
Gu Haiming*, Li Hongwei and Xie Bing

Department of Mathematics, Qingdao University of Science and Technology, Qingdao 266061, P.R. China

Abstract: A least-squares mixed finite element (LSMFE) method for the numerical solution of fourth-order elliptic equations is analyzed and developed in this paper. The a posteriori error estimator which is needed in the adaptive refinement algorithm is proposed. The local evaluation of the least-squares functional serves as a posteriori error estimator. The a posteriori errors are effectively estimated.

Keywords: least-squares mixed finite element method, fourth-order elliptic equations, least-squares functional, a posteriori error.

I. INTRODUCTION

A general theory of the least-squares method has been developed by A K Aziz, R B Kellogg and A B Stephens in [1]. The most important advantage of the least-squares method leads to a symmetric positive definite problem. The least-squares mixed finite element method approaches a least-squares residual minimization is introduced. This method has an advantage which is not subject to the LBB condition [2]. Finite element methods of least-squares type have been studied in many fields recently (see, e.g., Stokes equation [2], Elliptic problem [3], Newtonian fluid flow problem [4], Transmission problems [5].

An adaptive least-squares mixed finite element method has been studied (see, e.g., the linear elasticity [6]). But the research about fourth-order elliptic equations which are widely used in hydrodynamics is not common. This paper mainly puts emphasis on an adaptive least-squares mixed finite element method for fourth-order elliptic equations. Our emphasis in this paper is on the performance of an adaptive refinement strategy based on the a posteriori error estimator inherent in the least-squares formulation by the local evaluation of the functional.

This paper is organized as follows. The least-squares formulation of the fourth-order elliptic equations is described in Section 2. It includes the coercivity properties of the least-squares variational formulation. Appropriate spaces for the finite element approximation and a generalization of the coercivity are shown in Section 2 to the discrete form is discussed in Section 3. The error estimates of the fourth-order elliptic equations are derived in Section 4. In Section 5, a posteriori error estimators which are needed in an adaptive refinement algorithm are composed with the least-squares functional, and the posteriori errors are effectively estimated. Finally, we summarize our findings and present conclusions in Section 6. In this paper, we define $c$ to be a generic positive constant, $\varepsilon$ be a generic small positive constant.

II. A LEAST-SQUARES FORMULATION OF FOURTH-ORDER ELLIPTIC EQUATIONS

We start from the equations of fourth-order elliptic in the form [7]:

$$
\begin{align*}
\Delta^2 u &= f \quad \text{in} \Omega, \\
u &= 0 \quad \text{on} \partial \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \partial \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, with boundary $\partial \Omega$. We shall consider an adaptive least-squares mixed finite element method for (1)-(3).

Now we set $\Delta u = -\sigma$, then, we have:

$$
\begin{align*}
-\Delta \sigma &= f \quad \text{in} \Omega, \\
\Delta u + \sigma &= 0 \quad \text{in} \Omega, \\
u &= 0 \quad \text{on} \partial \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \partial \Omega,
\end{align*}
$$

We introduce the Sobolev spaces:

$$
\begin{align*}
H^1(\Omega) &= \{ p \in L^2(\Omega) : \nabla p \in L^2(\Omega) \}, \\
H_0^1(\Omega) &= \{ v \in H^1(\Omega) : \nabla v \mid_{\partial \Omega} = 0, \| \alpha \| < m \}.
\end{align*}
$$

Now, let us define the least-squares problem: find $(\sigma, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$
J(\sigma, u) = \inf_{q \in H^1(\Omega) \times H_0^1(\Omega)} J(q, v),
$$

where

$$
J(q, v) = \langle \Delta q + f, \Delta q + f \rangle_{0, \Omega} + \langle \Delta v + q, \Delta v + q \rangle_{0, \Omega}.
$$

*Address correspondence to this author at the Department of Mathematics, Qingdao University of Science and Technology, Qingdao 266061, P.R. China; Tel: 8653288956987; Fax: 8653288956987; E-mail: guhm@public.qd.sd.cn

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We introduce the least-squares functional:

$$F(\sigma, u) = \|\Delta \sigma + f\|_{L^2(\Omega)}^2 + \|\Delta u + \sigma\|_{H^1(\Omega)}^2. \tag{10}$$

Taking variations in (9) with respect to $q$ and $v$, the weak statement becomes: find

$$(\sigma, u) \in H^1(\Omega) \times H^1(\Omega)$$

such that:

$$B(\sigma, u, q, v) = -(f, \Delta v), \quad (\forall v \in H^1(\Omega), \forall q \in H^1(\Omega)), \tag{11}$$

where

$$B(\sigma, u, q, v) = (\Delta \sigma, \Delta q)_{0,\Omega} + (\Delta u + \sigma, \Delta v + q)_{0,\Omega}. \tag{12}$$

**Theorem 2.1.** The bilinear form $B(\cdot, \cdot; \cdot, \cdot)$ is continuous and coercive. In other words, there exist positive constants $\alpha$ and $\beta$, such that

$$B(\sigma, u, q, v) \leq \beta (\|\Delta \sigma\|_{L^2(\Omega)}^2 + \|\sigma\|_{L^2(\Omega)}^2 + \|\Delta u\|_{H^1(\Omega)}^2)^{\frac{1}{2}},$$

$$B(q, v; q, v) \geq \alpha (\|\Delta q\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2). \tag{13}$$

Proof: i) For the upper bound we have:

$$B(q, v; q, v) = (\Delta \sigma, \Delta q)_{0,\Omega} + (\Delta u + \sigma, \Delta v + q)_{0,\Omega} = \|\Delta \sigma\|_{L^2(\Omega)}^2 + \|\sigma\|_{L^2(\Omega)}^2 + \|\Delta u\|_{H^1(\Omega)}^2 \leq C(\|\Delta \sigma\|_{L^2(\Omega)}^2 + \|\sigma\|_{L^2(\Omega)}^2 + \|\Delta u\|_{H^1(\Omega)}^2).$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

ii) For the lower bound,

$$B(q, v; q, v) = (\Delta \sigma, \Delta q)_{0,\Omega} + (\Delta u + \sigma, \Delta v + q)_{0,\Omega} = \|\Delta \sigma\|_{L^2(\Omega)}^2 + \|\Delta u + \sigma\|_{H^1(\Omega)}^2 \geq \|\Delta \sigma\|_{L^2(\Omega)}^2 + \|\Delta u + \sigma\|_{H^1(\Omega)}^2.$$ 

So we can select the positive constants $\varepsilon$ and $\delta$, satisfying

$$1 - \varepsilon \delta > 0, 1 - \frac{\varepsilon}{\delta} > 0.$$

So we obtain

$$B(q, v; q, v) \geq \alpha (\|\Delta q\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2).$$

Then, we complete the proof.

**Theorem 2.2.** Let $f \in H^{-1}(\Omega)$. Then, (8) has a unique solution, and the solution is $(\sigma, u) \in H^1(\Omega) \times H^1(\Omega)$.

**Proof:** From Theorem 2.1, we know that the bilinear form $B(\cdot, \cdot; \cdot, \cdot)$ is coercive and bounded on $H^1(\Omega) \times H^1(\Omega)$. Then the result follows from Lax-Milgram theorem.

**III. FINITE ELEMENT APPROXIMATION**

In principle, the least-squares mixed finite element approach simply consists of minimizing (10) in finite-dimensional subspaces $H^1(\Omega) \subset H^1(\Omega)$ and $M(\Omega) \subset H^1(\Omega)$.

Suitable spaces are based on a triangulation $T_h$ of $\Omega$ and consist of piecewise polynomials with sufficient continuity conditions.

Let $T_h$ be a class quasi-uniform regular partition of $\Omega$.

$$H_h(\Omega) = \text{span}(\Phi(-X_1), \cdots, \Phi(-X_N)) + P^d \tag{15}$$

where $\Phi : R^d \to R$ is a radial basis function, $P^d$ denotes the space of polynomials of degree less than $m$ and $X = (X_1, \cdots, X_N) \subseteq \Omega$ is a set of distinct nodes.

Consider $\Phi$ whose Fourier transform $\hat{\Phi}$ has the property in [8]:

$$C_1(1 + ||\omega||)^{-\gamma} \leq \hat{\Phi} \leq C_2(1 + ||\omega||)^{-\gamma}, \tag{16}$$

with positive constants $C_1$ and $C_2$.

The least-squares functional:

$$F_h(\sigma, u) = \sum_{T \in T_h} (\|\Delta \sigma + f\|_{L^2(T)}^2 + \|\Delta u + \sigma\|_{L^2(T)}^2). \tag{17}$$

Minimizing the functional (17) is equivalent to the following variational problem: find $\sigma_h \in H_h$ and $u_h \in M_h$ such that

$$B_h(\sigma_h, u_h; q, v) = -(f, \Delta v), \tag{18}$$

holds for all $(q, v) \in H_h(\Omega) \times M_h(\Omega)$.

The discrete bilinear form $B_h(\cdot, \cdot; \cdot, \cdot)$ is defined as follows:

$$B_h(\sigma_h, u_h; q, v) = \sum_{T \in T_h} ((\Delta \sigma_h, \Delta q)_{0,T} + (\Delta u_h + \sigma_h, \Delta v + q)_{0,T}), \tag{19}$$

where

$$(\sigma_h, u_h) \in H_h(\Omega) \times M_h(\Omega), \quad (q, v) \in H_h(\Omega) \times M_h(\Omega).$$

**Theorem 3.1.** The bilinear form $B_h(\cdot, \cdot; \cdot, \cdot)$ is continuous and coercive, i.e. there exist positive constants $\alpha_h$ and $\beta_h$ such that

$$B(\sigma_h, u_h; q, v) \leq \beta_h (\sum_{T \in T_h} (\|\Delta \sigma_h\|_{L^2(T)}^2 + \|\sigma_h\|_{L^2(T)}^2 + \|\Delta u_h\|_{L^2(T)}^2)^{\frac{1}{2}}$$

$$\left(\sum_{T \in T_h} (\|\Delta q\|_{L^2(T)}^2 + \|q\|_{L^2(T)}^2 + \|\Delta v\|_{L^2(T)}^2)^{\frac{1}{2}}\right)^2, \tag{20}$$

$$B(q, v; q, v) \geq \alpha_h (\sum_{T \in T_h} (\|\Delta q\|_{L^2(T)}^2 + \|q\|_{L^2(T)}^2 + \|\Delta v\|_{L^2(T)}^2), \tag{21}$$
which holds for all 
\((\sigma, u_h) \in H^1(\Omega) \times M_h(\Omega), (q, v) \in H^1(\Omega) \times M_h(\Omega)\).

**Proof:** i) For the upper bound we have 
\[B_h(q, v; q, v) = \sum_{r \in T_h}(\Delta q, \Delta q)_{0,r} + (\Delta v + q, \Delta v + q)_{0,r}\]
\[= \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q + \Delta v ||^2_{0,r})\]
\[\leq C \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r}).\]

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 3.1.

ii) For the lower bound,
\[B_h(q, v; q, v) = \sum_{r \in T_h}(\Delta q, \Delta q)_{0,r} + (\Delta v + q, \Delta v + q)_{0,r}\]
\[= \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q + \Delta v ||^2_{0,r})\]
\[\geq \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q + \Delta v ||^2_{0,r} - 2(\Delta q, q)_{0,r})\]
\[\geq \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r} - \epsilon_1 || q ||^2_{0,r})\]
\[= \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + (1 - \epsilon_1) || \Delta v ||^2_{0,r} + (1 - \epsilon_1) || q ||^2_{0,r}).\]

So, we can select the positive constants \(\epsilon_1\) and \(\delta_1\), satisfying

\[1 - \epsilon_1 \delta_1 > 0, 1 - \frac{\epsilon_1}{\delta_1} > 0.\]

We obtain 
\[B(q, v; q, v) \geq \alpha \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r}).\]

Then we complete the proof.

**Theorem 3.2.** Let \(f \in H^1(\Omega)\). Then, (18) has a unique solution, and the solution is \((\sigma, u_h) \in H^1(\Omega) \times M_h(\Omega)\).

**Proof:** From Theorem 3.1, we know that the bilinear form 
\(B_h(\cdot, \cdot; \cdot, \cdot)\) is coercive and bounded on 
\(H^1(\Omega) \times M_h(\Omega)\). Then the result follows from Lax-Milgram theorem.

**IV. ERROR ESTIMATES**

The error estimates of the second-order elliptic problem have studied by Kim et al. [9]. In this section, we discuss the error estimates of the fourth-order elliptic equations.

Assume the domain \(\Omega\) is convex, from the general finite element approximation theory we have the estimate [8]:

**Lemma 4.1.** Assume \(u \in H^1(\Omega), \ \Phi\) satisfies (16) with \(\zeta \geq k > d + m\). Let \(H^1(\Omega)\) be given by (15). Then there exists a function \(s \in H^1(\Omega)\) such that for \(x \in \Omega\), the estimate

\[|| u - s ||_{0,\Omega} \leq c h^{k-w} || u ||_{0,\Omega} \quad (22)\]

is valid if \(h\) is sufficiently small.

We defined the:

\[B(\sigma, u_h; q, v) = (\Delta \sigma, q)_{0,\Omega} + (\Delta u_h + \sigma, \Delta v + q)_{0,\Omega}. \quad (23)\]

Since the exact solution \((u, \sigma)\) satisfy (12), using the condition (18), we get the following property:

\[B(\sigma - \sigma_h, u_h - u_h; q, v) = (\Delta (\sigma - \sigma_h), q)_{0,\Omega} + (\Delta (u - u_h) + (\sigma - \sigma_h), \Delta v + q)_{0,\Omega}\]
\[= 0, (\forall q \in H^1(\Omega), \forall v \in M_h(\Omega))\]

Now we are ready to derive the following error estimation.

**Theorem 4.2.** Suppose that \(u \in H^1(\Omega)\) and \(\sigma \in H^1(\Omega)\) are the solutions of (12), and \(u_h \in H^1(\Omega)\) and \(\sigma_h \in H^1(\Omega)\) are the solutions of (23). Then for sufficiently small \(h\), we have the error estimation

\[|| \sigma - \sigma_h ||^2_{0,\Omega} + || \Delta (\sigma - \sigma_h) ||^2_{0,\Omega} + || \Delta (u - u_h) ||^2_{0,\Omega} \leq c h^{2k-2} || u ||^2_{0,\Omega} + || \sigma ||^2_{0,\Omega}. \quad (24)\]

**Proof:** From (12), we have:

\[B(\sigma - \sigma_h, u_h - u_h; \sigma - \sigma_h, u_h - u_h) = (\Delta (u - u_h) + (\sigma - \sigma_h), \Delta (u - u_h) + (\sigma - \sigma_h))_{0,\Omega}\]
\[= (\Delta (u - u_h) + (\sigma - \sigma_h), \Delta (u - u_h) + (\sigma - \sigma_h))_{0,\Omega}\]
\[+ \Delta (\sigma - \sigma_h) ||^2_{0,\Omega}\]
\[\leq C \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r}).\]

From (14), we can obtain the following inequality:

\[|| \Delta (\sigma - \sigma_h) ||^2_{0,\Omega} + || \sigma - \sigma_h ||^2_{0,\Omega} + || \Delta (u - u_h) ||^2_{0,\Omega} \leq C \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r}).\]

So we have

\[|| \Delta (\sigma - \sigma_h) ||^2_{0,\Omega} + || \Delta (u - u_h) ||^2_{0,\Omega} + || \sigma - \sigma_h ||^2_{0,\Omega}\]
\[\leq C \sum_{r \in T_h}(|| \Delta q ||^2_{0,r} + || q ||^2_{0,r} + || \Delta v ||^2_{0,r}).\]

From above the inequalities, we have:
Now we defined the least-squares functional:

\[ \| \Delta (\sigma - \sigma_h) \|_{0,\Omega}^2 + \| \sigma - \sigma_h \|_{0,\Omega}^2 + \| \Delta (u - u_h) \|_{0,\Omega}^2 \]

\[ \leq \| \Delta (\sigma - \sigma_j) \|_{0,\Omega}^2 + \| \Delta (u - u_j) \|_{0,\Omega}^2 + \| \sigma - \sigma_j \|_{0,\Omega}^2 + \| \Delta (u - u_j) \|_{0,\Omega}^2 + \| \sigma - \sigma_j \|_{0,\Omega}^2 \]

\[ + \| \Delta (\sigma - \sigma_j) \|_{0,\Omega}^2 + \| \Delta (u - u_j) \|_{0,\Omega}^2 + \| \sigma - \sigma_j \|_{0,\Omega}^2 \]

\[ \leq 2 \left( \| \Delta (\sigma - \sigma_j) \|_{0,\Omega}^2 + \| \Delta (u - u_j) \|_{0,\Omega}^2 + \| \sigma - \sigma_j \|_{0,\Omega}^2 \right) \]

where we used Lemma 4.1, we have the following inequality:

\[ \| \sigma - \sigma_h \|_{0,\Omega}^2 + \| \Delta (\sigma - \sigma_h) \|_{0,\Omega}^2 + \| \Delta (u - u_h) \|_{0,\Omega}^2 \leq c h^{2k-2} \left( \| u \|_{2,\Omega}^2 + \| \sigma \|_{4,\Omega}^2 \right) \]

Then we complete the proof.

V. POSTERIORI ERROR ESTIMATION

One of the main motivations for using least-squares finite element approaches is the fact that the element-wise evaluation of the functional serves as an \textit{a posteriori} error estimator.

\textit{A posteriori} estimate attempt to provide quantitatively accurate measures of the discretization error through the so-called \textit{a posteriori} error estimators which are derived by using the information obtained during the solution process. In recent years, the use of \textit{a posteriori} error estimators has become an efficient tool for assessing and controlling computational errors in adaptive computations [10].

Now we defined the least-squares functional:

\[ F_h(\sigma_h, u_h) = \sum_{T \in T_h} \left( \| \Delta \sigma_h + f \|_{0,T}^2 + \| \Delta u_h + \sigma_h \|_{0,T}^2 \right) \]

where \((\sigma_h, u_h) \in H_0(\Omega) \times M_0(\Omega)\).

We have

\[ F_h(\sigma - \sigma_h, u - u_h) = \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) + f \|_{0,T}^2 + \| \Delta (u - u_h) + \sigma - \sigma_h \|_{0,T}^2 \right) \]

So we define the posteriori estimator as following:

\[ F_h(\sigma - \sigma_h, u - u_h) = \sum_{T \in T_h} \eta_h^2 \]

\textbf{Theorem 5.1.} Let \( f \in H^{-1}(\Omega) \). The least-squares functional constitutes an \textit{a posteriori} error estimator. In other words, for

\[ \eta_h^2 = \| \Delta (\sigma - \sigma_h) + f \|_{0,T}^2 + \| \Delta (u - u_h) + \sigma - \sigma_h \|_{0,T}^2 \]

there exist positive constants \( \alpha_T \) and \( \beta_T \) such that

\[ \sum_{T \in T_h} \eta_h^2 \leq \beta_T \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) \|_{0,T}^2 + \| \sigma - \sigma_h \|_{0,T}^2 \right) \]

\[ + \| \Delta (u - u_h) \|_{0,T}^2 \],

\[ \sum_{T \in T_h} \eta_h^2 \geq \alpha_T \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) \|_{0,T}^2 + \| \sigma - \sigma_h \|_{0,T}^2 \right) \]

\[ + \| \Delta (u - u_h) \|_{0,T}^2 \].

which holds for all \((\sigma_h, u_h) \in H_0(\Omega) \times M_0(\Omega)\).

\textbf{Proof:} From (26) and \( f \in H^{-1}(\Omega) \), we know

\[ \sum_{T \in T_h} \eta_h^2 = F_h(\sigma - \sigma_h, u - u_h) \]

\[ = \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) + f \|_{0,T}^2 + \| \Delta (u - u_h) + \sigma - \sigma_h \|_{0,T}^2 \right) \]

\[ = C \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) \|_{0,T}^2 + \| \Delta (u - u_h) + \sigma - \sigma_h \|_{0,T}^2 \right) \]

\[ = CB_h(\sigma - \sigma_h, u - u_h). \]

From Theorem 3.1, we have:

\[ B_h(\sigma - \sigma_h, u - u_h, \sigma - \sigma_h, u - u_h) \leq \beta_T \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) \|_{0,T}^2 \right) \]

\[ + \| \sigma - \sigma_h \|_{0,T}^2 + \| u - u_h \|_{0,T}^2 \],

\[ B_h(\sigma - \sigma_h, u - u_h, \sigma - \sigma_h, u - u_h) \geq \alpha_T \sum_{T \in T_h} \left( \| \Delta (\sigma - \sigma_h) \|_{0,T}^2 \right) \]

\[ + \| \sigma - \sigma_h \|_{0,T}^2 + \| u - u_h \|_{0,T}^2 \].

The positive constants \( \alpha_T = c \alpha_h \) and \( \beta_T = c \beta_h \), this completes the proof.

\textbf{Remark:} The mesh is adapted based on \textit{a posteriori} error estimate of the fourth-order elliptic equations. We use a mesh optimization procedure to compute the size of elements in the new mesh, based on the computed \textit{a posteriori} error estimate \( \eta \).

The mesh is adapted using the mesh modification procedures developed by Li et al. [11]. This requires the specification of a mesh metric field to define the desired element size and shape distribution from the computed \( \eta \). The mesh is then adapted to satisfy the prescribed metric field by the processes of refinement, coarsening and re-alignment.

Adaptive refinement strategies consist in refining those triangles with the largest values of \( \eta \).

VI. SUMMARY AND CONCLUSIONS

As the fourth-order elliptic equations belong to high-order partial differential equations which possess complex numerical structure, and the select of finite element spaces is difficult, so the research about the fourth-order elliptic equations is still quite few. This paper describes an adaptive least-squares mixed finite element methods for the fourth-order elliptic equations for the first time, constructs a \textit{posteriori} error estimator by the least-squares functional, and estimates the posterior errors effectively by composed bilinear form.

We describe an adaptive least-squares mixed finite element procedure for solving the fourth-order elliptic equations in this paper. The procedure uses a least-squares mixed finite element formulation and adaptive refinement based on \textit{a posteriori} error estimate. The method is applied to study the continuous and coercivity of the fourth-order elliptic equations.

In this paper, we applied relatively standard \textit{a posteriori} error estimation technique to solve the fourth-order elliptic equations adaptively.
This paper provides theory foundation for numerical computation in plate bending and fluid dynamics.

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