RECURSIVE FORMULAS RELATED TO THE SUMMATION OF THE MÖBIUS FUNCTION

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Abstract. For positive integers n, let $\mu(n)$ be the Möbius function, and $M(n)$ its sum $M(n) = \sum_{k=1}^{n} \mu(k)$. We find some identities and recursive formulas for computing $M(n)$; in particular, we present a two-parametric family of recursive formulas.

1. Introduction

The well-known Möbius function $\mu(n)$ is defined, for positive integers n, as

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ different prime numbers}, \\ 0 & \text{if there exists a prime } p \text{ such that } p^2 \text{ divides } n \end{cases}$$

(see [1, Chapter 2]). Then, for every real number $x \geq 0$, the summation of the Möbius function is defined by taking

$$M(x) = M(\lfloor x \rfloor) := \sum_{k=1}^{\lfloor x \rfloor} \mu(k).$$

In what follows, and as usually, we refer to $M(x)$ as the Mertens function, although, before F. Mertens (who used it in 1897, see [2]), T. J. Stieltjes already had introduced this function in his attempts to prove the Riemann Hypothesis (see [3, Lettre 79, p. 160–164], dated in 1885).

The behaviour of $M(x)$ is rather erratic and difficult of analyze, but it is very important in analytic number theory. In 1912, J. E. Littlewood [4] proved that the Riemann Hypothesis is equivalent to this fact:

$$|M(x)| = O(x^{1/2+\varepsilon}), \quad \text{when } x \to \infty, \quad \text{for every } \varepsilon > 0; \tag{1}$$

in relation to this subject, see also [5]. Of course, it is not yet known if (1) is true or false. Previously, in 1897, Mertens [2] had given a table of values of $M(n)$ for $1 \leq n \leq 10000$. Relying on this table, he conjectured that, for $x > 1$,

$$|M(x)| < \sqrt{x}.$$

This conjecture was disproved, in 1985, by A. M. Odlyzko and H. te Riele [6], but they did not find an explicit counterexample. Actually, for every value of $M(n)$ computed up to that date, always happened $|M(n)| < 0.6\sqrt{n}$. In 1987, J. Pintz [7] proved that the Mertens conjecture is false for some $n < \exp(3.21 \times 10^{64})$; and this was improved further recently in 2006 by T. Kotnik and H. te Riele [8], who showed that the Mertens conjecture is false for some $n < \exp(1.59 \times 10^{40})$. More studies about the order of the Mertens function can

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be found in [9] and [10]. Nowadays, to find an explicit counterexample of the Mertens conjecture is yet a very pursued result in number theory, and it generally believed that no counterexample will be found for $n < 10^{20}$.

To evaluate $M(n)$, a big quantity of recursive formulas appear in the mathematical literature. For instance, Stieltjes [3, Letter 79, p. 163] proved the expression

$$\sum_{k\le\sqrt{n}} (-1)^{k-1} M(n/k) = -1 + M(\sqrt{n})z(\sqrt{n}) - \sum_{k\le\sqrt{n}} z(n/k)\mu(k) \tag{2}$$

where $z(x) = 0$ if $\lfloor x \rfloor$ is even and 1 if it is odd; some other recursive formulas appear in the famous *Primzahlen* of E. Landau [11]. In 1996, M. Deléglise and J. Rivat [12], used an algorithm derived from the recurrence formula

$$M(x) = M(u) - \sum_{a\le u} \mu(a) \sum_{\frac{u}{a}<b\le\frac{x}{a}} M\left(\frac{x}{ab}\right) \tag{3}$$

(being $1 \le u \le x$) to evaluate $M(10^{16}) = -3195437$. More recursive formulas can be found in [13], and [14]; also, a large number of further references to related studies, including a nice historical review, are given in [15].

The aim of this paper is to prove different identities and recursive formulas satisfied by the Mertens function M. We devote to this end sections 2, 3 and 4; see Theorems 2, 3, 6, 9, and 10. For instance, in Theorem 3 we present a formula to evaluate $M(n)$ similar to the one given by its definition, but with only $\left\lfloor \frac{n}{3} \right\rfloor$ summands. Also, let us note the interesting expansion for $2M(n)+3$ that appears in Theorem 6, as well as the properties of the involved coefficients, studied below; they will lead us to Theorems 9 and 10. In particular, Theorem 10 gives a two-parametric family of recursive formulas for computing the Mertens function. As long as we know, all the "theorems" that we present in these sections are new; however, some of the "propositions" are already known, and we have included them by completeness.

Finally, in section 5, we study some properties of a function (that we will denote $H(n,m)$) related with the ones that appear in the previous sections; in particular, we prove the periodicity of this function.

2. Formulas in which only M appears

Let us begin by recalling the following well-known property of the Möbius function:

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases} \tag{4}$$

Indeed, it is trivial for $n = 1$. And, for $n > 1$, if $n = \prod_{j=1}^{k} p_j^{\alpha_j} > 1$ (p_j primes, $p_j \ne p_i$ for $j \ne i$), then

$$\sum_{d\mid n} \mu(d) = \binom{k}{0} - \binom{k}{1} + \cdots + (-1)^k \binom{k}{k} = (1-1)^k = 0.$$

The identity (4) allows to find a way of relating the value $M(n)$ with the values of $M(m)$, with m less than n. This result, also known (and whose proof we reproduce by completeness), is the following:

Proposition 1. *For every positive n, the Mertens function verifies*

$$1 = \sum_{a=1}^{n} M\left(\frac{n}{a}\right). \tag{5}$$

Proof. Actually, we will prove (5) also for real numbers $x \geq 1$. From the definition $M(x) = \sum_{k \leq x} \mu(k)$, we have

$$\sum_{a=1}^{\lfloor x \rfloor} M\left(\frac{x}{a}\right) = \sum_{a=1}^{\lfloor x \rfloor} \sum_{b=1}^{\lfloor \frac{x}{a} \rfloor} \mu(b).$$

If $ab = k$, then $a|k$ and, moreover, when the values of a and b vary, k takes the values $1, 2, \ldots, \lfloor x \rfloor$. Then, we have

$$\sum_{a=1}^{\lfloor x \rfloor} \sum_{b=1}^{\lfloor \frac{x}{a} \rfloor} \mu(b) = \sum_{1 \leq k \leq \lfloor x \rfloor} \sum_{a|k} \mu(a).$$

By applying (4), we get (5). □

Of course, from (5) we obtain the following recursive formula satisfied by $M(n)$:

$$M(n) = 1 - \sum_{a=2}^{n} M\left(\frac{n}{a}\right), \tag{6}$$

which is essentially one of the recursive formulae used by Neubauer [13] to compute $M(n)$ up to 10^{10}. Moreover, let us note that (4) and (5) were used by Deléglise and Rivat [12] to find the identity (3).

In (6), n summands appear. In the following theorem, we reduce the number of summands up to $\lfloor \frac{n-1}{2} \rfloor$.

Theorem 2. *If $n \geq 3$, then*

$$M(n) = -\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} M\left(\frac{n}{2a+1}\right). \tag{7}$$

Proof. If $n = 2m$ with $m > 1$, by applying (6) and (5), we get

$$\begin{aligned} M(2m) &= 1 - \sum_{a=2}^{2m} M\left(\frac{2m}{a}\right) = \sum_{a=1}^{m} M\left(\frac{m}{a}\right) - \sum_{a=2}^{2m} M\left(\frac{2m}{a}\right) \\ &= -\sum_{a=1}^{m-1} M\left(\frac{2m}{2a+1}\right) = -\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} M\left(\frac{n}{2a+1}\right). \end{aligned}$$

For the case $n = 2m+1$, let us first note that the greatest remainder that can be obtained when m is divided by a is $a-1$, and, moreover

$$\frac{a-1}{a} + \frac{1}{2a} = \frac{2a-2+1}{2a} = \frac{2a-1}{2a} < 1.$$

Thus, it is clear that

$$M\left(\frac{2m+1}{2a}\right) = M\left(\frac{m}{a} + \frac{1}{2a}\right) = M\left(\frac{m}{a}\right).$$

Then, by applying (6), (5), and this fact, we get

$$\begin{aligned} M(2m+1) &= 1 - \sum_{a=2}^{2m+1} M\left(\frac{2m+1}{a}\right) = \sum_{a=1}^{m} M\left(\frac{m}{a}\right) - \sum_{a=2}^{2m+1} M\left(\frac{2m+1}{a}\right) \\ &= -\sum_{a=1}^{m} M\left(\frac{2m+1}{2a+1}\right) = -\sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} M\left(\frac{n}{2a+1}\right). \end{aligned}$$

□

3. Formulas in which only μ appears

In the following theorem, we expand $M(n)$ as a sum with $\lfloor \frac{n}{3} \rfloor$ summands, in which only μ and the integer-part function appear. In particular, this result provides a more efficient way to compute $M(n)$ than just to use its definition $M(n) = \sum_{k=1}^{n} \mu(k)$.

Theorem 3. *If $n \geq 3$, then*

$$M(n) = -\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n-k}{2k} \right\rfloor \mu(k). \tag{8}$$

Proof. Let us remind (7) in Theorem 2. The greatest value achieved by $\left\lfloor \frac{n}{2a+1} \right\rfloor$ is $\lfloor \frac{n}{3} \rfloor$. Moreover, $\left\lfloor \frac{n}{2a+1} \right\rfloor$ takes value k if

$$k \leq \frac{n}{2a+1} < k+1,$$

i.e.,

$$\frac{n-(k+1)}{2(k+1)} < a \leq \frac{n-k}{2k}.$$

In this way, $\left\lfloor \frac{n}{2a+1} \right\rfloor = k$ for $\lfloor \frac{n-k}{2k} \rfloor - \left\lfloor \frac{n-(k+1)}{2(k+1)} \right\rfloor$ values of a.

As a consequence,

$$\begin{aligned}
M(n) &= -\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left(\left\lfloor \frac{n-k}{2k} \right\rfloor - \left\lfloor \frac{n-(k+1)}{2(k+1)} \right\rfloor \right) M(k) \\
&= -\Bigg(\left(\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n-2}{2\cdot 2} \right\rfloor \right) M(1) + \left(\left\lfloor \frac{n-2}{2\cdot 2} \right\rfloor - \left\lfloor \frac{n-3}{2\cdot 3} \right\rfloor \right) M(2) + \cdots \\
&\qquad + \left(\left\lfloor \frac{n-\lfloor \frac{n}{3} \rfloor}{2\lfloor \frac{n}{3} \rfloor} \right\rfloor - \left\lfloor \frac{n-\lfloor \frac{n}{3} \rfloor - 1}{2\left(\lfloor \frac{n}{3} \rfloor + 1\right)} \right\rfloor \right) M\left(\left\lfloor \frac{n}{3} \right\rfloor \right) \Bigg) \\
&= -\Bigg(\left\lfloor \frac{n-1}{2} \right\rfloor \mu(1) + \left\lfloor \frac{n-2}{2\cdot 2} \right\rfloor \mu(2) + \left\lfloor \frac{n-3}{2\cdot 3} \right\rfloor \mu(3) + \cdots \\
&\qquad + \left\lfloor \frac{n-\lfloor \frac{n}{3} \rfloor}{2\lfloor \frac{n}{3} \rfloor} \right\rfloor \mu\left(\left\lfloor \frac{n}{3} \right\rfloor \right) - \left\lfloor \frac{n-\lfloor \frac{n}{3} \rfloor - 1}{2\left(\lfloor \frac{n}{3} \rfloor + 1\right)} \right\rfloor M\left(\left\lfloor \frac{n}{3} \right\rfloor \right) \Bigg).
\end{aligned}$$

Now, let us observe

$$\frac{n-\lfloor \frac{n}{3} \rfloor - 1}{2\left(\lfloor \frac{n}{3} \rfloor + 1\right)} = \begin{cases} \frac{3m-m-1}{2m+2} = \frac{2m-1}{2m+2} < 1 & \text{if} \quad n = 3m, \\ \frac{3m+1-m-1}{2m+2} = \frac{2m}{2m+2} < 1 & \text{if} \quad n = 3m+1, \\ \frac{3m+2-m-1}{2m+2} = \frac{2m+1}{2m+2} < 1 & \text{if} \quad n = 3m+2, \end{cases}$$

and so

$$\left\lfloor \frac{n-\lfloor \frac{n}{3} \rfloor - 1}{2\left(\lfloor \frac{n}{3} \rfloor + 1\right)} \right\rfloor = 0.$$

Then, (8) follows. $\square$

The following result relates the value of $\mu(n)$ to the values of $\mu(m)$ for $1 \leq m < n$. Actually, this result is already known (see [1, Theorem 3.12]), although the proof that we make in this paper is different and, perhaps, new; here, we use an argument similar to the one used in the proof of Theorem 3.

Proposition 4. *The Möbius function satisfies*

$$1 = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \mu(k). \tag{9}$$

Proof. By Proposition 1, $1 = \sum_{a=1}^{n} M\left(\frac{n}{a}\right)$. Here, we have $\left\lfloor \frac{n}{a} \right\rfloor = k$ if and only if

$$k \le \frac{n}{a} < k+1,$$

i.e.,

$$\frac{n}{k+1} < a \le \frac{n}{k},$$

and so $M\left(\frac{n}{a}\right) = M\left(\left\lfloor \frac{n}{a} \right\rfloor\right) = M(k)$ for $\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n}{k+1} \right\rfloor$ values of a.

Then

$$\begin{aligned}
1 &= \sum_{a=1}^{n} M\left(\frac{n}{a}\right) = \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n}{k+1} \right\rfloor \right) M(k) \\
&= \left(\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right) M(1) + \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \right) M(2) + \cdots + \left(\left\lfloor \frac{n}{n} \right\rfloor - \left\lfloor \frac{n}{n+1} \right\rfloor \right) M(n) \\
&= \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor (M(2) - M(1)) + \cdots + \left\lfloor \frac{n}{n} \right\rfloor (M(n) - M(n-1)) - \left\lfloor \frac{n}{n+1} \right\rfloor M(n) \\
&= \left\lfloor \frac{n}{1} \right\rfloor \mu(1) + \left\lfloor \frac{n}{2} \right\rfloor \mu(2) + \cdots + \left\lfloor \frac{n}{n} \right\rfloor \mu(n) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \mu(k).
\end{aligned}$$

□

Now, let us prove another expansion of 1 as a sum of μ's. We will use this result in the proof of Theorem 6.

Proposition 5. *For $n \ge 3$, the Möbius function satisfies*

$$1 = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n}{3k} \right\rfloor \mu(k).$$

Proof. First, let us suppose $n = 3m$. By Proposition 4, we have

$$1 = \sum_{k=1}^{m} \left\lfloor \frac{m}{k} \right\rfloor \mu(k) = \sum_{k=1}^{\frac{n}{3}} \left\lfloor \frac{3m}{3k} \right\rfloor \mu(k) = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n}{3k} \right\rfloor \mu(k).$$

If $n = 3m+1$,

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n}{3k} \right\rfloor \mu(k) = \sum_{k=1}^{m} \left\lfloor \frac{3m+1}{3k} \right\rfloor \mu(k) = \sum_{k=1}^{m} \left\lfloor \frac{m}{k} \right\rfloor \mu(k) = 1,$$

because, in

$$\left\lfloor \frac{3m+1}{3k} \right\rfloor = \left\lfloor \frac{m}{k} + \frac{1}{3k} \right\rfloor,$$

the remainder when m is divided by k is always less or equal than $k-1$; and, by being

$$\frac{k-1}{k} + \frac{1}{3k} = \frac{3k-2}{3k} < 1,$$

we have

$$\left\lfloor \frac{3m+1}{3k} \right\rfloor = \left\lfloor \frac{m}{k} \right\rfloor.$$

If $n = 3m+2$,

$$\left\lfloor \frac{3m+2}{3k} \right\rfloor = \left\lfloor \frac{m}{k} + \frac{2}{3k} \right\rfloor = \left\lfloor \frac{m}{k} \right\rfloor$$

by being

$$\frac{k-1}{k} + \frac{2}{3k} = \frac{3k-1}{3k} < 1;$$

thus,

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left\lfloor \frac{n}{3k} \right\rfloor \mu(k) = \sum_{k=1}^{m} \left\lfloor \frac{3m+2}{3k} \right\rfloor \mu(k) = \sum_{k=1}^{m} \left\lfloor \frac{m}{k} \right\rfloor \mu(k) = 1. \qquad \square$$

Then, we establish the following result, in which, as in Theorem 3, we show an expansion of $M(n)$ as a sum of $\lfloor \frac{n}{3} \rfloor$ summands; in every summand, only the Möbius function and a coefficient (related to the integer-part function) appear. This will be a fruitful result, because, later in this paper, we will find some alternative formulas and interesting properties for the coefficients.

Theorem 6. *For $n \geq 3$, we have*

$$2M(n) + 3 = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} g(n,k)\mu(k) \tag{10}$$

with

$$g(n,k) = 3\left\lfloor \frac{n}{3k} \right\rfloor - 2\left\lfloor \frac{n}{2k} - \frac{1}{2} \right\rfloor. \tag{11}$$

Proof. Let us add 2 times the expansion for $M(n)$ in Theorem 3 plus 3 times the expansion for 1 in Proposition 5. $\square$

The next result presents an alternative way for computing $g(n,k)$:

Proposition 7. *For $k > 0$ and $n \geq 0$, let us take n_0 such that*

$$n \equiv n_0 \mod 6k, \quad 0 \leq n_0 < 6k.$$

Then

$$g(n,k) = \begin{cases} 2 & \text{if} \quad 0 \leq n_0 < k, \\ 0 & \text{if} \quad k \leq n_0 < 3k, \\ 1 & \text{if} \quad 3k \leq n_0 < 5k, \\ -1 & \text{if} \quad 5k \leq n_0 < 6k. \end{cases} \tag{12}$$

Proof. Le us decompose $n = n_0 + 6kn_1$, with $0 \leq n_0 < 6k$. By (11),

$$g(n,k) = 3\left\lfloor \frac{n_0 + 6kn_1}{3k} \right\rfloor - 2\left\lfloor \frac{n_0 + 6kn_1 - k}{2k} \right\rfloor.$$

Then, it is clear that

$$\begin{array}{lll} \text{if} \quad 0 \leq n_0 < k, & & g(n,k) = 6n_1 - 2\left\lfloor \frac{6k(n_1-1)}{2k} + \frac{5k+n_0}{2k} \right\rfloor = 2; \\ \text{if} \quad k \leq n_0 < 3k, & & g(n,k) = 6n_1 - 6n_1 = 0; \\ \text{if} \quad 3k \leq n_0 < 5k, & & g(n,k) = 3 + 6n_1 - 6n_1 - 2 = 1; \\ \text{if} \quad 5k \leq n_0 < 6k, & & g(n,k) = 3 + 6n_1 - 6n_1 - 4 = -1. \end{array} \qquad \square$$

4. Formulas in which both M and μ appear

Let us consider the function $g(n,k)$ for fixed n, i.e., as a function of k. In the following proposition, we show how $g(n,k)$ is constant when k varies a certain interval.

Proposition 8. *Let a and n be positive integers, with $a < n$. When k varies in the interval*

$$\frac{n}{a+1} < k \le \frac{n}{a}$$

the value of $g(n,k)$ remains constant. This value depends only upon the remainder of a modulus 6.

Proof. Let us decompose $a = a_0 + 6a_1$ with $0 \le a_0 < 6$. If $\frac{n}{a+1} < k \le \frac{n}{a}$, then $ka \le n < k(a+1)$ and so

$$ka_0 + 6ka_1 \le n < k(a_0 + 6a_1 + 1). \tag{13}$$

Thus, $n = n_0 + 6ka_1$ for some n_0 verifying $0 \le n_0 < 6k$. By substituting this value of n in (13), it becomes

$$ka_0 \le n_0 < ka_0 + k. \tag{14}$$

By (12), $g(n,k)$ takes the same value for all n_0 that satisfies (14), and this value of $g(n,k)$ depends only on a_0. □

As a consequence of Proposition 8, we can define the function

$$h(a) = g(n,k) \quad \text{for} \quad \frac{n}{a+1} < k \le \frac{n}{a}. \tag{15}$$

By using (12) (pay attention to a_0 in the proof of Proposition 8), $h(a)$ takes these values:

$$h(a) = \begin{cases} 2, & \text{if} \quad a \equiv 0 \mod 6, \\ 0, & \text{if} \quad a \equiv 1 \mod 6, \\ 0, & \text{if} \quad a \equiv 2 \mod 6, \\ 1, & \text{if} \quad a \equiv 3 \mod 6, \\ 1, & \text{if} \quad a \equiv 4 \mod 6, \\ -1, & \text{if} \quad a \equiv 5 \mod 6. \end{cases} \tag{16}$$

Now, we will split the sum in (10) in two parts, introducing a parameter r. The first part will consist in the $\lfloor \frac{n}{r+1} \rfloor$ first terms of the sum in (10). The second part will be, of course, the summands that remain; they will be manipulated in such way that we will get r sumands in which only the functions h and M appear. We can say that this is a *mixed* recursive formula for computing M: $M(n)$ is obtained from $\mu(m)$ and $M(m)$ with $m < n$.

Theorem 9. *Let n and r two integers satisfying $3 \le r \le n-1$. Then*

$$2M(n) + 3 = \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + \sum_{a=3}^{r} h(a) \left(M\left(\frac{n}{a}\right) - M\left(\frac{n}{a+1}\right) \right). \tag{17}$$

Proof. By (10) and (15),

$$\begin{aligned} 2M(n) + 3 = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} g(n,k)\mu(k) &= \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + \sum_{a=3}^{r} \sum_{k=\lfloor \frac{n}{a+1} \rfloor + 1}^{\lfloor \frac{n}{a} \rfloor} g(n,k)\mu(k) \\ &= \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + \sum_{a=3}^{r} h(a) \sum_{k=\lfloor \frac{n}{a+1} \rfloor + 1}^{\lfloor \frac{n}{a} \rfloor} \mu(k) \end{aligned}$$

$$= \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + \sum_{a=3}^{r} h(a) \left(M\left(\frac{n}{a}\right) - M\left(\frac{n}{a+1}\right) \right). \qquad \square$$

Prior to continue, let us note which would be the two limit cases: of course, $r = 2$ is Theorem 6; and, when $r = n$, the sum indexed by k in (9) disappears. Another particular case appears by taking $r = \lfloor \sqrt{n} \rfloor$; thus (17) becomes

$$M(n) = \frac{1}{2} \Bigg(-3 + \sum_{k=1}^{\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \rfloor} g(n,k)\mu(k) + M\left(\frac{n}{3}\right) + \sum_{a=4}^{\lfloor \sqrt{n} \rfloor} (h(a) - h(a-1)) M\left(\frac{n}{a}\right) - h(\lfloor \sqrt{n} \rfloor) M\left(\frac{n}{\lfloor \sqrt{n} \rfloor + 1}\right) \Bigg),$$

a formula that resembles (2) after isolating $M(n)$, but starting in $M(\frac{n}{3})$ instead of $M(\frac{n}{2})$.

On the other hand, by splitting again the summand on the right in (17), we can introduce a new parameter s:

Theorem 10. *Let n, r and s be three integers such that $s \geq 0$ and $6s+9 \leq r \leq n-1$. Then*

$$2M(n) + 3 = \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + \sum_{b=0}^{s} \left(M\left(\frac{n}{3+6b}\right) - 2M\left(\frac{n}{5+6b}\right) + 3M\left(\frac{n}{6+6b}\right) - 2M\left(\frac{n}{7+6b}\right) \right) + \sum_{a=6s+9}^{r} h(a) \left(M\left(\frac{n}{a}\right) - M\left(\frac{n}{a+1}\right) \right).$$

Proof. Let us expand the summand $\sum_{a=3}^{r}$ in (17). In this way,

$$\begin{aligned} 2M(n) + 3 = {} & \sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor} g(n,k)\mu(k) + h(3)M\left(\frac{n}{3}\right) - h(3)M\left(\frac{n}{4}\right) \\ & + h(4)M\left(\frac{n}{4}\right) - h(4)M\left(\frac{n}{5}\right) + h(5)M\left(\frac{n}{5}\right) - h(5)M\left(\frac{n}{6}\right) \\ & + h(6)M\left(\frac{n}{6}\right) - h(6)M\left(\frac{n}{7}\right) + h(7)M\left(\frac{n}{7}\right) - h(7)M\left(\frac{n}{8}\right) \\ & + h(8)M\left(\frac{n}{8}\right) - h(8)M\left(\frac{n}{9}\right) + \cdots \\ & + h(6s+8)M\left(\frac{n}{6s+8}\right) - h(6s+8)M\left(\frac{n}{6s+9}\right) \\ & + \sum_{a=6s+9}^{r} h(a) \left(M\left(\frac{n}{a}\right) - M\left(\frac{n}{a+1}\right) \right). \end{aligned}$$

By applying the values of h according (16), the result follows. $\square$

Thus, in Theorem 10 we have presented a two-parametric family of recurrence relation for computing an isolated value of $M(n)$. They provide mixed ways to calculate $M(n)$ using, in part, previously computed (and stored) values of $M(m)$ for a certain values of m, and another part that must be explicitly computed. Eventually, a suitable election of parameters

r and s (that may depend on n) will allow to get efficient methods of running this algorithm in a computer; at this point, it is clear that a careful implementation must be performed, taking into account the machine to be used. The idea to use this expansion is as follows: The terms in the first sum $\sum_{k=1}^{\lfloor \frac{n}{r+1} \rfloor}$ are directly evaluated. The second sum $\sum_{b=0}^{s}$ is computed by using previously computed and stored values of M. And the third sum

$$\sum_{a=6s+9}^{r} h(a) \left(M\left(\frac{n}{a}\right) - M\left(\frac{n}{a+1}\right) \right) = \sum_{a=6s+9}^{r} h(a) \sum_{k=\lfloor \frac{n}{a+1} \rfloor + 1}^{\lfloor \frac{n}{a} \rfloor} \mu(k)$$

can be computed by using both methods, according the size of n, r and s.

Finally, let us note which are the limit cases of the identity established by Theorem 10: $s = -1$ (Theorem 9); $s = -1$ and $r = 2$ (Theorem 6); $r = n$ ($\sum_b$ disappears); and $r = 6s+8$ ($\sum_k$ disappears).

5. The function $H(n,m)$. Periodicity

In the previous sections (see Theorems 6, 9 and 10), we often obtain expressions with the form $\sum_{k=1}^{m} g(n,k)\mu(k)$. Thus, in this section, we define a new function $H(n,m)$ (for non-negative integers n and positive integers m) by taking

$$H(n,m) := \sum_{k=1}^{m} g(n,k)\mu(k),$$

and we are going to study some of its properties. Also, we will use the following notation:

$$C_m := 6 \cdot \operatorname{lcm}\{1, 2, \ldots, m\}.$$

First, let us see that, when we fix m in the second variable of H, the function is periodic with period C_m.

Proposition 11. *For every non-negative integer t, we have*

$$H(n + tC_m, m) = H(n,m).$$

Proof. By being $g(n+6kt, k) = g(n,k)$ for $k = 1, 2, \ldots, m$, we have $g(n + C_m t, k) = g(n,k)$. Thus, the result follows. □

The following result gives the value of $H(n,m)$ as a function of $M(m)$.

Proposition 12. *For every non-negative integer t, we have*

$$\begin{aligned} H(0 + tC_m, m) &= 2M(m), \\ H(1 + tC_m, m) &= 2M(m) - 2, \\ H(2 + tC_m, m) &= 2M(m), \\ H(n + tC_m, m) &= 2M(m) + 3, \qquad \textit{if } \ 2 < n \le m. \end{aligned}$$

Proof. By Proposition 11, without loss of generality, we can suposse $t = 0$. Then, it is enough for computing $H(n,m)$ for $0 \le n \le m$. First, let us analyze the cases $n = 0, 1, 2, 3$. By applying (12), we have

$$\begin{aligned} H(0,m) &= \sum_{k=1}^{m} g(0,k)\mu(k) = 2\sum_{k=1}^{m} \mu(k) = 2M(m), \\ H(1,m) &= \sum_{k=1}^{m} g(1,k)\mu(k) = 0 \cdot \mu(1) + 2\sum_{k=2}^{m} \mu(k) = 2M(m) - 2, \end{aligned}$$

$$H(2,m) = \sum_{k=1}^{m} g(2,k)\mu(k) = 0 \cdot \mu(1) + 0 \cdot \mu(2) + 2\sum_{k=3}^{m} \mu(k) = 2M(m).$$

For n verifying $2 < n \le m$, let us decompose

$$H(n,m) = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} g(n,k)\mu(k) + \sum_{k=\lfloor \frac{n}{3} \rfloor + 1}^{n} g(n,k)\mu(k) + \sum_{k=n+1}^{m} g(n,k)\mu(k). \tag{18}$$

Now, in the first sum, let us apply (10); in the second sum, let us use that, for $\lfloor \frac{n}{3} \rfloor < k \le n$ (i.e., $k \le n < 3k$) we have $g(n,k) = 0$ (see (12)); and, finally, for the third sum, let us note that $g(n,k) = 2$ for $0 \le n < n+1 \le k$. In this way, (18) becomes

$$H(n,m) = (2M(n) + 3) + 0 + 2\,(M(m) - M(n)) = 2M(m) + 3. \qquad \square$$

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