Nonlinear Spinor Field in Bianchi Type-II Spacetime

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Abstract: Within the scope of a Bianchi type-II (BII) cosmological model we study the role of a nonlinear spinor field in the evolution of the Universe. The system allows exact solutions only for some special choice of spinor field nonlinearity.

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1. INTRODUCTION

Recently, a number of authors showed the important role that spinor fields play on the evolution of the Universe [1-11]. In these papers the authors using the spinor field as the source of gravitational one answered to some fundamental questions of modern cosmology: (i) problem of initial singularity; (ii) problem of isotropization and (iii) late time acceleration of the Universe. But most of those works were done within the scope of either FRW or Bianchi type-I cosmological models. Some works on Bianchi type-III, V, VI$_0$ and VI were also done in recent time [6, 12]. As far as I know, there is still no work on Bianchi type-II model where the author used spinor field as a source. In this report we plan to fill up that gap.

2. BASIC EQUATIONS

We consider the simplest possible spinor field model within the framework of a BI cosmological gravitational field given by the Lagrangian density

$$\mathcal{L} = \frac{R}{2\varkappa} + \frac{i}{2}[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi] - m_{sp}\bar\psi\psi + F, \tag{1}$$

where $F(I,J)$, $I = S^2 = (\bar\psi\psi)^2$ and $J = P^2 = (i\bar\psi\gamma^5\psi)^2$ is the spinor field nonlinearity and R is the scalar curvature.

The gravitational field in our case is given by a Bianchi type-II (BII) metric:

$$ds^2 = dt^2 - a_1^2(dx + zdy)^2 - a_2^2dy^2 - a_3^2dz^2, \tag{2}$$

with a_1, a_2, a_3 being the functions of time only.

The spinor field equations corresponding to the metric (1) has the form

$$i\gamma^\mu\nabla_\mu\psi - m_{sp}\psi + F_S\psi + iF_P\gamma^5\psi = 0, \tag{3}$$

$$i\nabla_\mu\bar\psi\gamma^\mu + m_{sp}\bar\psi - F_S s\bar\psi - iF_P\bar\psi\gamma^5 = 0, \tag{4}$$

where $F_S = \dfrac{dF}{dS}$ and $F_P = \dfrac{dF}{dP}$. In (1), and (2) ∇_μ is the covariant derivative of spinor field:

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \quad \nabla_\mu\bar\psi = \frac{\partial\bar\psi}{\partial x^\mu} + \bar\psi\Gamma_\mu, \tag{5}$$

with Γ_μ being the spinor affine connection. The spinor affine connections for the metric (2) has the form

$$\Gamma_0 = 0,$$

$$\Gamma_1 = \frac{1}{2}\dot a_1\bar\gamma^1\bar\gamma^0 + \frac{1}{4}\frac{a_1^2}{a_2a_3}\bar\gamma^2\bar\gamma^3,$$

$$\Gamma_2 = \frac{1}{2}\dot a_2\bar\gamma^2\bar\gamma^0 + \frac{1}{2}z\dot a_1\bar\gamma^1\bar\gamma^0 + \frac{1}{4}\frac{a_1}{a_3}\bar\gamma^1\bar\gamma^3 + \frac{1}{4}\frac{za_1^2}{a_2a_3}\bar\gamma^2\bar\gamma^3,$$

$$\Gamma_3 = \frac{1}{2}\dot a_3\bar\gamma^3\bar\gamma^0 - \frac{1}{4}\frac{a_1^2}{a_2}\bar\gamma^1\bar\gamma^2. \tag{6}$$

It can be easily verified that

$$\gamma^\mu\Gamma_\mu = -\frac{1}{2}\left(\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\right)\bar\gamma^0 - \frac{1}{4}\frac{a_1^2}{a_2a_3}\bar\gamma^1\bar\gamma^2\bar\gamma^3,$$

$$\Gamma_\mu\gamma^\mu = \frac{1}{2}\left(\frac{\dot a_1}{a_1} + \frac{\dot a_2}{a_2} + \frac{\dot a_3}{a_3}\right)\bar\gamma^0 - \frac{1}{4}\frac{a_1^2}{a_2a_3}\bar\gamma^1\bar\gamma^2\bar\gamma^3,$$

Defining

$$V = a_1a_2a_3, \tag{7}$$

and taking into account that the spinor field is a function of t only, one finds

$$\bar\gamma^0\left(\dot\psi + \frac{1}{2}\frac{\dot V}{V}\psi + \frac{1}{4}\frac{a_1^2}{a_2a_3}i\bar\gamma^5\psi\right) + im_{sp}\psi - iF_S\psi + F_P\bar\gamma^5\psi = 0, \tag{8}$$

$$\left(\dot{\bar\psi} + \frac{1}{2}\frac{\dot V}{V}\bar\psi + \frac{1}{4}\frac{a_1^2}{a_2a_3}i\bar\psi\bar\gamma^5\right)\bar\gamma^0 - im_{sp}\bar\psi + iF_S\bar\psi - F_P\bar\psi\bar\gamma^5 = 0, \tag{9}$$

>From (2) one finds

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$$\dot{S}+\frac{\dot{V}}{V}S+\frac{1}{2}\frac{a_1^2}{a_2a_3}P-2F_PA^0=0, \tag{10}$$

$$\dot{P}+\frac{\dot{V}}{V}P-\frac{1}{2}\frac{a_1^2}{a_2a_3}S-2m_{sp}A^0+2F_SA^0=0, \tag{11}$$

$$\dot{A}^0+\frac{\dot{V}}{V}A^0+2m_{sp}P-2F_SP+2F_PS=0, \tag{12}$$

where, $A^0=\bar{\psi}\bar{\gamma}^5\bar{\gamma}^0\psi$. From (2) one finds

$$V^2(S^2+P^2+A^{02})=\mathrm{C}onst. \tag{13}$$

Note that, from (10) and (11) one finds

$$\frac{1}{2}\frac{\partial}{\partial t}(S^2+P^2)+\frac{\dot{V}}{V}(S^2+P^2)-2(F_PS-F_SP)A^0=0. \tag{14}$$

As one sees, the assumption

$$F_PS-F_SP=0, \tag{15}$$

leads to

$$V^2(S^2+P^2)=C_0^2,\quad C_0^2=\mathrm{C}onst. \tag{16}$$

It can be easily verified that the relation (15) holds, if one assumes that $F=F(S^2+P^2)$.

Let us now write the components of energy momentum tensor for the spinor field. In the case considered, one finds

$$T_0^0=m_{sp}S-F,\quad T_1^1=T_2^2=T_3^3=SF_S+PF_P-F. \tag{17}$$

Let us now write the Einstein field equations corresponding to BII metric (2). As it was shown in a recent paper [13], thanks to $T_1^1=T_2^2=T_3^3$ the off-diagonal component of the Einstein equation can be overlooked. As a result we now have the following system:

$$\frac{\ddot{a}_2}{a_2}+\frac{\ddot{a}_3}{a_3}+\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}-\frac{3}{4}\frac{a_1^2}{a_2^2a_3^2}=\kappa(SF_S+PF_P-F), \tag{18}$$

$$\frac{\ddot{a}_3}{a_3}+\frac{\ddot{a}_1}{a_1}+\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}+\frac{1}{4}\frac{a_1^2}{a_2^2a_3^2}=\kappa(SF_S+PF_P-F), \tag{19}$$

$$\frac{\ddot{a}_1}{a_1}+\frac{\ddot{a}_2}{a_2}+\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}+\frac{1}{4}\frac{a_1^2}{a_2^2a_3^2}=\kappa(SF_S+PF_P-F), \tag{20}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}+\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}+\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}-\frac{1}{4}\frac{a_1^2}{a_2^2a_3^2}=\kappa(m_{sp}S-F). \tag{21}$$

Subtracting (19) from (20) one finds

$$\frac{\ddot{a}_2}{a_2}-\frac{\ddot{a}_3}{a_3}+\frac{\dot{a}_1}{a_1}(\frac{\dot{a}_2}{a_2}-\frac{\dot{a}_3}{a_3})=0, \tag{22}$$

which yields

$$\frac{a_3}{a_2}=D_2\exp[X_2\int V^{-1}dt], \tag{23}$$

with D_2 and X_2 being some arbitrary constants.

Summation of (18), (19), (20) and 3 times (21) gives the equation for V :

$$2\frac{\ddot{V}}{V}=\frac{a_1^2}{a_2^2a_3^2}+\kappa[m_{sp}S+3(SF_S+PF_P-2F)]. \tag{24}$$

The right hand side of (24) explicitly depends on a_1, a_2 and a_3. We need some additional conditions to overcome it. Following many authors we assume the expansion ϑ is proportional to any of the components (say σ_1^1) of the shear tensor σ. We will choose a comoving frame of reference so that $u_\mu=(1,0,0,0)$ and $u_\mu u^\mu=1$. In this case we find

$$\vartheta=\Gamma_{\mu 0}^{\mu}=\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}=\frac{\dot{V}}{V}, \tag{25}$$

and

$$\sigma_1^1=\frac{\dot{a}}{a}-\frac{1}{3}\vartheta. \tag{26}$$

The proportionality condition

$$\sigma_1^1=q_1\vartheta,\qquad q_1=const. \tag{27}$$

leads to

$$a_1=(a_2a_3)^{(1+3q_1)/(2-3q_1)}. \tag{28}$$

On account of (28), (7) and (23) one finds

$$a_1=V^{(1+3q_1)/3}, \tag{29}$$

$$a_2=(1/\sqrt{D_2})V^{(2-3q_1)/6}e^{-\frac{X_2}{2}\int\frac{dt}{V}}, \tag{30}$$

$$a_3=\sqrt{D_2}V^{(2-3q_1)/6}e^{\frac{X_2}{2}\int\frac{dt}{V}}. \tag{31}$$

The Eq. (24) now can be written as

$$2\ddot{V}=V^{(12q_1+1)/3}+\kappa[m_{sp}S+3(SF_S+PF_P-2F)]V. \tag{32}$$

To this end we assume that the spinor field be a massless one and the spinor field nonlinearity is given by $F=F(K)$ with $K=S^2+P^2$. In this case $F_S=2SF_K$ and $F_P=2PF_K$, hence $SF_S+PF_P=2(S^2+P^2)F_K=2KF_K$. The Eq. (32) then reads

$$2\ddot{V}=V^{(12q_1+1)/3}+6\kappa[KF_K-F]V. \tag{33}$$

Let us now choose the spinor field nonlinearity in some concrete form. We will consider the case when F is a power law of K, namely, $F=K^n$. Taking into account that $K=S^2+P^2=C_0^2/V^2$ we rewrite (33) as

$$2\ddot{V}=V^{(12q_1+1)/3}+6\kappa(n-1)C_0^{2n}V^{1-2n}, \tag{34}$$

with the solution in quadrature

$$\int\frac{dV}{\sqrt{[3/(12q_1+4)]V^{(12q_1+4)/3}-3\kappa C_0^{2n}V^{2-2n}+C_1}}=t+t_0, \tag{35}$$

with C_1 and t_0 being integration constants. As one sees, in the model considered, the Heisenberg-Ivanenko type nonlinearity with $n = 1$ has no influence on the evolution of the Universe.

3. CONCLUSION

Within the scope of a Bianchi type-II cosmological model the role of a nonlinear spinor field on the evolution of the Universe is studied. It is shown that the model allows exact solutions only for some special choice of nonlinearity. In the case considered the isotropization process of the initially anisotropic spacetime does not take place.

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