The Lie Algebra of Local Killing Fields

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Abstract: We present an algebraic procedure that finds the Lie algebra of the local Killing fields of a smooth metric. In particular, we determine the number of independent local Killing fields about a given point on the manifold. As an application, we provide a local classification of the types of surfaces that admit the various possible Lie algebras of local Killing fields, in terms of the Gaussian curvature.

Keywords: Lie algebra, local Killing field, derived flag, curvature, locally symmetric space.

1. INTRODUCTION

Killing fields describe the infinitesimal isometries of a metric and as such play a significant role in differential geometry and general relativity. In this paper we present an algebraic method that finds the Lie algebra of the local Killing fields of a smooth metric g. In particular, we determine the number of independent local Killing fields of g about any given point. In the section following, we identify the local Killing fields of a metric with local parallel sections of an associated vector bundle W, endowed with a connection ∇. An examination of the form of the curvature of ∇ leads to a characterization of spaces of constant curvature by means of a system of linear equations. In Section 3 we investigate the Lie algebra structure of Killing fields. Section 4 includes an overview of the procedure developed in [1]. Therein the bundle generated by the local parallel sections of W is found by calculating a derived flag of subsets of W. The number of independent Killing fields of g about a point $x \in M$ is then equal to the dimension of the fibre $\widetilde{W}_x$ over x of the terminal subset of the derived flag. Associated to $\widetilde{W}_x$ is a Lie algebra canonically isomorphic to the Lie algebra K_x of local Killing fields about x. The method is illustrated by providing a short proof of a classical theorem that gives a necessary condition for a space to be locally symmetric, expressed by the vanishing of a set of quadratic homogeneous polynomials in the curvature. Section 5 considers the derived flag for surfaces. We obtain a complete classification of the local Riemannian metrics corresponding to the various possible kinds of Lie algebra K_x. Furthermore, we show that if a regular surface with non-constant curvature possesses a Killing field then the integral curves of the curvature vector field are geodesic paths.

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2. KILLING FIELDS AND CONSTANT CURVATURE

We associate to Killing fields parallel sections of a suitable vector bundle in the manner put forward by Kostant [2]. The utility of such a framework is two-fold: first, it permits us to apply algebraic techniques adapted to finding the subbundle generated by local parallel sections. Second, it enables a purely algebraic description of the Lie bracket of two Killing fields, avoiding the explicit appearance of derivatives.

Let g be a metric on a differentiable manifold M of dimension n; g is assumed to be pseudo-Riemannian of signature (p,q) unless otherwise stated. K is a Killing field of g if and only if,

$$K_{a;b} + K_{b;a} = 0 \tag{1}$$

where the semi-colon indicates covariant differentiation with respect to the Levi-Civita connection of g. It is straightforward to verify that,

$$K_{a;bc} = R_{abc}{}^{d}\, K_d \tag{2}$$

for Killing fields K, where $R_{abc}{}^{d}$ is the Riemann curvature tensor of g, defined according to,

$$A_{c;ba} - A_{c;ab} = R_{abc}{}^{d} A_d$$

The summation convention shall be used throughout.

Let W be the Whitney sum $W := T^*M \oplus \Lambda^2 T^*M$. A local section of W has the form $X = K + L$, where $K = K_a dx^a$ is a local section of T^*M and $L = L_{ab} dx^a \wedge dx^b$ is a local section of $\Lambda^2 T^*M$. Define a connection ∇ on W by,

$$\nabla_i X = (K_{a;i} - L_{ai})dx^a + (L_{ab;i} - R^{c}_{abi} K_c)dx^a \wedge dx^b \tag{3}$$

For an open subset $U \subseteq M$, let K_U denote the local Killing fields $K : U \to T^*M$ and let P_U denote the local parallel sections $X : U \to W$; the subscript U shall be omitted when $U = M$. Define the map $\phi_U : K_U \to P_U$ by,

$$\phi_U(K_a) := K_a + K_{a;b} \tag{4}$$

It is clear that the image of ϕ_U does, in fact, lie in P_U. The inverse $\psi_U : P_U \to K_U$ of ϕ_U is the projection of W onto T^*M: $\psi_U(K_a + L_{ab}) := K_a$. This establishes a vector space isomorphism,

$$K_U \quad \leftrightarrow \quad P_U \tag{5}$$

Consider a vector space V with a non-degenerate, symmetric bilinear form h. Let $B = B_{abcd}$ be a covariant 4-tensor on V satisfying the following relations common to a Riemann curvature tensor:

$$B_{abcd} = B_{cdab} = -B_{bacd} = -B_{abdc} \tag{6}$$

and let $T = T_{ab\cdots}$ be an n-tensor on V with $n \geq 2$. The derivation $B \star T$ is the $(n+2)$-tensor defined by,

$$B \star T_{abcd\cdots} := B_{sbcd}T^s{}_{a\cdots} + B_{ascd}T^s{}_{b\cdots} + B_{absd}T^s{}_{c\cdots} + B_{abcs}T^s{}_{d\cdots} \tag{7}$$

Indices are raised by h.

Lemma 1. If V is 2-dimensional then $B \star L = 0$ for all $L \in \Lambda^2 V^*$.

Proof:

It shall be convenient to work in an orthonormal basis of V in which $h = diag(\eta_1, \eta_2)$, where $\eta_i = \pm 1$. Then $L^i{}_j = \eta_i L_{ij}$. Owing to the symmetries (6), there are effectively two cases to consider.

(i) $a = b = 1$ case:

$$B \star L_{abcd} = \eta_2 (B_{21cd} + B_{12cd}) L_{21} = 0$$

(ii) $a = c = 1$, $b = d = 2$ case:

$$B \star L_{abcd} = \eta_2 B_{2212} L_{21} + \eta_1 B_{1112} L_{12} + \eta_2 B_{1222} L_{21} + \eta_1 B_{1211} L_{12} = 0$$

q.e.d.

Applying the Bianchi identities, the curvature $F(i,j) := \nabla_{[i}\nabla_{j]}$ of ∇ takes the form,

$$F(i,j)(X) = (R_{ijkl;s}K^s + R \star L_{ijkl})dx^k \wedge dx^l \tag{8}$$

where $X = K_a dx^a + L_{ab} dx^a \wedge dx^b$ [3]. In the sequel, it shall be convenient to view the curvature F as a map $F : W \to \Lambda^2 T^*M \otimes W$ given by $w \mapsto F(,)(w)$. F is composed of two pieces: a K-part and an L-part. The K-part provides a description of locally symmetric spaces: g is locally symmetric if and only if $T^*M \subseteq \ker F$. The L-part, on the other hand, provides a characterization of metrics of constant sectional curvature by means of a system of homogeneous linear equations.

Proposition 2. Let g be Riemannian and $n \geq 3$. Then M is a space of constant curvature if and only if,

$$R \star L = 0$$

for all $L \in \Lambda^2 T^*M$

Expressed in terms of indices, M has constant curvature (for $n \geq 3$) if and only if for all $L \in \Lambda^2 T^*M$,

$$R_{sjkl}\, L^s{}_i + R_{iskl}\, L^s{}_j + R_{ijsl}\, L^s{}_k + R_{ijks}\, L^s{}_l = 0 \tag{9}$$

It is evident from the lemma that the proposition does not hold for $n = 2$.

Proof:

$\Rightarrow$ If g has constant curvature then $R_{ijkl} = \kappa_0(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$ with respect to an orthonormal frame, where κ_0 is a constant. Substitution of this expression into the left hand side of (9) gives zero for all skew-symmetric $L = L_{ab}$.

$\Leftarrow$ Suppose that (9) holds for all $L \in \Lambda^2 T^*M$. We shall work in an orthonormal frame $X_1, \ldots, X_n$ for g; this will allow us to deal with lowered indices throughout: $L^i{}_j = L_{ij}$. Let $i = k, j$ and l be three distinct indices in (9). This gives,

$$R_{sjil}L_{si} + R_{isil}L_{sj} + R_{ijsl}L_{si} + R_{ijis}L_{sl} = 0 \tag{10}$$

Put $L_{rs} := \delta_{rl}\delta_{sj} - \delta_{rj}\delta_{sl}$ into (10) to obtain $R_{ijij} = R_{ilil}$. It follows that for any two pairs of distinct indices $i \neq j$ and $a \neq b$, $R_{ijij} = R_{abab}$. Thus,

$$R_{ijij} = \kappa(x) \quad for\ i \neq j \tag{11}$$

where κ is some function on M.

Next, let $i = k$ and $j = l$ be two distinct indices in (9). This gives:

$$R_{ijis}L_{sj} + R_{ijsj}L_{si} = 0 \tag{12}$$

Let m be any index distinct from i and j and put $L_{rs} := \delta_{rm}\delta_{sj} - \delta_{rj}\delta_{sm}$ into (12). We obtain,

$$R_{ijim} = 0 \quad for\ i, j\ and\ m\ distinct \tag{13}$$

Consider a pair Y_1, Y_2 of orthonormal vectors in T_xM. If X_1, X_2 span the same plane as Y_1, Y_2 at x then $R(Y_1, Y_2, Y_1, Y_2) = \kappa(x)$, by (11). If Y_1, Y_2 span a plane orthogonal to X_1, X_2 then we may as well suppose $X_3 = Y_1$ and $X_4 = Y_2$, whence $R(Y_1, Y_2, Y_1, Y_2) = \kappa(x)$, from (11) again.

The last possibility is that Y_1, Y_2 and X_1, X_2 span planes that intersect through a line, which for the purpose of calculating sectional curvature we may take to be generated by $X_1 = Y_1$, by means of appropriate rotations of the pairs X_1, X_2 and Y_1, Y_2 within the respective planes they span. We may suppose, furthermore, that X_3 is the normalized component of Y_2 orthogonal to X_2; thus,

$$Y_2 = aX_2 + bX_3,$$

where $a^2 + b^2 = 1$. From (11) and (13) this gives,

$R(Y_1, Y_2, Y_1, Y_2) = R(X_1, aX_2 + bX_3, X_1, aX_2 + bX_3)$
$= a^2 R_{1212} + b^2 R_{1313}$
$= \kappa(x)$

Therefore g has constant curvature at each point $x \in M$. By Schur's Theorem, g has constant curvature.

q.e.d.

3. THE LIE ALGEBRA STRUCTURE OF K_U

Let V be an n-dimensional vector space equipped with a non-degenerate, symmetric bilinear form h, of signature (p,q), and let $B = B_{abcd}$ be a covariant 4-tensor on V satisfying the usual algebraic relations of a Riemann curvature:

$$B_{abcd} = -B_{bacd} \quad (14)$$

$$B_{abcd} = -B_{abdc} \quad (15)$$

$$B_{abcd} + B_{acdb} + B_{adbc} = 0, \quad \textit{and afortiori} \quad (16)$$

$$B_{abcd} = B_{cdab}$$

By virtue of (14) and (15) we may define a skew-symmetric, bilinear bracket operation on $V^* \oplus \Lambda^2 V^*$ by,

$$[K_a + L_{ab}, K'_a + L'_{ab}]: = L'_{ab}K^b - L_{ab}K'^b + L'_a{}^c L_{cb} - L_a{}^c L'_{cb} + B_{abcd}K^c K'^d \quad (17)$$

where indices are raised and lowered with h. If a subspace W of $V^* \oplus \Lambda^2 V^*$ is closed with respect to the bracket and satisfies the Jacobi identity then we denote the associated Lie algebra by $A(W, B, h)$.

Lemma 3. Let W be a subspace of $V^* \oplus \Lambda^2 V^*$, closed with respect to the bracket operation. The Jacobi identity holds on W if and only if for all $X = K + L, X' = K' + L'$ and $X'' = K'' + L''$ in W, where $K, K', K'' \in V^*$ and $L, L', L'' \in \Lambda^2 V^*$,

$$B \star L_{abcd}K'^c K''^d + B \star L'_{abcd}K''^c K^d + B \star L''_{abcd}K^c K'^d = 0 \quad (18)$$

Proof:

Let $K, K', K'' \in V^*$ and $L, L', L'' \in \Lambda^2 V^*$. There are four cases to consider.

(i) $K - K' - K''$ case. We have,

$$[K,[K',K'']] = [K, B_{abcd}K'^c K''^d] = B_{abcd}K^b K'^c K''^d$$

Therefore,

$$[K,[K',K'']] + [K',[K'',K]] + [K'',[K,K']]$$
$$= (B_{abcd} + B_{acdb} + B_{adbc})K^b K'^c K''^d$$
$$= 0 \quad (19)$$

by equation (16).

(ii) $K - K' - L$ case. First,

$$[K,[K',L]] = [K, L_{ab}K'^b] = B_{abcd}K^c L^d_s K'^s$$

Also,

$$[L,[K,K']] = [L, B_{abcd}K^c K'^d]$$
$$= B_{ascd}K^c K'^d L^s{}_b - L_a{}^s B_{sbcd}K^c K'^d$$

Combining these with (15) and the fact that $L = L_{ab}$ is skew-symmetric, we obtain,

$$[K,[K',L]] + [K',[L,K]] + [L,[K,K']] = B \star L_{abcd}K^c K'^d \quad (20)$$

(iii) $K - L - L'$ case. Observe that,

$$[K,[L,L']] = [K, L'L - LL'] = L'LK - LL'K$$

and,

$$[L,[L',K]] = -[L, L'K] = LL'K$$

Using these equations gives,

$$[K,[L,L']] + [L,[L',K]] + [L',[K,L]] = 0 \quad (21)$$

(iv) $L - L' - L''$ case. It is elementary to verify that,

$$[L,[L',L'']] + [L',[L'',L]] + [L'',[L,L']] = 0 \quad (22)$$

After applying (19)-(22),

$$[X,[X',X'']] + [X',[X'',X]] + [X'',[X,X']]$$

simplifies to,

$$B \star L_{abcd}K'^c K''^d + B \star L'_{abcd}K''^c K^d + B \star L''_{abcd}K^c K'^d$$

q.e.d.

Proposition 4. (i) If V is 2-dimensional then any subspace W of $V^* \oplus \Lambda^2 V^*$, closed with respect to the bracket, defines a Lie algebra $A(W, B, h)$.

(ii) If V is arbitrary and $B = 0$ then $V^* \oplus \Lambda^2 V^*$ defines a Lie algebra $A(V^* \oplus \Lambda^2 V^*, B = 0, h)$.

Proof:

(i) Follows from Lemmas 1 and 3. (ii) is immediate.

q.e.d.

For local Killing fields $K, K' \in K_U$, the Lie bracket $K'' := [K, K']$ is,

$$K''_a = L'_{ab}K^b - L_{ab}K'^b \quad (23)$$

where we have written $L_{ab} := K_{a;b}$ and $L'_{ab} := K'_{a;b}$. Furthermore, $L''_{ab} := K''_{a;b}$ is given by,

$$L''_{ab} = L'^c_a L_{cb} - L_{ac}L'_{cb} + R_{abcd}K^c K'^d \quad (24)$$

Defining a bracket on P_U by,

$$[K + L, K' + L']: = L'_{ab}K^b - L_{ab}K'^b + L'^c_a L_{cb} - L_{ac}L'_{cb} + R_{abcd}K^c K'^d \quad (25)$$

gives an isomorphism $\phi_U : K_U \to P_U$ of Lie algebras:

$$\phi_U([K, K']) = [\phi_U(K), \phi_U(K')] \quad (26)$$

For $x \in U$, define the subspace $W_{U,x}$ of W_x by,

$$W_{U,x} := \{w \in W_x : w = X(x), \quad \textit{for some} \quad X \in P_U\} \tag{27}$$

Since parallel sections of a vector bundle are determined by their value at a single point, P_U and $W_{U,x}$ are isomorphic as vector spaces via the restriction map $r_x : P_U \to W_{U,x}$, given by $r_x(X) := X(x)$. Comparing (17) and (25), we have, in fact, a Lie algebra isomorphism $r_x : P_U \to A(W_{U,x}, R_x, g_x)$. Composing this with ϕ_U characterizes the Lie algebra of K_U.

Lemma 5.

$$r_x \circ \phi_U : K_U \to A(W_{U,x}, R_x, g_x) \tag{28}$$

is an isomorphism of Lie algebras.

In order to find the Lie algebra of the local Killing fields of g about the point x it remains to calculate $W_{U,x}$ for a sufficiently small neighbourhood U of x. This is accomplished in the following section.

4. PARALLEL FIELDS AND LOCALLY SYMMETRIC SPACES

We begin by briefly reviewing the method from [1]. This describes an algebraic procedure for determining the number of independent local parallel sections of a smooth vector bundle $\pi : W \to M$ with a connection ∇. Since the existence theory is based upon the Frobenius Theorem, smooth data are required.

Let W' be a subset of W satisfying the following two properties:

P1: the fibre of W' over each $x \in M$ is a linear subspace of the fibre of W over x, and,

P2: W' is *level* in the sense that each element $w \in W'$ is contained in the image of a local smooth section of W', defined in some neighbourhood of $\pi(w)$ in M.

Let X be a local section of W'. The covariant derivative of X is a local section of $W \otimes T^*M$. Define $\widetilde{\alpha}$ by,

$$\widetilde{\alpha}(X) := \phi \circ \nabla(X)$$

where $\phi : W \otimes T^*M \to (W/W') \otimes T^*M$ denotes the natural projection taken fibrewise. If f is any differentiable function with the same domain as X then,

$$\widetilde{\alpha}(fX) = f\widetilde{\alpha}(X)$$

This means that $\widetilde{\alpha}$ defines a map,

$$\alpha_{W'} : W' \to (W/W') \otimes T^*M$$

which is linear on each fibre of W'.

The kernel of $\alpha_{W'}$ is a subset of W', which satisfies property P1 but not necessarily property P2. In order to carry out the above constructions to $\ker \alpha_{W'}$, as we did to W', the non-level points in $\ker \alpha_{W'}$ must be removed. To this end we define a leveling map S as follows. For any subset V of W satisfying P1 let $S(V)$ be the subset of V consisting of all elements v for which there exists a smooth local section $s : U \subseteq M \to V \subseteq W$ such that $v = s(\pi(v))$. Then $S(V)$ satisfies both P1 and P2.

We may now describe the construction of the maximal flat subset $\widetilde{W}$, of W. Let,

$$\begin{aligned} V^{(0)} &:= \{w \in W \mid F(,)(w) = 0\} \\ W^{(i)} &:= S(V^{(i)}) \\ V^{(i+1)} &:= \ker \alpha_{W^{(i)}} \end{aligned}$$

where, as before, $F : TM \otimes TM \otimes W \to W$ is the curvature tensor of ∇. This gives a sequence,

$$W \supseteq W^{(0)} \supseteq W^{(1)} \supseteq \cdots \supseteq W^{(k)} \supseteq \cdots$$

of subsets of W. For some $k \in N$, $W^{(l)} = W^{(k)}$ for all $l \geq k$. Define $\widetilde{W} = W^{(k)}$, with projection $\widetilde{\pi} : \widetilde{W} \to M$.

We say that the connection ∇ is *regular at* $x \in M$ if there exists a neighbourhood U of x such that $\widetilde{\pi}^{-1}(U) \subseteq \widetilde{W}$ is a vector bundle over U. $\widetilde{W}_x$ shall denote the fibre of $\widetilde{W}$ over $x \in M$.

Lemma 6. Let ∇ be a connection on the smooth vector bundle $\pi : W \to M$.

(i) If $X : U \subseteq M \to W$ is a local parallel section then the image of X lies in $\widetilde{W}$.

(ii) Suppose that ∇ is regular at $x \in M$. Then for every $w \in \widetilde{W}_x$ there exists a local parallel section $X : U \subseteq M \to \widetilde{W}$ with $X(x) = w$.

We may now describe the Lie algebra of local Killing fields about a point x.

Theorem 7. Let g be a smooth metric on a manifold M with associated connection ∇ on $W = T^*M \oplus \Lambda^2 T^*M$, which is assumed to be regular at $x \in M$. Then g has $\dim \widetilde{W}_x$ independent local Killing fields in a sufficiently small neighbourhood U of $x \in M$. Moreover, the Lie algebra of Killing fields on U is canonically isomorphic to the Lie algebra $A(\widetilde{W}_x, R_x, g_x)$.

Proof:

By Lemma 6 there exists a sufficiently small open neighbourhood U of x such that $W_{U,x} = \widetilde{W}_x$. The theorem now follows from Lemma 5.

q.e.d.

As an illustration, we provide a short algebraic proof of the classical theorem that locally symmetric spaces satisfy,

$$R \star R = 0 \tag{29}$$

[4].

Lemma 8. If M is locally symmetric then $T^*M \subseteq \widetilde{W}$.

Proof:

Suppose M is locally symmetric and let $x \in M$. Then there is an open neighbourhood U of x such that the space of Killing fields on U, whose covariant derivative vanishes at x, has dimension n. By the isomorphism given in Lemma 5, $T_x^*M \subseteq W_{U,x}$. From Lemma 6 (i) we have $W_{U,x} \subseteq \widetilde{W}_x$ and so $T_x^*M \subseteq \widetilde{W}_x$.

q.e.d.

Let $T = T_{a\cdots}$ be an n-tensor with $n \geq 1$. Define $R \cdot T$ to be the $(n+2)$-tensor obtained by contracting the rightmost index of R with the leftmost index of T:

$$R \cdot T_{abc\cdots} := R_{abc}{}^{s} T_{s\cdots} \tag{30}$$

Let $p := T^*M$, the cotangent space of M and let $p^{(1)}$ be defined as the set of all elements $v \in T^*M$ satisfying,

$$R \star R \cdot v = 0 \tag{31}$$

Define t to be the set of all $L \in \Lambda^2 T^*M$ such that,

$$R \star L = 0 \tag{32}$$

We shall assume that t has constant rank.

Proof of (29):

Let us calculate the derived flag of W supposing M to be locally symmetric. From (8), $W^{(0)} = V^{(0)} := \ker F = p \oplus t$. Let $X = K + L$ be a local section of $W^{(0)}$, where K and L are local sections of p and t, respectively. By definition, $X(x) \in V_x^{(1)}$ if and only if $\nabla_i X(x) \in W_x^{(0)}$, for all i. This is equivalent to,

$$(*):\ L_{ab;(i)} - R_{ab(i)}{}^{c} K_c \in t \text{ at } x.$$

Taking the covariant derivative of $R \star L = 0$ gives $R \star L_{;(i)} = 0$. Thus $L_{ab;(i)}$ is a local section of t. This means that $(*)$ holds if and only if $R \cdot K_x$ lies in t. This is the case precisely when $K \in p^{(1)}$ at x, and so $V^{(1)} = p^{(1)} \oplus t$. By Lemma 8, we must have $V^{(1)} = T^*M \oplus t$, whence it follows that $p^{(1)} = T^*M$ and $W^{(1)} = V^{(1)} = W^{(0)}$. Therefore $R \star R = 0$.

q.e.d.

We have also shown that for M locally symmetric, $\widetilde{W} = W^{(0)} = p \oplus t$. Conversely, if $\widetilde{W} = p \oplus t$ then $T^*M \subseteq \ker F$, from which we conclude that M is locally symmetric. The terminal subbundle of the derived flag therefore characterizes locally symmetric spaces:

$$M \text{ is a locally symmetric space if and only if } \widetilde{W} = p \oplus t. \tag{33}$$

In particular, the derived flag computes the local canonical decomposition.

As observed above, a Riemannian manifold with $\dim M \geq 3$ has constant curvature if and only if $R \star L = 0$ for all (Proposition 2). Furthermore, all manifolds of dimension $n = 1$ or 2 satisfy $R \star L = 0$ for $L \in \Lambda^2 T^*M$ (cf. Lemma 1). Since spaces of constant curvature are locally symmetric the curvature F vanishes for such manifolds. It is not difficult to see that the converse holds (for the 2-dimensional case use the canonical form of the Riemann curvature: $R_{ijkl} = c(g_{il}g_{jk} - g_{ik}g_{jl})$, where c is the Gaussian curvature). Consequently,

$$M \text{ is a space of constant curvature if and only if } \widetilde{W} = W. \tag{34}$$

Employing Theorem 7, we obtain as a corollary the familiar result that a Riemannian manifold possesses the maximal possible number $\frac{1}{2}n(n+1)$ of independent local Killing fields if and only if it is a space of constant curvature.

5. CLASSIFICATION FOR RIEMANNIAN SURFACES

In this final section we shall determine which Riemannian surfaces correspond to the various types of Lie algebra K_x. In the process, a necessary condition for a Riemannian surface to possess a Killing field is obtained. First, we recall the situation involving the maximal number of local Killing fields:

Let M be a Riemannian surface. The following are equivalent:

(i) $\dim K_x = 3$ for all $x \in M$.

(ii) M has constant Gaussian curvature c.

(iii) M is locally symmetric.

In this case,

(a) if $c = 0$ then K_x is isomorphic to the Lie algebra of $\Re^2 \times_{sd} SO(2)$;

(b) if $c > 0$ then $K_x \cong sl_2\Re$;

(c) if $c < 0$ then $K_x \cong su_2$.

The equivalence of (i)-(iii) follows from Lemma 1, (34) and the observation below (34). Assume that these conditions hold. $c = 0$ corresponds, locally, to flat Euclidean space, for which K_x is isomorphic to the Lie algebra of the semidirect product of translations and rotations. Suppose $c \neq 0$ and let X and Y be orthogonal vectors in T_x^*M with norm,

$$X^2 = Y^2 = \frac{1}{|c|}$$

Define $H \in \Lambda^2 T_x^* M$ by,

$$[X, Y] := H$$

Then,

$$[H, X] = -sg(c)Y \quad \textit{and} \quad [H, Y] = sg(c)X$$

where $sg(c)$ denotes the sign of c. Appealing to Lemma 5, this identifies K_x with $sl_2\Re$ for $c>0$ and with su_2 for $c<0$ [5].

Next, we calculate the derived flag for W assuming that the surface is *regular*: $W^{(i)} = V^{(i)}$ has constant rank for all i. The elements $K \in T^*M$ satisfying $R_{ijkl}{}^{;s}K_s$ are those for which $c^{;s}K_s = 0$. Therefore, by Lemma 1,

$$W^{(0)} = \ker\partial c \oplus \Lambda^2 T^* M \tag{35}$$

where ∂c denotes the vector field $c^{;a}$. The case $\partial c = 0$ has been handled above. Suppose therefore that $\dim\ker\partial c = 1$; that is, ∂c is non-vanishing. Then $W^{(0)}$ is a rank-two fibre bundle over M. Let $X = K + L$ be a local section of $W^{(0)}$, where K is a local section of $\ker\partial c$ and L is a local section of $\Lambda^2 T^* M$. $X(x) \in W_x^{(1)}$ is equivalent to $\nabla_i X(x) \in W_x^{(0)}$ for all i, by the definition of the derived flag. Since $\Lambda^2 T^* M \subseteq W^{(0)}$, $X(x) \in W_x^{(1)}$ if and only if,

$$K_{a;(i)} - L_{a(i)} \in \ker\partial c \tag{36}$$

At x. Taking the covariant derivative of $c^{;s}K_s = 0$ gives the equation $c^{;a}K_{a;b} = -c_{;ab}K^a$. Substituting this into (36) determines $W^{(1)}$ as the subset of all $K + L \in W^{(0)}$ satisfying,

$$c_{;ab}K^a + L_{ab}c^{;a} = 0 \tag{37}$$

By contracting (37) with K and ∂c, it is evident that $W^{(1)}$ consists of the zero elements in W along with the solutions of,

$$c_{;ab}K^a K^b + L_{ab}c^{;a}K^b = 0 \tag{38}$$

$$c_{;ab}K^a c^{;b} = 0 \tag{39}$$

where $0 \neq K \in \ker\partial c$ and $L \in \Lambda^2 T^* M$. Equation (39) has a solution $0 \neq K \in \ker\partial c$ if and only if,

$$c_{;ab}c^{;b} = fc_a \tag{40}$$

for some f. (40) may be written in terms of differential forms as,

$$dc \wedge D_{\partial c}dc = 0 \tag{41}$$

where D denotes covariant differentiation. This leads to a necessary condition for the existence of a Killing field on a Riemannian surface.

Theorem 9. If a regular Riemannian surface possesses a Killing field then,

$$dc \wedge D_{\partial c}dc = 0$$

Proof:

By Theorem 7, if a regular Riemannian surface has a Killing field then $\widetilde{W}$ has rank at least one. Since $\widetilde{W} \subseteq W^{(1)}$, equation (39) must have a non-trivial solution $K \in \ker\partial c$ at each $x \in M$. The theorem now follows from the fact that (39) is equivalent to (41).

q.e.d.

Corollary 10. Let M be a regular Riemannian surface with non-constant curvature. If M possesses a Killing field then the integral curves of ∂c are geodesic paths.

By a *geodesic path* we mean a curve that is a geodesic when appropriately parameterized.

Proof:

Equation (41) is equivalent to $D_{\partial c}\partial c = f\partial c$, which implies that integral curves of the non-vanishing vector field ∂c may be parametrized so as to give geodesics of M.

q.e.d.

An example would be the punctured paraboloid $z = x^2 + y^2$; $(x, y) \neq (0,0)$, with the induced metric from its embedding into 3-dimensional Euclidean space. The integral curves of ∂c are described by the geodesic paths $\gamma_\theta(t) = (t\cos\theta, t\sin\theta, t^2)$, up to reparametrization.

Now let us return to calculating $W^{(1)}$. If $dc \wedge D_{\partial c}dc = 0$ on the surface then the non-zero elements of $W^{(1)}$ are the solutions to (38) for which $0 \neq K \in \ker\partial c$. For any choice of non-trivial $K \in \ker\partial c$, (38) uniquely determines an element $L = L(K) \in \Lambda^2 T^* M$. Therefore $W^{(1)}$, in this case, is a rank one vector bundle over M. If, on the other hand, $dc \wedge D_{\partial c}dc \neq 0$ on M then $W^{(1)}$ is the zero bundle and there do not exist any local Killing fields. As a consequence, K_x cannot be 2-dimensional; this may also be seen directly by considering the Lie bracket operation. Henceforth we shall assume that $dc \wedge D_{\partial c}dc = 0$.

To find $W^{(2)}$, let $X = K + L$ be a local section of $W^{(1)}$. By definition, $X(x) \in W_x^{(2)}$ if and only if $\nabla_i X(x) \in W_x^{(1)}$ for all i. Owing to (37), this is equivalent to

$$c_{;ab}K'^a_{(i)} + L'_{ab(i)}c^{;a} = 0 \tag{42}$$

Where,

$$K'_{a(i)} := K_{a;(i)} - L_{a(i)}$$

$$L'_{ab(i)} := L_{ab;(i)} - R_{ab(i)}{}^{c}K_c$$

(Note that from the description of $W^{(1)}$ contained in (36) it follows that $K'_{a(i)} + L'_{ab(i)} \in W^{(0)}$). Taking the covariant derivative of (37) gives,

$$c_{;ab}K^a_{;(i)} + L_{ab;(i)}c^{;a} = -c_{;ab(i)}K^a - L_{ab}c^{;a}_{(i)} \tag{43}$$

Substituting (43) into (42) defines $W^{(2)}$ as the subset of all $K + L \in W^{(1)}$ such that,

$$c_{;abc}K^a + L_{ab}c^{;a}{}_c + L_{ac}c^{;a}{}_b + R_{abcd}c^{;a}K^d = 0 \tag{44}$$

If this has only the trivial solution then $\widetilde{W} = W^{(2)}$ is the zero bundle. Otherwise, $\widetilde{W} = W^{(2)} = W^{(1)}$ has rank one.

Theorem 11. Let M be a regular Riemannian surface. Then $\dim K_x = 1$ for all $x \in M$ if and only if,

(i) $dc \neq 0$,

(ii) $dc \wedge D_{\partial c} dc = 0$, and

(iii) equation (44) holds for all $K + L \in W^{(1)}$.

Proof:

Conditions (i)-(iii) are equivalent to $rank\ \widetilde{W} = 1$. The result now follows from Theorem 7.

q.e.d.

We summarize the discussion in this section with the following corollary.

Corollary 12. For regular Riemannian surfaces, K_x may be one of five possible Lie algebras. It is isomorphic to either $sl_2\Re$, su_2 or the Lie algebra of $\Re^2 \times_{sd} SO(2)$ when the Gaussian curvature is constant and positive, negative or zero, respectively. K_x is the 1-dimensional Lie algebra when the conditions of Theorem 11 are met. Otherwise, there do not exist any local Killing fields and K_x is trivial.

REFERENCES

[1] Atkins R. Existence of Parallel Sections of a Vector Bundle. JGP 2011; 61: 309-11.

[2] Kostant B. Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold. Trans Amer Math Soc 1955; 80.

[3] Console S, Olmos C. Curvature invariants, Killing vector fields, connections and cohomogeneity. Proc Am Math Soc 2009; 137(3): 1069-72.

[4] Helgason S. Differential geometry, lie groups, and symmetric spaces. Graduate studies in mathematics American Mathematical Society 2001.

[5] Fulton W, Harris J. Representation theory. Graduate texts in mathematics. Springer 1991.

Received: November 04, 2009 Revised: December 28, 2010 Accepted: February 28, 2011