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Numerical Detection of the Lowest "Efficient Dimensions" for Chaotic Fractional Differential Systems

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Abstract: In this paper we numerically investigate the dynamical behavior of fractional differential systems. By utilizing the fractional Adams method, we numerically find the smallest "efficient dimensions" of the fractional Lorenz system and Rössler system such that they are chaotic.

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1. Introduction

Fractional calculus is one of the active research fields in mathematic analysis, which deals with the investigation and application of integrals and derivatives of arbitrary orders. Although they are often real in applications, these order can be complex in viewpoint of pure mathematics. The subject of fractional calculus is by no means new. The idea goes back to the Leibniz's note in his letter to L'Hospital, dated 30 September 1695. For three centuries the theory developed mainly as a pure theoretical field of mathematics [1-6]. However, in the last few decades it was found that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials. In recent years it has turned out that many phenomena in engineering, physics, chemistry, and other sciences can be described very successfully by models using mathematical tools of fractional calculus, such as viscoelastic system, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves [7-11].

As the interdisciplinary applications are characterized elegantly by fractional calculus, many authors also begin to investigate the chaotic dynamics of fractional nonlinear dynamical systems [12-21]. Since no general theory is available for studying the chaotic dynamical systems, all the investigations rely on numerical simulations. The typical method to do numerical computations of fractional ordinary equations is to approximate fractional operators by using standard integer operators, this can be realized via utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of a fractional-order integrator with any desired accuracy over any frequency band, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitudes of the corresponding Bode diagrams. But this frequency method seems not to be suitable for numerical detection of the chaotic attractors of chaotic fractional differential systems. Detailed reports in this respect can be found in [22]. Actually, the first choice of numerical calculations for fractional differential equation is the time-domain method. One of the typical numerical methods for fractional ordinary differential equa-

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tions is the fractional Adams method, constructed by Diethelm, et al. [23]. This method was sufficiently applied to calculation of periodic orbits and/or chaotic attractors for fractional differential equations. For example, see [14,15,17,18,24,25].

The general fractional system reads as

$$\begin{cases} {}_C D_{0,t}^{\alpha} x(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{1}$$

in which $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$, $f(t, x(t)) = (f_1(t, x(t)), \cdots, f_n(t, x(t)))^T \in R^n$, ${}_C D_{0,t}^{\alpha} x(t) = ({}_C D_{0,t}^{\alpha_1} x_1(t), \cdots, {}_C D_{0,t}^{\alpha_n} x_n(t))^T \in R^n$, $0 < \alpha_i \leq 1$, $i = 1, \cdots, n$. Its "efficient dimensions" is defined by $\mid q \mid = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, [14]. The fractional derivative is in the sense of Caputo, [1-5, 21], that is, the α-th Caputo ($n - 1 < \alpha < n \in Z^+$) fractional derivative of $y(t)$ is as follows,

$$ {}_C D_{0,t}^{\alpha} y(t) = \frac{\int_0^t (t-\tau)^{n-\alpha-1} y^{(n)}(\tau) d\tau}{\Gamma(n-\alpha)}. \tag{2}$$

In equation (2), $y^{(n)}(\tau)$ is the classical derivative with respect to τ. If $\alpha = 1$, ${}_C D_{0,t}^{\alpha} y(t)$ is the typical derivative $y'(t)$. This means that in (1), if $(\alpha_1, \alpha_2, \cdots, \alpha_n) = (1, 1, \cdots, 1)$, then system (1) is the ordinary differential system.

Based on the reliable numerical approach and numerical tracking technique, this letter studies the evolution of chaotic dynamics of fractional Lorenz system and Rössler system. The "efficient dimensions", which play the role of damping, for these systems to have complete chaotic attractors at different parameter values are obtained. The rest of the paper is arranged as follows. In Section 2, we introduce how to find such a smallest bound by computational method and give numerical examples. Finally, we draw conclusions in the last section.

2. Numerical exploration of the smallest efficient dimension

First, we consider the scalar form of (1). This system has a unique analytic solution under suitable conditions. This solution solves the following Volterra integral equation,

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{3}$$

In order to numerically compute system (1), we apply the fractional Adams method [23]. Set $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$. Then (3) can be discretized as follows,

$$x_h(t_{n+1}) = x(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j)), \tag{4}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & \\ \quad - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n, \\ 1, & j = n+1, \end{cases}$$

$$x_h^p(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j)),$$

and

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha}).$$

If ${}_C D_{0,t}^{\alpha} x(t) \in C^2[0, T]$, $\alpha \in (0, 1)$, then the truncated error estimate is

$$\max_{j=0,1\cdots N} |x(t_j) - x_n(t_j)| = O(h^{1+\alpha}).$$

In this article, we firstly study the following fractional Lorenz system

$$\begin{cases} {}_C D_{0,t}^{\alpha_1} x_1(t) &=& (25\gamma + 10)(x_2 - x_1), \\ {}_C D_{0,t}^{\alpha_2} x_2(t) &=& (28 - 35\gamma)x_1 - x_1 x_3 \\ && +(29\gamma - 1)x_2, \\ {}_C D_{0,t}^{\alpha_3} x_3(t) &=& x_1 x_2 - \frac{\gamma+8}{3} x_3. \end{cases} \tag{5}$$

When $\gamma \in [0, 0.8)$, system (5) belongs to fractional generalized Lorenz system [25]. I this paper, based on the scheme (4) we apply numerical tracking technique to find the smallest value of efficient dimension of (5) for fixed γ, say, $\gamma = 0.78$.

In the following, we introduce the calculation procedure.

1^0 First, fix $\alpha_2 = \alpha_3 = 1$, let α_1 decrease with step length $\triangle s = 0.1$. After k steps, $\alpha_1(k) = 1 - 0.1k$. If system (5) has no chaotic attractor for $\alpha_1(k)$ and α_2, α_3, but has a chaotic attractor for $\alpha_1(k-1) = 1 - 0.1(k-1)$ and the same α_2, α_3 values. Next we increase $\alpha_1(k-1) = 1 - 0.1(k-1)$ with a new step length $\triangle' s = 0.01$. After k' steps, $\alpha_1 = 1 - 0.1(k-1) - 0.01k'$. If system (5) has no chaotic attractor for $\alpha_1 = 1 - 0.1(k-1) - 0.01k'$ and α_2, α_3 but has one for $\alpha_1^0 = 1 - 0.1(k-1) - 0.01(k'-1)$ and the same α_2, α_3 values. We think the smaller efficient dimension $\mid q \mid = \alpha_2 + \alpha_3 + 1 - 0.1(k-1) - 0.01(k'-1)$ such that system (5) has a chaotic attractor. Now we can stop here.

2^0 Fix $\alpha_1 = 1 - 0.1(k-1) - 0.01(k'-1)$, $\alpha_3 = 1$, now let α_2 decrease as above and find a smaller value α_2^0 such that system (5) is chaotic.

3^0 Fix $\alpha_1 = 1 - 0.1(k-1) - 0.01(k'-1)$, $\alpha_2 = \alpha_2^0$, now let α_3 decrease as above and find a smaller value α_3^0 such that system (5) is chaotic.

4^0 The smallest efficient dimension $|q| = \alpha_1^0 + \alpha_2^0 + \alpha_3^0$ is what we find.

It is surprisingly found that chaos is generated when $\alpha_1^0 = 0.52$, $\alpha_2^0 = 0.36$, $\alpha_3^0 = 0.44$ for fixed $\gamma = 0.78$, in other words, the efficient dimension $\mid q \mid = \alpha_1^0 + \alpha_2^0 + \alpha_3^0 = 1.32$. In what follows, we list partial figures for fractional Lorenz system, see Figs. 1-2.

Another example is the fractional Rössler system, described by

$$\begin{cases} {}_C D_{0,t}^{\alpha_1} x_1(t) = -(x_2 + x_3), \\ {}_C D_{0,t}^{\alpha_2} x_2(t) = x_1 + p x_2, \\ {}_C D_{0,t}^{\alpha_3} x_3(t) = x_3(x_1 - 10) + 0.2, \end{cases} \tag{6}$$

where system parameter p is allowed to vary.

By almost the same procedure as that of system (5), we can get the smallest efficient dimension $\mid q \mid = \alpha_1 + \alpha_2 + \alpha_3 = 2.04$.

3. Conclusions

To model and/or analyze fractional differential equation [26], etc, especially to disclose the dynamics of fractional differential systems, have been become more and more important due to their real-world applications [4,7]. For chaotic fractional differential systems, the definition of efficient dimension was firstly mentioned in [13], where a novel but not mathematically rigorous numerical method was applied to the fractional Lorenz system, and where to seek the smallest efficient dimension for the chaotic fractional system was also proposed. In this paper, we apply the fractional Adams method and numerical tracking technique to find the smallest efficient dimensions of chaotic Lorenz and Rössler systems. The numerical results and computer graphics show that the tracking technique is efficient. The algorithm and technique presented in this paper can be applied to other fractional differential systems.

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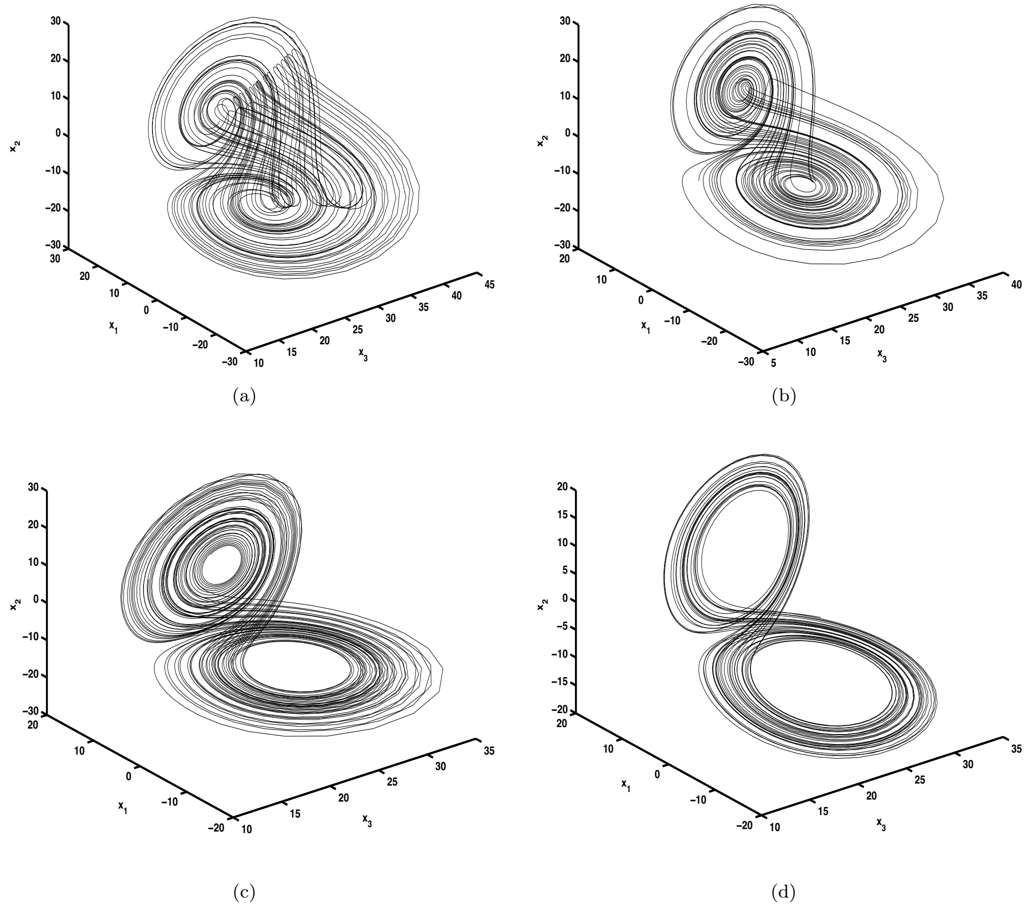

Fig. 1. (a), (b), (c), (d) The fractional Lorenz system. Initial value $(x_{10}, x_{20}, x_{30}) = (-15.5, -17.48, 35.64)$, $\gamma = 0.78$, time step length $h = 0.004$, the number $N = 3500$, the first 100 points are removed. (a) $(\alpha_1, \alpha_2, \alpha_3,) = (0.99, 0.99, 0.99)$, x_1 vs x_2 vs x_3. (b) $(\alpha_1, \alpha_2, \alpha_3,) = (0.9, 0.9, 0.9)$, x_1 vs x_2 vs x_3. (c) $(\alpha_1, \alpha_2, \alpha_3,) = (0.85, 0.85, 0.85)$, x_1 vs x_2 vs x_3. (d) $(\alpha_1, \alpha_2, \alpha_3,) = (0.84, 0.8, 0.8)$, x_1 vs x_2 vs x_3.

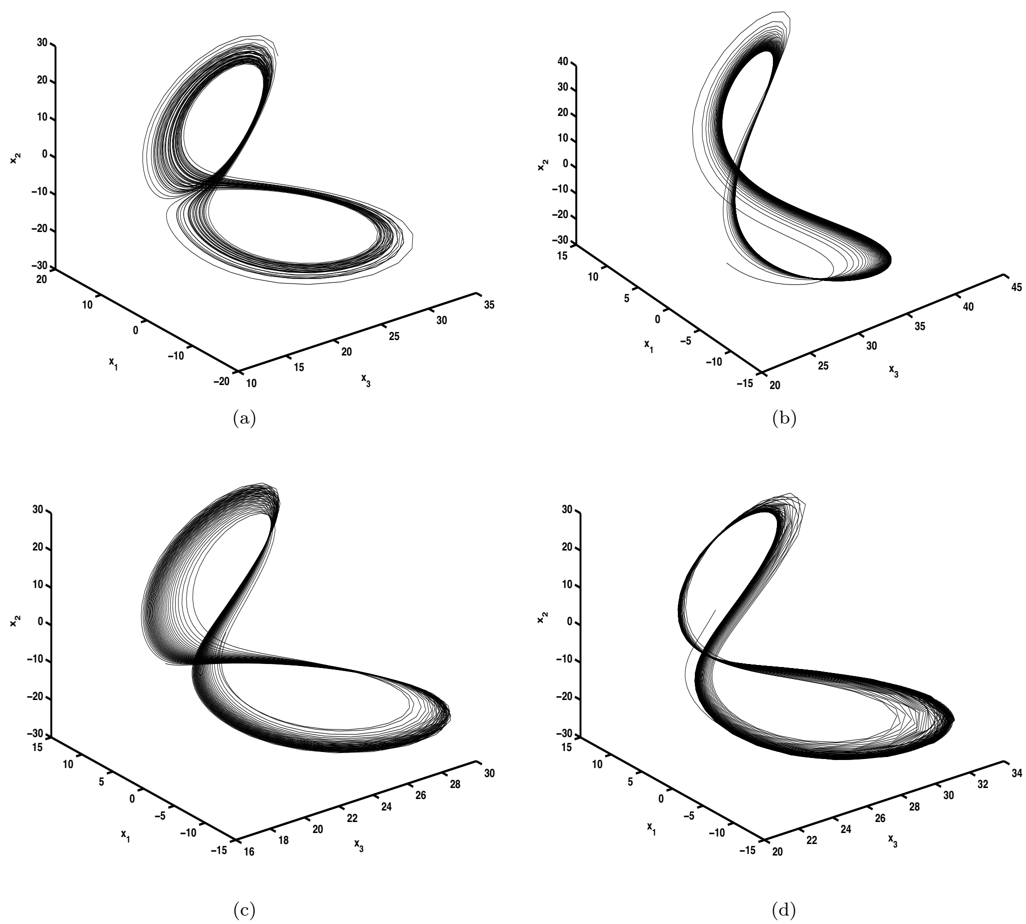

Fig. 2. (a), (b), (c), (d) The fractional Lorenz system. Initial value $(x_{10}, x_{20}, x_{30}) = (-4.755, -1.4273, 21.922)$, $\gamma = 0.78$, , the number $N = 3800$, the first 100 points are removed. (a) time step length $h = 0.003$ $(\alpha_1, \alpha_2, \alpha_3,) = (0.81, 0.68, 0.68)$, x_1 vs x_2 vs x_3. (b) time step length $h = 0.0005$ $(\alpha_1, \alpha_2, \alpha_3,) = (0.72, 0.5, 0.55)$, x_1 vs x_2 vs x_3. (c) time step length $h = 0.0005$ $(\alpha_1, \alpha_2, \alpha_3,) = (0.61, 0.45, 0.55)$, x_1 vs x_2 vs x_3. (d) time step length $h = 0.00018$ $(\alpha_1, \alpha_2, \alpha_3,) = (0.52, 0.36, 0.44)$, x_1 vs x_2 vs x_3.

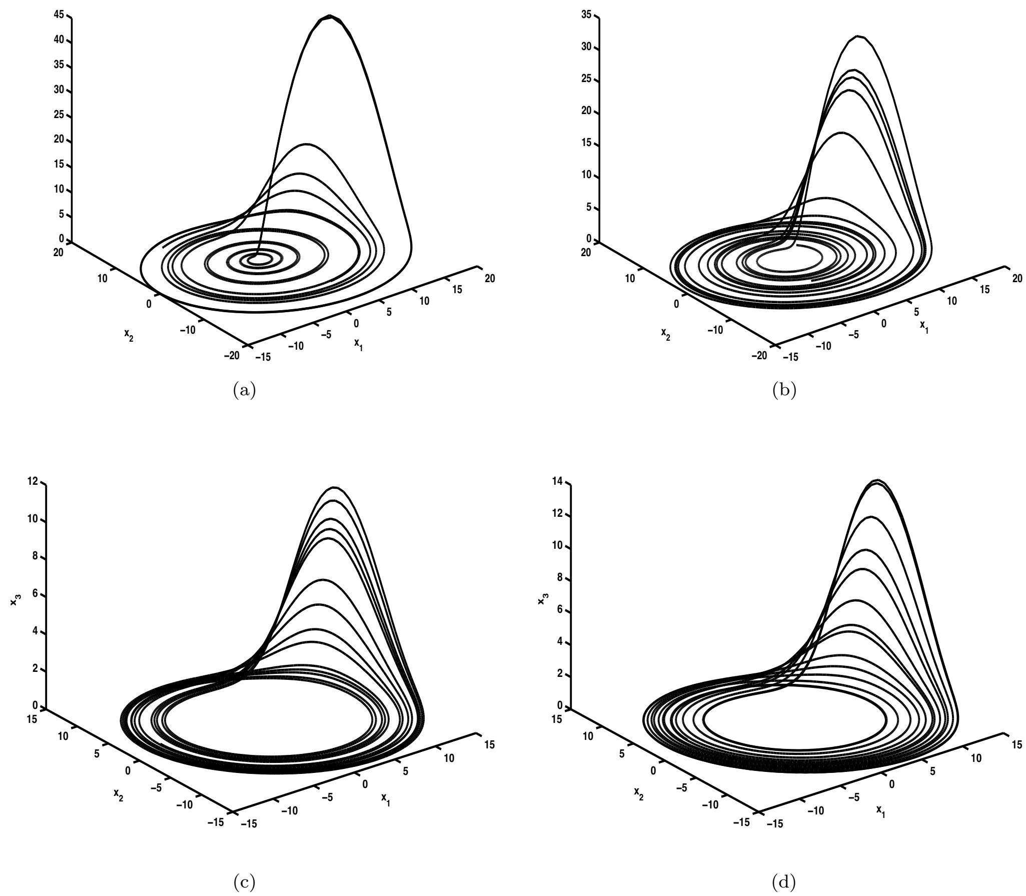

Fig. 3. (a), (b), (c), (d) The fractional Rössler system, Initial value $(x_{10}, x_{20}, x_{30}) = (-15, -15, 0)$, time step length $h = 0.03$, the number $N = 3500$, the first 500 points are removed. (a) $p = 0.2$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.99, 0.99, 0.99)$ x_1 vs x_2 vs x_3. (b) $p = 0.3$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.95, 0.95, 0.95)$ x_1 vs x_2 vs x_3. (c) $p = 0.4$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.9, 0.9, 0.9)$ x_1 vs x_2 vs x_3.. (d) $p = 0.5$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.87, 0.87, 0.87)$ x_1 vs x_2 vs x_3.

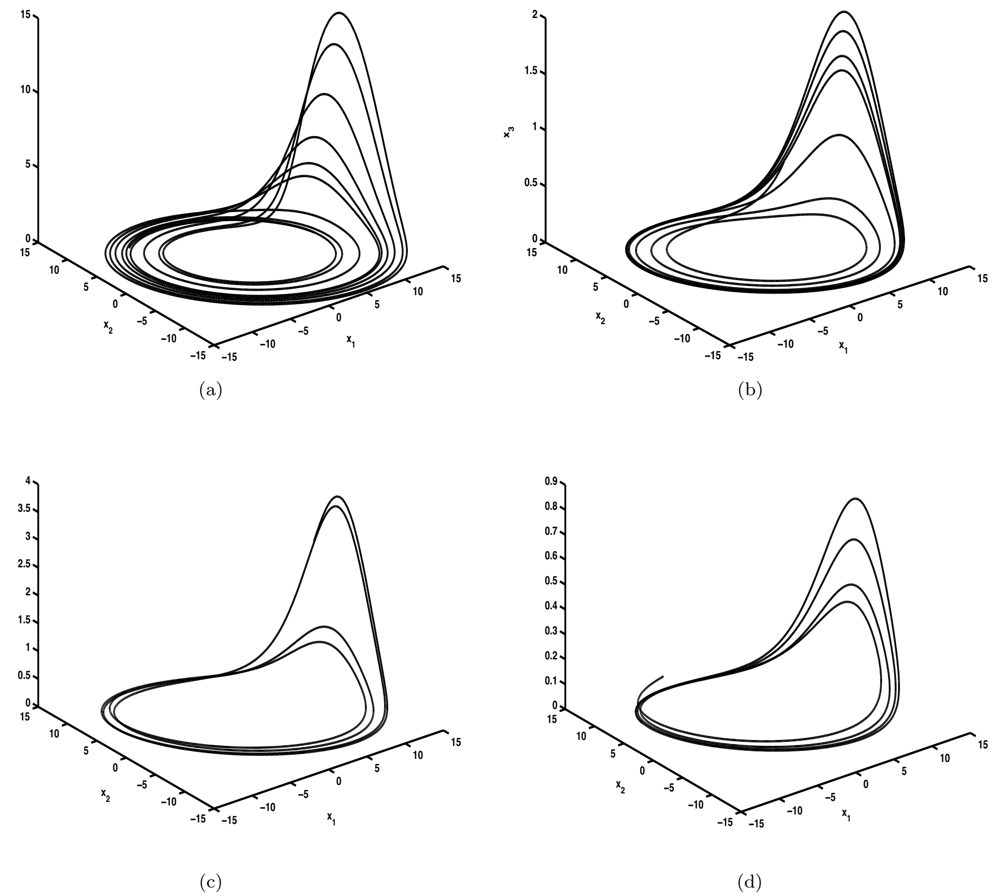

Fig. 4. (a), (b), (c), (d) The fractional Rössler system, Initial value $(x_{10}, x_{20}, x_{30}) = (-10.359, 2.4176, 0.011131)$, the number $N = 3800$, the first 500 points are removed. (a) $h = 0.03$, $p = 0.6$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.84, 0.84, 0.84)$ x_1 vs x_2 vs x_3. (b) $h = 0.02$, $p = 0.8$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.75, 0.75, 0.75)$ x_1 vs x_2 vs x_3. (c) $h = 0.015$, $p = 0.9$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.7, 0.7, 0.7)$ x_1 vs x_2 vs x_3. (d) $h = 0.008$, $p = 0.9$, $(\alpha_1, \alpha_2, \alpha_3,) = (0.68, 0.68, 0.68)$ x_1 vs x_2 vs x_3.

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