \cdots, N$, in (4), we have

$$V(x_i) \leq V(x_i), \quad t \geq 0,$$

(8)

where

$$V(x_i) = \left(x(0) - Dx(-\tau)\right)^T P \left(x(0) - Dx(-\tau)\right)$$

$$+ \int_{-\tau}^0 e^{2a s} \cdot x^T(s) Q x(s) ds$$

$$+ h_0 \cdot \left(\int_{-\tau}^0 e^{2a s} \cdot (s + h_0) \cdot x^T(s) R x(s) ds\right)$$

$$+ \int_{-\tau}^0 e^{2a s} \cdot x^T(s) S x(s) ds + \int_{-\tau}^0 e^{2a s} \cdot \dot{x}^T(s) T \dot{x}(s) ds,$$

(6b)

and

$$\delta_1 = \lambda_{\text{max}}(P)(1 + \|D\|^2) + h_0 \cdot \lambda_{\text{max}}(Q) + h_0 \cdot \lambda_{\text{max}}(R)$$

$$+ \tau \cdot \left(\lambda_{\text{max}}(S) + \lambda_{\text{max}}(T)\right).$$
On the other hand, we have
\[ \lambda_{\min}(P) \cdot e^{\alpha t} \cdot \| \varphi(t) \| \leq e^{\alpha t} \cdot x(t) P \varphi(t) \leq V(x(t)) \leq \delta_t \cdot \| x_0 \|, \]  
where \( \varphi(t) = x(t) - D x(t - \tau) \). From (9), we can obtain the following result
\[ \| x(t) \| = \| \varphi(t) \| + \| x(t - \tau) \| + \| \varphi(t) \| \leq \| D \| \cdot \| x(t - \tau) \| + \| x_0 \|, \]
where \( \delta_t = \sqrt{\frac{\lambda_1(P) \cdot \| x_0 \|}{\| D \|}} \). Since \( \| D \| < 1 \) and \( \tau > 0 \), we can choose a sufficiently small positive constant \( \xi = \alpha - (\ln \| D \|)/\tau \) satisfying \( \| D \| e^{\alpha t} < 1 \). By the Razumikhin-like approach of [8], we obtain the following result
\[ \| x(t) \| \leq \left[ \sup_{-\xi \leq \theta < 0} \| x(\theta) \| + \frac{\delta_t}{1 - \| D \| e^{\alpha t}} \right] e^{-\xi t}, \]
where \( \delta_t \) represents the symmetric form in the matrix and
\[ \Xi_i = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \ast & \Xi_{22} & \Xi_{23} \\ \ast & \ast & \Xi_{33} \end{bmatrix} < 0, \]
where \( \ast \) represents the symmetric form in the matrix and
\[ \Xi_{11} = 2\alpha \cdot P + PA_{01} + A_{01}'P + Q - e^{-2\alpha \theta_u} \cdot R, \]
\[ \Xi_{12} = PA_{01} + e^{-2\alpha \theta_u} \cdot R, \quad \Xi_{13} = A_{01}'U^T, \]
\[ \Xi_{22} = e^{-2\alpha \theta_u} \cdot (1 - h_d) \cdot Q + R^T, \quad \Xi_{23} = A_{01}'U^T, \]
\[ \Xi_{33} = h_{d\mu} \cdot R - U - U^T. \]

This completes the proof.

**Corollary 1:** System (1) with \( D = 0 \) is global exponentially stable with convergence rate \( \alpha > 0 \), if there exist some \( n \times n \) matrices \( P, Q, R > 0 \), a matrix \( U \), such that the following LMI conditions hold for all \( i = 1, 2, \ldots, N \)
\[ \Xi_i = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \ast & \Xi_{22} & \Xi_{23} \\ \ast & \ast & \Xi_{33} \end{bmatrix} < 0, \]

where \( \ast \) represents the symmetric form in the matrix and
\[ \Xi_{11} = 2\alpha \cdot P + PA_{01} + A_{01}'P + Q - e^{-2\alpha \theta_u} \cdot R, \]
\[ \Xi_{12} = PA_{01} + e^{-2\alpha \theta_u} \cdot R, \quad \Xi_{13} = A_{01}'U^T, \]
\[ \Xi_{22} = e^{-2\alpha \theta_u} \cdot (1 - h_d) \cdot Q + R^T, \quad \Xi_{23} = A_{01}'U^T, \]
\[ \Xi_{33} = h_{d\mu} \cdot R - U - U^T. \]

### Table 1. Comparing Some Previous Results with this Paper

<table>
<thead>
<tr>
<th>Results</th>
<th>Convergence Rate ( \alpha )</th>
<th>Upper Bounds of Delay ( h_d ) and ( h_{d\mu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7]</td>
<td>( \alpha = 0.3 )</td>
<td>( h_d = 0.1 ), ( h_{d\mu} = 0.4 )</td>
</tr>
<tr>
<td>Our results</td>
<td>( \alpha = 0.3 )</td>
<td>( h_d = 0.44 ) (( h_d = 1 ) or unknown)</td>
</tr>
<tr>
<td>[7]</td>
<td>( \alpha = 0 )</td>
<td>( h_d = 0.1 ), ( h_{d\mu} = 1 )</td>
</tr>
<tr>
<td>Our results</td>
<td>( \alpha = 0 )</td>
<td>( h_d = 0.1 ), ( h_{d\mu} = 2.58 )</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 0 )</td>
<td>( h_{d\mu} = 0.68 ) (( h_d = 1 ) or unknown)</td>
</tr>
</tbody>
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REFERENCES