Tensor Products of the Gassner Representation of The Pure Braid Group†

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Abstract: The reduced Gassner representation is a multi-parameter representation of P_n, the pure braid group on n strings. Specializing the parameters $t_1, t_2,...,t_n$ to nonzero complex numbers $x_1,x_2,...,x_n$ gives a representation $G_n(x_1,...,x_n)$: $P_n \to GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $x_1...x_n \neq 1$.We find a sufficient condition that guarantees that the tensor product of an irreducible $G_n(x_1,...,x_n)$with an irreducible $G_n(y_1, ..., y_n)$ is irreducible. We fall short of finding a necessary and sufficient condition for irreducibility of the tensor product. Our work is a continuation of a previous one regarding the tensor product of complex specializations of the Burau representation of the braid group.

Keywords: Braid group, Pure braid group, Gassner representation.

1. INTRODUCTION

The pure braid group, P_n, is a normal subgroup of the braid group, B_n, on n strings. It has a lot of linear representations. One of them is the Gassner representation which comes from the embedding $P_n \to \text{Aut}(F_n)$, by means of Magnus representation [1, p.119]. According to Artin, the automorphism corresponding to the braid generator σ_i takes x_i to $x_ix_{i+1}x_i^{-1}$,x_{i+1} to x_i, and fixes all other free generators. Applying this standard Artin representation to the generator of the pure braid group, we get a representation of the pure braid group by automorphisms. The Gassner representation is obtained from the Artin representation via the free differential calculus, as is explained in chapter 3 of Birman's book [1]. Such a representation has a composition factor, the reduced Gassner representation $G_n(t_1,...,t_n)$: $P_n \to GL_{n-1}(\mathbb{C}[t_1^{\pm1},...,t_n^{\pm1}])$, where $t_1,...,t_n$ are indeterminates. Specializing $t_1,...,t_n$ to nonzero complex numbers $x_1,...,x_n$ defines a representation $G_n(x_1,...,x_n)$: $P_n \to GL_{n-1}(\mathbb{C}) = \text{GL}(\mathbb{C}^{n-1})$ which is irreducible if and only if $x_1...x_n \neq 1$ [2]. For some $j \in \{1,...,n\}$, we consider a free normal subgroup of rank $n - 1$, namely, U_j and consider the complex specialization of the reduced Gassner representation restricted to this free normal subgroup.

In [3], we considered the tensor product of complex specializations of the Burau representation of the braid group and found a necessary and sufficient condition that guarantees irreducibility. Here, we consider the tensor product of the following irreducible representation restricted to a free normal subgroup of the pure braid group, namely, U_j:

$$G_n(x_1,...,x_n) \otimes G_n(y_1,...,y_n):U_j \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}).$$

It is a known fact that tensor products of the classical irreducible representations of the pure braid groups are " generically" irreducible, as they can be obtained by a mono dromy construction. For more details, see [4]. As a consequence, the question left to be answered is: For which values of the parameters is the representation above irreducible?

Our main result, Theorem 1, is that for $n \geq 3$ and $x_1,...,x_n,y_1,...,y_n \in \mathbb{C} - \{0,1\}$, the representation above is irreducible under the following condition: for some i and $j \in \{1,...,n\}$, $i < j$, $x_ix_j \neq y_iy_j$ and $\beta_\alpha x_j\gamma_\alpha y_j \neq 1$, for every $\alpha \in \{1,...,n - 1\}$.Here $\beta_\alpha = x_\alpha$, $\gamma_\alpha = y_\alpha$ for $\alpha < j$ and $\beta_\alpha = x_{\alpha+1}$, $\gamma_\alpha = y_{\alpha+1}$ for $\alpha \geq j$. However, we fail to resolve the question whether or not the tensor product obtained is reducible under each of the following conditions specified because there is no general principle to imply reducibility. In other words, we will only find a sufficient condition for irreducibility.

The idea of the proof is similar to that in [3]. We consider a free normal subgroup of the pure braid group of rank $n - 1$ denoted by U_j where $1 \leq j \leq n$. Let $\mathbb{C}[U_j]$ be the group algebra of U_j over $\mathbb{C}$, and let Abe the augmentation ideal of $\mathbb{C}[U_j]$. If M is any U_j-module, then $\mathcal{A}M$ is a U_j-submodule of M. We first show (Lemma 2(b)) that if $\mathbb{C}^{n-1}$ is made into a U_j-module via $G_n(x_1,...,x_n)$: $U_j \to GL(\mathbb{C}^{n-1})$,then $\mathcal{A}\mathbb{C}^{n-1}$ is its unique minimal nonzero U_j-submodule. Of course $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$ when $G_n(x_1,...,x_n)$ is irreducible.

Now let $x_1,...,x_n,y_1,...,y_n \in \mathbb{C}-\{0\}$, so that $G_n(x_1,...,x_n) \otimes G_n(y_1,...,y_n)$ defines a diagonal action of U_j on $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. The main technical result is Proposition 1, which gives a sufficient condition for $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$ to be the unique minimal nonzero U_j-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. This implies the irreducibility of the tensor product above.

2. DEFINITIONS AND NOTATION

Notation 1. *The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i + 1)$, $1 \leq i \leq n - 1$. It has the following generators*:

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},\ 1 \leq i < j \leq n$$

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†Mathematics Subject Classification. Primary: 20F36.

We will construct for each $j = 1,\dots,n$ a free normal subgroup of rank $n-1$, namely, U_j. Let U_j be the subgroup generated by the elements

$$A_{1,j}, A_{2,j}, \dots, A_{j-1,j}, A_{j,j+1}, \dots, A_{j,n},$$

where $A_{i,j}$ are those generators of P_n that become trivial after the deletion of the j-th strand. For a fixed value j, the image of A_{ij} under the reduced Gassner representation is denoted by τ_i, where $\tau_i = I - P_iQ_i$. In other words, the generators of U_j are $A_{i,j}$ where $A_{i,j} = A_{j,i}$ whenever $i > j$ and $j = 1,2,\dots,n$. It is known that U_j generates a free subgroup of P_n which is isomorphic to the subgroup U_n freely generated by $\{A_{1,n}, A_{2,n}, \dots, A_{n-1,n}\}$. This is intuitively clear because it is quite arbitrary how we assign indices to the braid "strings".

For a fixed choice of j, we denote the image of the generator of U_j, namely, $A_{i,j}$, under the reduced Gassner representation by τ_i. That is

$$\tau_1 = A_{1,j}, \dots, \tau_{j-1} = A_{j-1,j}, \quad \tau_j = A_{j,j+1}, \quad \tau_{j+1} = A_{j,j+2}, \dots, \tau_{n-1} = A_{j,n},$$

Definition 1. *The reduced Gassner representation restricted to U_j is defined as follows: $\tau_i = I - P_iQ_i$ for $1 \le i \le n-1$. For $i < j$, P_i is the column vector given by*

$$(1-t_1, \dots, 1-t_{i-1}, \underbrace{I - t_it_j}_{i}, t_j(1-t_{i+1}), \dots, t_j(1-t_{j-1}),$$

$$\underbrace{t_{j+1}-1}_{j}, t_{j+2}-1, \dots, t_n-1)^T,$$

and for $i \ge j$, P_i is the column vector given by

$$(t_j(t_1-1), \dots, t_j(t_{j-1}-1), \underbrace{1-t_{j+1}, \dots, 1-t_i}_{i-j},$$

$$1-t_{i+1}t_j, t_j(1-t_{i+2}), \dots, t_j(1-t_n))^T.$$

Here T is the transpose and Q_i is the row vector given by

$$Q_i = (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0), \; 1 \le i \le n-1.$$

The definition of the reduced Gassner representation restricted to a free normal subgroup is the same, up to equivalence, as the definition in [2]. Representations given by pseudo reflections $I - A_iB_i$ and $I - C_iD_i$ are equivalent if the inner products (B_iA_j) and (D_iC_j) are conjugate by a diagonal matrix. For more details, see [5].

We identify $\mathbb{C}^{n-1}$ with $(n-1) \times 1$ column vectors. We let $e_1, \dots, e_{n-1}$ denote the standard basis for $\mathbb{C}^{n-1}$, and we consider matrices to act by left multiplication on column vectors.

Definition 2. *If $r = a_1e_1 + \cdots + a_{n-1}e_{n-1} \in \mathbb{C}^{n-1}$, the support of r, denoted supp(r), is the set $\{e_i \mid a_i \neq 0\}$. If $s = \Sigma a_{ij}(e_i \otimes e_j) \in \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$, the support of s, also denoted supp(s), is the set $\{e_i \otimes e_j \mid a_{ij} \neq 0\}$, and a_{ij} is called the coefficient of $e_i \otimes e_j$ in s.*

Definition 3. *For $t = (t_1, \dots, t_n)$, we define $v_i(t) = e_i - \tau_i(t)(e_i) = (I - \tau_i(t))(e_i)$. In other words, we have the following:*

For $1 \le i \le j-1$, we have $v_i(t)$=

$$(1-t_1, \dots, 1-t_{i-1}, \underbrace{1 - t_it_j}_{i}, t_j(1-t_{i+1}), \dots, t_j(1-t_{j-1}),$$

$$\underbrace{t_{j+1}-1}_{j}, \; t_{j+2}-1, \dots, t_n-1)^T.$$

and for $i \ge j$, we have $v_i(t)$ =

$$(t_j(t_1-1), \dots, t_j(t_{j-1}-1), \underbrace{1-t_{j+1}, \dots, 1-t_i}_{i-j},$$

$$1-t_{i+1}t_j, t_j(1-t_{i+2}), \dots, t_j(1-t_n))^T.$$

Let $v_1(t), \dots, v_{n-1}(t)$ be the columns of the matrix $N(t_1, \dots, t_n)$ defined as follows:

$$N(t) = \begin{pmatrix} 1-t_1t_j & \cdots & 1-t_1 & | & t_j(t_1-1) & \cdots & t_j(t_1-1) \\ t_j(1-t_2) & \cdots & 1-t_2 & | & t_j(t_2-1) & \cdots & t_j(t_2-1) \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ t_j(1-t_{j-1}) & \cdots & 1-t_{j-1}t_j & | & t_j(t_{j-1}-1) & \cdots & t_j(t_{j-1}-1) \\ -- & -- & -- & - & & & \\ t_{j+1}-1 & \cdots & t_{j+1}-1 & & 1-t_jt_{j+1} & \cdots & 1-t_{j+1} \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ t_n-1 & \cdots & t_n-1 & & t_j(1-t_n) & \cdots & 1-t_nt_j \end{pmatrix}$$

For $x_1, \dots, x_n \in \mathbb{C}-\{0\}$ and $x = (x_1, \dots, x_n)$, the representation $U_j \to GL_{n-1}(\mathbb{C}) = GL(\mathbb{C}^{n-1})$ obtained by specializing $t_i \to x_i$ is denoted by $G_n(x)$ and the representation $G_n(y)$ is defined in the same manner by specializing $t_i \to y_i$ and having $y = (y_1, \dots, y_n)$. Also $v_i(x), v_i(y)$ are defined analogously.

3. PRELIMINARIES

Lemma 1 records for future use the action of $\tau_1(t), \dots, \tau_{n-1}(t), v_1(t), \dots, v_{n-1}(t)$, and is proved by direct computation.

Lemma 1. *For $t = (t_1, \dots, t_n)$, we have*

(1)	$\tau_i(t)(v_i(t)) = t_it_jv_i(t)$	*for* $1 \le i \le j-1$,
	$\tau_i(t)(v_i(t)) = t_{i+1}t_jv_i(t)$	*for* $1 \le j \le i$,
(2)	$\tau_i(t)(v_k(t)) = v_k(t) + (t_{i+1}-1)v_i(t)$	*for* $j \le i < k$,
	$\tau_i(t)(v_k(t)) = v_k(t) + (t_i-1)v_i(t)$	*for* $i < k < j$,
	$\tau_i(t)(v_k(t)) = v_k(t) + t_j(1-t_i)v_i(t)$	*for* $i < j \le k$,
(3)	$\tau_i(t)(v_k(t)) = v_k(t) + (1-t_{i+1})v_i(t)$	*for* $k < j \le i$,
	$\tau_i(t)(v_k(t)) = v_k(t) + t_j(t_i-1)v_i(t)$	*for* $k < i < j$,
	$\tau_i(t)(v_k(t)) = v_k(t) + t_j(t_{i+1}-1)v_i(t)$	*for* $j \le k < i$.

Note that Lemma 1 remains true for any specialization $t_i \to x_i$, where $x_i \in \mathbb{C}^*$. For simplicity, we denote $(x_1, \dots, x_n)$ by the vector x.

Lemma 2. *Having U_j a free normal subgroup of the pure braid group, we let $G_n(x)$: $U_j \to GL(\mathbb{C}^{n-1})$ be a specialization of the Gassner representation restricted to U_j making $\mathbb{C}^{n-1}$ into a U_j-module, where $n \ge 3$. Then*

(a) *Let $\mathcal{A}$ be the kernel of the homomorphism $\mathbb{C}[U_j] \to \mathbb{C}$ induced by $\tau_i \to 1$ (the augmentation ideal). Then $\mathcal{A}\mathbb{C}^{n-1}$ is equal to the $\mathbb{C}$-vector space spanned by $v_1(x), \dots, v_{n-1}(x)$.*

(b) *If M is a nonzero U_j-submodule of $\mathbb{C}^{n-1}$, then $\mathcal{A}\mathbb{C}^{n-1} \subseteq M$. Hence $\mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal nonzero U_j-submodule of $\mathbb{C}^{n-1}$.*

(c) *If $p(x) = (x_j-1)^{n-2}(x_1x_2\dots x_n - 1) \neq 0$, then $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$, and $G_n(x_1, x_2, \dots, x_n)$ is irreducible.*

Proof. The proof is similar to that in [3, p.107]. Here, we will take the free normal subgroup, U_j, of rank $n-1$ instead of the braid group, B_n. Notice that, in the proof of (b), we need the fact that if $v_j \in M$ for some j then all $v_i \in M$. This is due to Lemma 1. As for (c), the determinant of $N(x_1,...,x_n)$ is $p(x) = (x_j - 1)^{n-2}(x_1x_2...x_n - 1)$, so if $p(x) \neq 0$ then $v_1(x),...v_{n-1}(x)$ is a basis for $\mathbb{C}^{n-1}$ and $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$. For more details, see [2].

4. PROOF OF THE MAIN THEOREM

Proposition 1. Suppose that $x = (x_1,...,x_n)$ and $y = (y_1,...,y_n) \in \mathbb{C}^n$, where $x_s,y_s \in \mathbb{C} - \{0,1\}$ for $1 \leq s \leq n$, and for some i and j $(i < j)$, we have $x_ix_j \neq y_iy_j, x_\alpha x_jy_\alpha y_j \neq 1$ for every $\alpha \in \{1,...,j-1\}$ and $x_{\alpha+1}x_jy_{\alpha+1}y_j \neq 1$ for every $\alpha \in \{j,...,n-1\}$. Let M be a nonzero U_j-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$ under the action of $G_n(x) \otimes G_n(y) : U_j \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, where $n \geq 3$. Then M contains all $v_i(x) \otimes v_j(y)$ for $1 \leq i, j \leq n-1$. Thus M contains $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$, where the action of U_j on the first factor is induced by $G_n(x_1,...,x_n)$ and the action of U_j on the second factor is induced by $G_n(y_1,...,y_n)$.

Proof. For $1 \leq j \leq n$, we consider the normal free subgroup of rank $n-1$, namely, U_j, defined as before.

First, we observe that if $e_u \otimes e_v \in supp(m)$ for some $m \in M$ then $e_\alpha \otimes e_v \in supp(\tau_u(e_u \otimes e_v))$ for every choice of $\alpha = 1,...,n-1$ and $v \neq u$. This is clear because of our assumption that none of the parameters t_i's is equal to zero or one.

Claim 1. There exists an $s \in \{1,...,n-1\}$such that $e_s \otimes e_s \in supp(m)$for some $m \in M$.

Proof of Claim 1.

Case 1: Suppose that there exists an s and $m \in M$ such that $e_s \otimes e_s \in supp(m)$, then we are done.

Case 2: Suppose that there exists (s,t) with $1 \leq s,t \leq n-1$ and $s \neq t$ such that

$$m = a(e_s \otimes e_t) + W, \tag{1}$$

where $a \in \mathbb{C}^*$ and $supp(W)$ does not contain $e_s \otimes e_t, e_t \otimes e_s$. We also assume that $supp(W)$ does not contain $e_\alpha \otimes e_\alpha$ for any α.

Then $\tau_t(m) = a(e_s \otimes e_t - v_t) + \tau_t(W)$, which implies that $e_s \otimes e_s \in supp(\tau_t(m))$ and so we are done .

Case 3: Suppose that for any pair (s,t) and any $m \in M$ such that $e_s \otimes e_t \in supp(m)$, we have that $e_t \otimes e_s \in supp(m)$ as well. That is, consider $m \in M$ such that

$$m = a(e_s \otimes e_t) + b(e_t \otimes e_s) + W, \text{ where}$$

$supp(W)$ does not contain $e_s \otimes e_t, e_t \otimes e_s$ and $e_\alpha \otimes e_\alpha$ for any α. In this case, W is either zero or its elements are of the form $\sum_{k,l}(c_{k,l}\ e_k \otimes e_l + d_{l,k}\ e_l \otimes e_k)$

Here the constants $a, b, c_{k,l}, d_{l,k} \in \mathbb{C}^*$.

Applying τ_t ,we observe that $e_i \in supp(\tau_t(e_t))$, where i is the integer given by the hypothesis of Proposition 1. Then

$$\tau_t(m) = a(e_s \otimes e_i) + b(e_i \otimes e_s) + W, \text{ where}$$

$supp(W)$ does not contain $e_s \otimes e_i, e_i \otimes e_s$, and both of a, b are not zeros. For simplicity, we denote $\tau_t(m)$ by m.

If $e_\alpha \otimes e_\alpha \in supp(W)$ for some α , then we are done. If not, we see that

$aM + bN =$ coefficient of $e_s \otimes e_s$ in $\tau_i(m)$ and $aM(1 + y_iy_j) + bN(1 + x_ix_j) =$ coefficient of $e_s \otimes e_s$ in $\tau_i^2(m)$.

The values of M and N are not zeros and can be obtained directly from Definition 3. The determinant

$$det\begin{pmatrix} M & N \\ M(1+y_iy_j) & N(1+x_ix_j) \end{pmatrix} = MN(x_ix_j - y_iy_j)$$

is nonzero, since $x_ix_j - y_iy_j \neq 0$ by hypothesis. Then one of $\tau_i(m), (\tau_i)^2(m)$ has $e_s \otimes e_s$ in its support.

Claim 2. Suppose that $e_\alpha \otimes e_\alpha \in supp(m)$ for some $m \in M$. Then $v_\alpha(x) \otimes v_\alpha(y) \in M$ if $x_\alpha x_jy_\alpha y_j \neq 1$ for $\alpha = 1,...,j-1$ and $x_{\alpha+1}x_jy_{\alpha+1}y_j \neq 1$ for $\alpha = j,...,n-1$.

Proof of Claim 2. A calculation shows that

$$(\tau_\alpha - 1)(\tau_\alpha - \gamma_\alpha y_j)(\tau_\alpha - \beta_\alpha x_j)(e_\alpha \otimes e_\alpha) = \gamma_\alpha\beta_\alpha x_jy_j(\beta_\alpha x_j\gamma_\alpha y_j - 1)(v_\alpha(x) \otimes v_\alpha(y))$$

and

$$(\tau_\alpha - 1)(\tau_\alpha - \gamma_\alpha y_j)(\tau_\alpha - \beta_\alpha x_j)(e_u \otimes e_v) = 0 \text{ if } (u, v) \neq (\alpha, \alpha).$$

Here we have

$$\beta_\alpha = x_\alpha, \gamma_\alpha = y_\alpha \text{ for } \alpha = 1,...,j-1$$

and

$$\beta_\alpha = x_{\alpha+1}, \gamma_\alpha = y_{\alpha+1} \text{ for } \alpha = j,...,n-1.$$

Claim 3. There exists an s such that $v_s(x) \otimes v_s(y) \in M$.

Proof of Claim 3. There exists, by claim1, an element $e_s \otimes e_s \in supp(m)$ for some $m \in M$. By Proposition 1, we have $\beta_\alpha x_j\gamma_\alpha y_j \neq 1$ for all $\alpha = 1,...,n-1$. It follows, by claim 2, that $v_s \otimes v_s \in M$.

Claim 4. For $\alpha = 1,...,n-1$, we have that $v_\alpha(x) \otimes v_\alpha(y) \in M$.

Proof of Claim 4. We have, by claim 3, an integer s such that $v_s(x) \otimes v_s(y) \in M$.

It follows that $(\sum_{k=1}^{n-1} A_ie_i) \otimes (\sum_{k=1}^{n-1} B_ie_i) \in M$, which implies that $e_\alpha(x) \otimes e_\alpha(y) \in supp(m)$ for some $m \in M$. Since $\beta_\alpha x_j\gamma_\alpha y_j \neq 1$, it follows, by claim 2, that for $\alpha = 1,...,n-1$,we have

$$v_\alpha(x) \otimes v_\alpha(y) \in M.$$

Here A_i and B_i are non zero numbers determined in Definition 3.

Claim 5. For the value i given by Proposition 1, we have that $v_i \otimes v_\alpha \in M$ and $v_\alpha \otimes v_i \in M$ for all $\alpha = 1,...,n-1$.

Proof of Claim 5. Given an integer $\alpha = 1,...,n-1$. If $\alpha = i$ then we are done by claim 4. Assume then that $\alpha \neq i$. Since $v_\alpha \otimes v_\alpha \in M$ for every α, it follows that $\tau_i(v_\alpha \otimes v_\alpha) \in M$. It follows that

$$av_i \otimes v_\alpha + bv_\alpha \otimes v_i \in M, \tag{1}$$

Applying τ_i again, we obtain that

$$ax_ix_jv_i \otimes v_\alpha + by_iy_jv_\alpha \otimes v_i \in M. \tag{2}$$

(1) and (2) imply that

$$b(x_ix_j - y_iy_j)v_\alpha \otimes v_i \in M.$$

By our hypothesis, we get that

$$v_\alpha \otimes v_i \in M \text{ and } v_i \otimes v_\alpha \in M.$$

Claim 6. $v_\alpha \otimes v_\beta \in M$ for all $\alpha,\beta \in \{1,2,...,n-1\}$.

Proof of Claim 6. Fix the value of β. Let $\alpha \in \{1,...,n-1\}$and $\alpha \neq \beta$. Then by claim 5, we get that $v_i \otimes v_\alpha, v_i \otimes v_\beta \in M$ and so

$$\tau_\alpha(v_i \otimes v_\beta) \in M.$$

This implies that

$$v_\alpha \otimes v_\beta \in M.$$

Having done this for every value of α, and consequently β, this would complete the proof.

Let $x = (x_1,...,x_n)$ and $y = (y_1,...,y_n) \in \mathbb{C}^n$, where $x_s, y_s \in \mathbb{C} - \{0,1\}$, $1 \leq s \leq n$.

Theorem 1. Let $G_n(x)$ and $G_n(y)$ *be the representations* : $U_j \to GL(\mathbb{C}^{n-1})$ *that denote the specializations of the reduced Gassner representation restricted to the free normal subgroup of the pure braid group, namely,* P_n, *where* $x_1,...,x_n,y_1,...,y_n \in \mathbb{C} - \{0,1\}$. *Let* $p(t) = (t_j - 1)^{n-2}(t_1t_2...t_n - 1)$. *For* $n \geq 3$, *consider the tensor product of irreducible representations* $G_n(x) \otimes G_n(y) : U_j \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, *where* $p(x) \neq 0$ *and* $p(y) \neq 0$.

If $x_ix_j \neq y_iy_j$ *for some* $i < j$, $x_\alpha x_j y_\alpha y_j \neq 1$ *for* $\alpha = 1,...,j-1$ *and* $x_{\alpha+1}x_j y_{\alpha+1} y_j \neq 1$ *for* $\alpha = j+1,...,n-1$ *then the above representation is irreducible.*

Proof. The proof is the same as in [3]. By Proposition 1, $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal nonzero U_j-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. In particular, it is an irreducible U_j-module. By Lemma 2 and the fact that $p(x) \neq 0$, $p(y) \neq 0$, the left factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n(x)$ and the right factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n(y)$.

Since irreducibility on a subgroup implies irreducibility on the group itself, it follows that Theorem 1 is also true for the tensor products of specializations of the Gassner representation of the pure braid group with n strings. Therefore, we get the following corollary.

Corollary 1. *Suppose that for* $x_1,...,x_n,y_1,...,y_n \in \mathbb{C} - \{0,1\}$ *and for some i,j, we have* $x_ix_j \neq y_iy_j$ *for* $1 \leq i < j \leq n$ *and* $\beta_\alpha x_j \gamma_\alpha y_j \neq 1$ *for all* $\alpha = 1,...,n-1$. *Here* $\beta_\alpha, \gamma_\alpha$ *are defined in claim* 2 *in the proof of Proposition* 1. *Then the following representation is irreducible*:

$$G_n(x) \otimes G_n(y) : P_n \to \mathrm{GL}(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}), \text{ where } p(x) \neq 0, p(y) \neq 0.$$

Our work is an extension of a previous one, where we have proved in [3] that the representation obtained, by tensoring irreducible complex specializations of the Burau representation, namely, $\beta_n(x)$ or $\hat{\beta}_n(x)$ and $\beta_n(y)$ or $\hat{\beta}_n(y)$ is irreducible if and only if $x \neq y$.

REMARKS

(1) In our work, We have found a sufficient condition for irreducibility of the representation obtained by tensoring complex specializations of the reduced Gassner representation restricted to a free normal subgroup, but we fall short of finding a necessary and sufficient condition for irreducibility of the tensor product. In the proof of our main theorem, we needed to find a pair (i,j)such that $x_ix_j \neq y_iy_j$; whereas for each $\alpha \neq j$, we needed the conditions $\beta_\alpha x_j \gamma_\alpha y_j \neq 1$ to show that $v_\alpha \otimes v_\alpha \in M$. (See claim 2 in the proof of Proposition 1). Based on the proof of Proposition 1, if we could find pure braid elements in U_j, namely, γ_m, such that $\gamma_m(v_m) = av_{m+1}$, $m = 1,...,n-1$then we could show, by induction , that if $v_m \otimes v_m \in M$ then $v_{m+1} \otimes v_{m+1} \in M$ for all values of m. In that case, the sufficient condition for irreducibility in Proposition 1 would be simpler and replaced by the following condition:

$$x_ix_j \neq y_iy_j \text{ and } \beta_\alpha x_j \gamma_\alpha y_j \neq 1 \text{ for some values of } i, j, \alpha$$

Some further work is needed to investigate whether or not there are pure braids in U_j, namely, γ_m such that $\gamma_m(v_m) = av_{m+1}$, $m = 1,...,n-1$.

(2) In [2], it was proved that if $p(x) = (x_j - 1)^{n-2}(x_1...x_n - 1) \neq 0$ then the representation $G_n(x_1,...,x_n) : U_j \to GL(\mathbb{C}^{n-1})$ is irreducible. In a future work, we will attempt to describe the composition factors of $G_n(x_1,...,x_n)$ when it is reducible, that is when $p(x) = 0$. In other words, it will be useful to answer the following question:

If $G_n(x) : P_n \to GL(\mathbb{C}^{n-1})$ is irreducible, when and how uniquely can it be extended to an irreducible representation $\tilde{G}_n(x_1,...,x_n) : P_{n+1} \to GL(\mathbb{C}^{n-1})$? This question was answered in the case of the braid group [5, p.284]. If we succeed to answer the question above regarding the pure braid group then the statements in Theorem 1 and Corollary 1 will also hold true for the tensor products of irreducible $G_n(x_1,...,x_n)$ or $\tilde{G}_{n-1}(x_1,...,x_n)$ with an irreducible $G_n(y_1,...,y_n)$ or $\tilde{G}_{n-1}(y_1,...,y_n)$.

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Received: October 29, 2008 Revised: December 26, 2008 Accepted: April 01, 2009