The Solubility of the Group of the Form $ABA$\textsuperscript{†}

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Abstract: In this paper, we show that if $A$ and $B$ are abelian subgroups of coprime orders and $A$ is self normalizing then $G = ABA$ possesses a normal complement to $A$. The proof presented here is direct and elementary.

1. INTRODUCTION

In [1] Gorenstein and Herstein have shown that if $G = ABA$ where $A$ and $B$ are both cyclic subgroups of relatively prime orders, then $G$ is solvable, moreover the Sylow $p$-subgroups of $G$, for odd $p$, are abelian and the Sylow 2-subgroups of $G$ are either abelian or isomorphic to the Quaternion group. Furthermore if $N_G(A) = A$ then $G$ contains a normal complement to $A$. In general, Gorenstein [2] has proved that if $G = ABA$ where $A$ and $B$ are both cyclic subgroups and $N_G(A) = A$, then $G$ is solvable. In this paper, we show that if $A$ and $B$ are abelian subgroups of coprime orders and $A$ is self normalizing then $G = ABA$ possesses a normal complement to $A$. The proof presented here is direct and elementary.

2. PRELIMINARIES

Theorem 2.1 (H.Wielandt). Let $H$ be an abelian Hall subgroup of a group $G$. Then there is a normal complement to $H$ in $G$ if and only if no two distinct elements of $H$ are conjugate in $G$.

Proof. See [4, Corollary 10.18].

Theorem 2.2. Let $G$ be a group which possesses a nilpotent Hall $\pi$-subgroup $H$. Then every $\pi$-subgroup of $G$ is contained in a conjugate of $H$. In particular, all Hall $\pi$-subgroups of $G$ are conjugate.

Proof. See [3, 9.1.10].

By using the Wieldant’s Theorem, we can generalize the Frattini Argument and Burnside normal $p$-complement Theorem.

Proposition 2.3. Let $K$ be a normal subgroup of $G$ and suppose that $H$ is a nilpotent Hall $\pi$-subgroup of $K$ then $G = N_G(H)K$.

Proof. Let $g \in G$. Then $H^g$ is a nilpotent Hall $\pi$-subgroup of $K$. By Theorem 2.2 $H^g$, $H$ are conjugate in $K$ so there exists $k \in K$ such that $H^g = H$, thus $gk \in N_G(H)$. Hence $g \in N_G(H)K$.

Proposition 2.4. Let $H$ be a nilpotent Hall $\pi$-subgroup of $G$ then the following hold

1. Any two elements of $Z(H)$ which are conjugate in $G$ are conjugate in $N_G(H)$.
2. If $H \leq Z(N_G(H))$ then $H$ has a normal complement.

Proof. (1) Choose $x, x^g$ in $Z(H)$ where $g \in G$. Now $H, H^{x^g} \leq C_G(x)$, so by Theorem 2.2 there exists $y \in C_G(x)$ such that $H^y = H^{x^g}$. Therefore $H^{y^g} = H$, thus $yg \in N_G(H)$ and $x^{yg} = x^g$.

(2) Choose $x, x^g$ in $H = Z(H)$. By (1) there exists $y \in N_G(H)$ such that $y^g = x^g$. But $x \in Z(N_G(H))$, so $x^{yg} = x$. Since no distinct elements of $H$ are conjugate then by Theorem 2.1 $H$ has normal complement.

3. ABA-GROUPS

Theorem 3.1. Let $G$ be a group that contains abelian subgroups $A$ and $B$ with the following properties

1. $G = ABA$.
2. $A$ and $B$ have coprime orders.
3. $A$ is its own normalizer.

Then $A$ is a Hall subgroup and $G$ possesses a normal complement to $A$.

Proof. Assume the theorem is false. Suppose $G$ is a counter example of minimal order.

Set $
\pi = \{p \in \pi(A) : O_p(A) \text{ is not normal in } G\}$
\(\sigma = \{p \in \pi(A) : O_p(A) \text{ is normal in } G\}\)

It is obvious that $A = O_\pi(A) \times O_{\sigma}(A)$. For the sake of clarity, we break up the proof into a sequence of steps.

Step 1. If $A_0 \leq A$, then $N_G(A_0) = \langle N_G(A_0) \cap B \rangle A$.

Proof. It is clear.

Step 2. $O_\pi(A)$ is a Hall Subgroup of $G$ and $G$ has a normal $\pi$-complement.

Proof. Choose a Sylow $p$-subgroup $P$ of $G$ such that $O_p(A) \leq P$.

By Step 1, we have $N_G(O_p(A)) = \langle N_G(O_p(A)) \cap B \rangle A$. 

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Since $N_G(O_p(A)) < G$ and by the minimality of $G$, there exists $K \trianglelefteq N_G(O_p(A))$ such that $N_G(O_p(A)) = AK$ and $A \cap K = 1$.

Now $A$ is a Hall subgroup of $N_G(O_p(A))$, so $O_p(A) \in Syl_p(N_G(O_p(A)))$. We have

$$N_P(O_p(A)) \trianglelefteq O_p(A).$$

Therefore $N_G(O_p(A)) = O_p(A)$, thus $O_p(A) = P$. In particular, $O_p(A) \in Syl_p(G)$. Now $O_p(A)$ is a Hall subgroup of $G$.

By Step 1,

$$N_G(O_p(A)) = AN_G(O_p(A))A.$$ Let $g \in N_G(O_p(A))$. Now $O_p(A)$ is a normal subgroup of $G$, so we have

$$g \in N_G(A) = A.$$ Therefore

$$N_G(O_p(A)) = A.$$ Since $O_p(A) \trianglelefteq N_G(O_p(A)) = A$ and $O_p(A)$ is a Hall subgroup of $G$ then by Proposition 2.4 (2) $G$ has a normal complement to $O_p(A)$.

**Step 3.** $O_p(A)$ is contained in $Z(G)$.

**Proof.** We have $O_p(A) \leq C_G(O_p(A))$. Since $C_G(O_p(A))$ is a normal subgroup of $G$ and $O_p(A)$ is a nilpotent Hall subgroup of $G$ then by Proposition 2.3, we have that

$$G = N_G(O_p(A))C_G(O_p(A)).$$

$$= AC_G(O_p(A)).$$

$$= C_G(O_p(A)).$$ Hence $O_p(A) \trianglelefteq Z(G)$.

**Step 4.** Every $\sigma$-element of $G$ is contained in $O_p(A)$.

**Proof.** Let $g$ be a $\sigma$-element of $G$. Then

$$g = a_{\pi} a_{\sigma} a_{\pi} a_{\sigma}^{-1}$$

for some $a_{\pi}, a_{\pi}^\prime, a_{\sigma}, a_{\sigma}^\prime \in A$ and $b \in B$. Since $O_p(A) \leq Z(G)$ then

$$g = a_{\pi} a_{\sigma}^\prime b a_{\sigma}^{-1}. $$ By Step 3, we have

$$G = O_p(A)O_{p'}(G).$$

So $a_{\pi} a_{\sigma}^\prime b \in O_{p'}(G)$. Set

$$\overline{G} = G/O_{p'}(G).$$ Therefore

$$\overline{g} = a_{\pi} a_{\sigma}^\prime.$$ But $\overline{g}$ is $\sigma$-element, so

$$a_{\pi} a_{\sigma}^\prime \in O_{p'}(G) \cap O_p(A) = 1.$$ Thus $a_{\pi} = a_{\pi}^\prime$. Hence $g = (a_{\pi} b a_{\sigma})^{a_{\pi}^{-1}}$. Since the orders of $A$ and $B$ are relatively prime and by Step 3 we deduce that $b = 1$. Hence

$$g = a_{\pi} a_{\sigma} a_{\pi}^{-1} = a_{\sigma} a_{\sigma}^\prime.$$ **Step 5.** $O_p(A)$ is a Hall subgroup and $G$ has a normal $\sigma$-complement.

**Proof.** By Steps 4 and 5, $O_p(A)$ is a Hall normal abelian subgroup of $G$. By Proposition 2.4, $O_p(A)$ has a normal $\sigma$-complement in $G$.

Finally, $O_p(A)$ and $O_{p'}(A)$ are Hall subgroups and have normal complements in $G$. By taking the intersection of the two complements, we get a normal complement to $A$ in $G$.

**REFERENCES**


