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Error Estimates of a Computational Method for Generalised Connecting Orbits

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Abstract: We provide error estimates for an approximation method to compute simultaneously solutions of two dynamical systems each with given asymptotic behaviour and both coupled only by conditions on initial values. The method applies to compute connecting orbits — point–to–point, point–to–periodic and periodic–to–periodic — as in the literature and in numerical applications. Since our set–up is more general, we call solutions of our systems generalised connecting orbits and provide further applications like Skiba points in economic models or solutions with a discontinuity. By specifying the asymptotic rates our method also applies to the computation of solutions converging in a strongly stable manifold. The numerical analysis shows that the error decays exponentially with the length of the approximation intervals even in the strongly stable case and for periodic solutions. For orbits connecting hyperbolic equilibria this is in agreement with known results in the literature. In our method we select appropriate asymptotic boundary conditions which depend typically on parameters. In order to solve these types of boundary value problems we set up an iterative procedure which is called boundary corrector method.

Keywords: Numerical method, point–to–periodic, periodic–to–periodic, generalised connecting orbits, asymptotic rate, asymptotic boundary condition, error estimates.

I. INTRODUCTION

A connecting orbit is a pair (x, λ) of parameters λ and a solution x of $\dot{x} = f(x, \lambda)$ converging for $t \to \pm\infty$ to given sets. Therefore, a connecting orbit converges to solutions $v_+(\lambda)$ or $v_-(\lambda)$ in these sets.

Computation and continuation of homoclinic and heteroclinic point–to–point connecting orbits is well established and implemented in recent versions of the continuation package Auto with Homcont, see [1-4]. Methods to compute point–to–periodic and periodic–to–periodic connecting orbits including the computation of boundary conditions and phase fixing conditions are introduced and implemented for several examples using Auto. In [5] point–to–periodic connecting orbits are computed for the Lorenz system, the electronic circuit model and for a three-level food chain model. For the latter model periodic-to-periodic orbits are computed in [6]. In [7] point–to–periodic and periodic–to–periodic connections are computed for the Lorenz system, a coupled Duffing system and a model with a saddle–node Hopf bifurcation and a detailed numerical analysis of this method is provided in [8].

This paper provides the numerical analysis framework for the methods used therein and generalises results on point–to–point connecting orbits and the work in [9] since it also applies to convergence in strongly stable or strongly unstable manifolds and allows for a discontinuity of the system. Usually in the literature a connecting orbit (x, λ) solves for $t \in \mathbb{R}$

$$\begin{aligned} &\dot{x} = f(x,\lambda), \\ &\lim_{t\to\infty} |x(t) - v_+(\lambda)(t)| = 0, \\ &\lim_{t\to-\infty} |x(t) - v_-(\lambda)(t)| = 0. \end{aligned} \tag{1}$$

Among others, solving (1) is a main application of the set–up analysed here. In this paper we solve for $t \in [0, \infty)$ systems of the form

$$\begin{aligned} &\dot{x}_+ = f_+(x_+, \lambda),\ x_+(t) \in \mathbb{R}^m, \\ &\dot{x}_- = f_-(x_-, \lambda),\ x_-(t) \in \mathbb{R}^m, \\ &g(x_+(0), x_-(0), \lambda) = 0, \\ &|x_+(t) - y_+(\lambda)(t)| \le Ce^{\gamma_+ t}, \\ &|x_-(t) - y_-(\lambda)(t)| \le Ce^{\gamma_- t} \end{aligned} \tag{2}$$

with appropriate constants $C > 0$, $\gamma_+ < 0$, $\gamma_- < 0$ and solutions $y_+(\lambda)(t)$ and $y_-(\lambda)(t)$ characterising the asymptotic behaviour.[1] It is straightforward to use the system (2) for solving (1) by defining $f_+ := f$, $f_- := -f$, $g(x_+(0), x_-(0), \lambda) := x_+(0) - x_-(0)$, $y_+(\lambda)(t) := v_+(\lambda)(t)$ and $y_-(\lambda)(t) := v_-(\lambda)(-t)$ (here the time is inverted). On the other hand, a solution (x_+, x_-, λ) of (2) defines by $x(t) := x_+(t)$ for $t \geq 0$ and $x(t) := x_-(-t)$ for $t < 0$ a solution of (1) which is smooth in $t = 0$.

However, the system (2) is rather flexible in the functions f_+, f_-, in the coupling condition g and in the asymptotic rates γ_+, γ_-. In Section II we illustrate this by embedding the usual connecting orbits in our context and by providing additional applications.

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[1]Even if the analysis in this paper formally applies to compact invariant sets we only apply it to stationary points and to periodic orbits. In more delicate problems it would be difficult to fix an orbit on the invariant set and this is essential for the method analysed here.

The constants γ_+, γ_- are upper bounds for asymptotic rates. For example, if y_+ is a hyperbolic equilibrium, then γ_+ is an upper bound for the real parts of the stable eigenvalues. Choosing the asymptotic rates γ_+, γ_- and appropriate asymptotic boundary conditions allows to select solutions converging in the strongly stable directions. In particular, $x_+(\cdot)$ and $y_+(\lambda)(\cdot)$ are called γ_+–asymptotic if $|x_+(t) - y_+(\lambda)(t)| \leq Ce^{\gamma_+ t}$ holds for t sufficiently large. For "−" the definition is analogous. With

$$\gamma := \max(\gamma_+, \gamma_-), \qquad x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \text{ and } y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}$$

the system (2) has the structure

$$\dot{x} = f(x, \lambda), \qquad x(t) \in \mathbb{R}^{2m}, \; g(x(0), \lambda)=0,$$
$$|x(t) - y(\lambda)(t)| \leq Ce^{\gamma t}$$

of the problems treated in [10]. Nevertheless it is not appropriate to apply the results in [10] directly since the asymptotic rates γ_+ and γ_- need not coincide. The method and the error estimates in [10] depend crucially on the possibility to separate manifolds with different asymptotic rates and therefore on a gap in the eigenvalue structure. Applying the results in [10] directly implies that the gap in the full eigenvalue structure is the intersection of the gaps of both parts. Since the solutions $y_+(\lambda)$ and $y_-(\lambda)$ have, in general, different asymptotic rates the gap in the eigenvalue structure of the full system might shrink or even vanish. Moreover, to use the information that both solutions are only coupled at the initial values we have to consider a block partitioning of the system and to take the different asymptotic rates into account.

In [11] a similar approximation method applied to point–to–periodic and periodic–to–periodic connecting orbits is analysed and implemented including the computation of periodic orbits and the asymptotic boundary conditions for the Lorenz system and with van der Pol oscillators. Indeed, the error estimates in [11] are similar to ours if inserting the corrections in [12]. For the proof of the central theorem the authors claim "The steps on the proof in [13, Theorem 3.1] apply here …" (or [10, Theorem 4] in the correction [12]). However, [13, Theorem 3.1] applies only to hyperbolic stationary points and [10, Theorem 4] applies only to one sided boundary value problems.

In this paper we provide error estimates of the computational method and give a detailed proof in IV.2 taking into account the structure of the system with two parts and the coupling condition. Indeed, we observe that this kind of proof needs some additional assumptions on the length of the intervals as provided in IV.4 We are not aware of any proof using such technique and avoiding restrictions on the length of the intervals.

In Section III we define a solutions of (2) as **generalised connecting orbit,** define **non-degeneracy** and relate the non-degeneracy to the non-singularity of a linear system. We compute a solution $z_+(t)$, $t \in J_+ := [0, T_+]$, $z_-(t)$, $t \in J_- := [0, T_-]$ and v of the boundary value problem

$$\begin{pmatrix} \dot{z}_+ - f_+(z_+, v) \\ \dot{z}_- - f_-(z_-, v) \\ g(z_+(0), z_-(0), v) \\ M_+(T_+, v)(z_+(T_+) - y_+(v)(T_+)) \\ M_-(T_-, v)(z_-(T_-) - y_-(v)(T_-)) \end{pmatrix} = 0$$

as an approximation of a generalised connecting orbit (x_+, x_-, λ) solving (2) and provide error estimates for $||z_+(t) - x_+(t)||$, $||z_-(t) - x_-(t)||$ and $|v - \lambda|$.

The matrices $M_+(T_+,v)$, $M_-(T_-,v)$ define appropriate asymptotic boundary conditions. Since we concentrate on the approximation of the connecting orbit, we assume further on that the matrices $M_+(T_+,v)$, $M_-(T_-,v)$ and the points $y_+(v)(T_+)$, $y_-(v)(T_-)$ are given as smooth functions of v. For a numerical implementation this belongs to the boundary value problem. Theoretical results to get a smooth parameterisation are in [9, 13]. Practical implementations are provided in [14] and in packages like Auto or Homcont, see e. g. [3].

In Section IV we analyse the error of this method. The error is shown to decay exponentially with the length of the intervals. In particular, the error estimates for both parts allow for choosing the length of the intervals[2] T_- and T_+ in an appropriate way. As corollary of our main theorem we show that the method applied to point–to–periodic and periodic-to-periodic connecting orbits and with appropriate asymptotic boundary matrices induces error estimates

$$\begin{aligned} |v_J - \bar{\lambda}| \quad &\leq Ce^{2\max(\gamma_+ T_+, \gamma_- T_-)}, \\ \| z_J^+ - \bar{x}_+ \mid J_+ \|_\infty &\leq Ce^{2\max(\gamma_+ T_+, \gamma_- T_-)}, \\ \| z_J^- - \bar{x}_- \mid J_- \|_\infty &\leq Ce^{2\max(\gamma_+ T_+, \gamma_- T_-)}. \end{aligned}$$

Recall that $\gamma_+ < 0$ and $\gamma_- < 0$.

To avoid the parameter-dependent computation of the asymptotic boundary matrices $M_+(T_+,v)$ and $M_-(T_-,v)$ we develop the iterative boundary corrector method for generalised connecting orbits. This method requires at most three iterations.

II. CONNECTING ORBITS AND OTHER APPLICATIONS

The system (2) is rather flexible in the functions f_+, f_-, in the coupling condition g and in the asymptotic rates γ_+ and γ_-. To illustrate this, we apply it first to point–to–point, point–to–periodic and periodic–to–periodic connecting orbits and then provide additional applications.

Connecting Orbits

A connecting orbit from one compact invariant set $\mathbf{V}_-(\lambda)$ to another $\mathbf{V}_+(\lambda)$ is a solution $(\bar{x}, \bar{\lambda})$ on $\mathbb{R}$ of a parametrised dynamical system ($\Lambda \subset \mathbb{R}^p$ is the parameter set)

$$\dot{x} = f(x, \lambda), \; x(t) \in \mathbb{R}^m, \; \lambda \in \Lambda \subset \mathbb{R}^p$$

which converges to $\mathbf{V}_+(\lambda)$ as $t \to \infty$ and to $\mathbf{V}_-(\lambda)$ as $t \to -\infty$. As seen in [9] a connecting orbit is non-degenerate if a transversality condition holds and the number of parameters

[2]Equivalenty one may use time scaling of both differential equations.

is $p = m_{+u} - m_{-u} - m_{-c} + 1 = m_{+u} + m_{-s} - m + 1$. The numbers $m_{+s} + m_{+c}$, $m_{+u} + m_{+c}$ are the dimensions of the centre stable and centre unstable manifolds of $\mathbf{V}_+(\lambda)$ and $m_{-s} + m_{-c}$, $m_{-u} + m_{-c}$ are the dimensions of the centre stable and centre unstable manifolds of $\mathbf{V}_-(\lambda)$. Note that $m = m_{+s} + m_{+c} + m_{+u} = m_{-s} + m_{-c} + m_{-u}$ holds. As in the introduction we reformulate this as generalised connecting orbit solving system (2). In our approach we deal with single orbits $y_+(\lambda)$ on $\mathbf{V}_+(\lambda)$ and $y_-(\lambda)$ on $\mathbf{V}_-(\lambda)$. In case of periodic solutions we fix the phase of the periodic orbit to get a single solution. We need additional conditions or parameters to get non-degeneracy as defined in Section III.

- For a point–to–point connecting orbit we use a phase fixing condition $\chi(x_+, x_-, \lambda) = 0$ in addition to $x_+ - x_- = 0$ to define
$$g(x_+, x_-, \lambda) := \begin{pmatrix} x_+ - x_- \\ \chi(x_+, x_-, \lambda) \end{pmatrix}.$$
- For a point–to–periodic connecting orbit the phase is fixed by fixing the phase of the periodic orbit. Therefore, $g(x_+, x_-, \lambda) := x_+ - x_-$ is an appropriate function g.
- For a periodic–to–periodic connecting orbit the phase of only one periodic orbit can be fixed. Thus, the phase of the second periodic orbit is used as additional "free" parameter included in the parameter vector $\lambda \in \Lambda \times \mathbb{R} \subset \mathbb{R}^{p+1}$ and we define $g(x_+, x_-, \lambda) := x_+ - x_-$.

The dimension[3] of the manifold $\mathbf{W}$ coupling the systems at $t = 0$ and defined by $g(x_+, x_-, \lambda) = 0$ is $m_{+u} + m_{+c} + m_{-c} + m_{-s}$ in each case.

The theory on connecting orbits with equilibria or periodic orbits as invariant sets is provided in [9] and [15]. In [15] "bifurcation functions" are defined and it is proved that connecting orbits exist if and only if the "bifurcation functions" are zero. In [9] the non-degeneracy is related to the regularity of an operator and the non-singularity of a linear map. A similar concept is used in this paper.

Numerical computations with periodic orbits using an approximation method similar to ours are in [16] and in [14]. In [14] periodic–to–periodic connecting orbits for a specific Hamiltonian system arising from a reduced water-wave problem are computed. An explicite error analysis is provide in neither of the papers.

An approximation method for connecting orbits of hyperbolic stationary points is analysed in [13, 17]. Applying to the set–up in [13, 17] we get the same results. In our generalisation we also approximate connecting orbits starting in the strongly unstable or ending strongly stable manifold. This might be applied to approximate orbit flip solutions as it is done in [1, 18].

Solutions with Discontinuity

The splitting at 0 allows for discontinuities, since f_+ and f_- do not necessarily have the form $f_+ = f$, $f_- = -f$ as in the connecting orbit cases before. In order to approximate solutions with "jumps" at 0, we define the condition at 0 by $x_+(0) - x_-(0) = v$ with $v \in \mathbb{R}^m$. This may also be used as heuristic approach to detect initial approximations for connecting orbits as follows: First compute solutions which satisfy all conditions of a connecting orbit except for $x_+(0) = x_-(0)$ and define the difference vector v. Then use the components of v as continuation parameters and try to continue v to 0. A similar method for locating connecting orbits is developed in [19]. It is called "successive continuation". A local convergence analysis for this method is presented in [19]. A similar technique allowing initially for discontinuities and then detecting connecting orbits by continuation is Lin's method implemented in [7] and with detailed numerical analysis in [8].

Skiba Points

The control problems discussed in [20] lead to $2n$–dimensional dynamical systems (state and costate system). The first n variables are state-variables with given initial value and the second n variables are costate-variables not fixed initially. We want to approximate simultaneously two solutions $x_+ \in \mathbb{R}^{2n}$, $x_- \in \mathbb{R}^{2n}$ of $\dot{x} = f(x)$ which converge to different solutions y_+, y_- and which satisfy $x_{+i}(0) = x_{-i}(0) = v_i$ for $i = 1, \cdots, n$ and given $v \in \mathbb{R}^n$. To get unique solutions converging either to equilibria or to periodic orbits the stable manifold has to be n–dimensional (an equilibrium y_+ has $m_b^+ := n$ unstable eigenvalues and a periodic orbit y_+ has $m_b^+ := n + 1$ centre unstable Floquet multipliers, same with "–"). We set $f_+ = f_- = f$, $g(x_+(0), x_-(0)) = \begin{pmatrix} x_+(0) - v \\ x_-(0) - v \end{pmatrix} \in \mathbb{R}^{2n}$ and in the periodic case we add the phase of each periodic orbit as "free" parameter. Therefore, we have $p = 0$ parameter if the y_+, y_- are both equilibria, $p = 1$ parameter if one of y_+, y_- is a periodic orbit and the other is an equilibrium and $p = 2$ parameters if both y_+ and y_- are periodic orbits. With $m = 2n$ we see that

$$\mathbf{W} = \{(x_+^0, x_-^0, \lambda) \in \mathbb{R}^{2m+p} \mid g(x_+^0, x_-^0) = 0\}$$

is a manifold in $\mathbb{R}^{2m+p}$ of dimension $2m + p - m = m_b^+ + m_b^-$. This is the key condition for non-degeneracy (see Section III). The aim is to compare the values of an objective function $\hat{U}$ for both trajectories. In particular, we want to "free" one component v_i and approximate solutions which satisfy $\hat{U}(x_+(0)) = \hat{U}(x_-(0))$ for an objective function $\hat{U}$. Thus, we substitute for a given index i the conditions $x_{+i}(0) - v_i = 0$ and $x_{-i}(0) - v_i = 0$ by the conditions $x_{+i}(0) - x_{-i}(0) = 0$ and $\hat{U}(x_+(0)) - \hat{U}(x_-(0)) = 0$. Such solutions are called Skiba points (see [20]).

III. NONDEGENERATE GENERALISED CONNECTING ORBITS

In this section we define the concept of a **generalised connecting orbit** as a solution of (2). We use a transversality condition to define the non-degeneracy of a

[3]The dimension of the centre manifold is $m_{+c} = 0$ for a hyperbolic equilibrium and $m_{+c} = 1$ for a hyperbolic periodic orbit y_+. Analogous for "–".

generalised connecting orbit. Moreover, we relate the non-degeneracy to the non-singularity of a linear operator.

Given a parameter set $\Lambda \subset \mathbb{R}^p$, two parametrised dynamical systems

$$\dot{x}_+ = f_+(x_+, \lambda),\ x_+(t) \in \mathbb{R}^m, \tag{3}$$

$$\dot{x}_- = f_-(x_-, \lambda),\ x_-(t) \in \mathbb{R}^m, \tag{4}$$

two families of solutions $\{y_+(\lambda)\}_{\lambda\in\Lambda}$ of (3) and $\{y_-(\lambda)\}_{\lambda\in\Lambda}$ of (4) and a manifold $\mathbf{W} \subset \mathbb{R}^{2m+p}$, a generalised connecting orbit (x_+, x_-, λ) consists of a solution x_+ of (3) on $[0, \infty)$ which converges with an exponential rate $\gamma_+ < 0$ to $y_+(\lambda)$ and a solution x_- of (4) on $[0, \infty)$ which converges with an exponential rate $\gamma_- < 0$ to $y_-(\lambda)$ so that the coupling condition $(x_+(0), x_-(0), \lambda) \in \mathbf{W}$ is satisfied. The manifold $\mathbf{W}$ is defined as the zero set of a function g. For stationary points and periodic orbits the set of points converging with an asymptotic rate is a manifold which can be foliated, see e. g. [21-23]. In particular, we define the unification of all leafs evaluated at $y_+(\lambda)(0)$, $\lambda \in \Lambda \subset \mathbb{R}^p$ by $M^{+0}_{\gamma_+} \subset \mathbb{R}^m \times \Lambda$ and call it γ_+–**stable manifold** of y_+ **evaluated at** 0. By this definition, any solution $u(t)$ of (3) with $(u(0), \lambda) \in M^{+0}_{\gamma_+}$ satisfies $|u(t) - y_+(\lambda)(t)| \le Ce^{\gamma_+ t}$ and therefore converges with an asymptotic rate $\gamma_+ < 0$ to $y_+(\lambda)$. Analogously, we define $M^{-0}_{\gamma_-}$. Summarising, a generalised connecting orbit ($\bar{x}_+$, $\bar{x}_-$, $\bar{\lambda}$) has the properties

$$(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \in \mathbf{W},$$
$$(\bar{x}_+(0), \bar{\lambda}) \in M^{+0}_{\gamma_+},\ (\bar{x}_-(0), \bar{\lambda}) \in M^{-0}_{\gamma_-}.$$

To linearise along the limiting orbit we assume

A1. The functions satisfy $f_+, f_- \in C^2(\mathbb{R}^{m+p}, \mathbb{R}^m)$ and the second derivatives $f_+^{(2)}$ and $f_-^{(2)}$ are locally Lipschitz with respect to x.

A2. The parameter set $\Lambda \subset \mathbb{R}^p$ is a bounded open set and $y_+, y_- \in C^2(\Lambda, BC^1(\mathbb{R}_+, \mathbb{R}^m)$ with the Banach space BC^1 of bounded, once differentiable functions. Moreover, $y_+(\lambda)$ is a solution of $\dot{x} = f_+(x, \lambda)$ and $y_-(\lambda)$ is a solution of $\dot{x} = f_-(x, \lambda)$ for each $\lambda \in \Lambda$. The functions y_+, y_- are bounded in $\lambda \in \Lambda$ and $t \in \mathbb{R}_+$.

A3. The linear operator

$$\bar{L}_+(\lambda) := \frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial}{\partial x} f_+(y_+(\lambda), \lambda), \lambda \in \Lambda$$

has a shifted exponential dichotomies with data ($\bar{K}_+$, $\bar{\alpha}_+$, $\bar{\beta}_+$, $\bar{P}^+_a(\lambda)$, $\bar{P}^+_b(\lambda)$), $\bar{\alpha}_+ < 0$ of class C^1 with respect to λ. The ranks of the projectors are independent of λ and given by the dimensions $m^+_a :=$ dim $\mathbf{R}$ ($\bar{P}^+_a(t)(\lambda)$) and $m^+_b :=$ dim $\mathbf{R}$ ($\bar{P}^+_b(t)(\lambda)$) of ranges $\mathbf{R}$ of the respective projectors. Analogous assumptions and notions hold for "–".

To precise **A3** and the notations therein, consider a linear differential operator $Lx = \dot{x} - A(\cdot)x$ with $A \in BC(J, \mathbb{R}^{m\times m})$, $x \in C^1(J, \mathbb{R}^m)$ and $J \subset \mathbb{R}$ is some interval. Denote the solution operator of $Lx = 0$ by $S(\cdot,\cdot)$, i. e. $LS(\cdot,s) = 0$ and $S(s,s) = \mathrm{Id}_{\mathbb{R}^m}$ for all $s \in J$. Using the concept of an exponential dichotomy (first developed in [24]) to separate fibres with different asymptotic rates we define a shifted exponential dichotomy as in [15] and its smoothness in the parameter as in [18, 25]. The assumption **A3** implies the existence of sufficiently smooth fibres.

Definition 3.1 (Shifted Exponential Dichotomy)

$L := \mathrm{d}/\mathrm{d}t - A(\cdot)$ has a **shifted exponential dichotomy** on J with exponents $\alpha < \beta$ if there exist a constant K and projectors $P_a(t)$, $P_b(t)$, $t \in J$ with $P_b(t) = \mathrm{Id}_{\mathbb{R}^m} - P_a(t)$ so that

$$S(t, s)P_\kappa(s) = P_\kappa(t)S(t, s),\ \kappa = a, b \text{ and } t, s \in J$$
$$||P_a(t)S(t, s)|| \le Ke^{\alpha(t-s)},\ t \ge s$$
$$||P_b(s)S(s, t)|| \le \mathrm{K}e^{-\beta(t-s)},\ t \ge s.$$

We call $(K, \alpha, \beta, P_a, P_b)$ the **dichotomy data**. The dichotomy data are of class C^l with respect to λ if $A \in BC^l(\Lambda, BC(J, \mathbb{R}^{m\times m}))$, $l \ge 0$ and $L(\lambda) = \mathrm{d}/\mathrm{d}t - A(\lambda)(\cdot)$ has a shifted exponential dichotomy with dichotomy data $(K, \alpha, \beta, P_a(\lambda), P_b(\lambda))$ for all $\lambda \in \Lambda$. The projectors $P_a(\cdot)(t)$, $P_b(\cdot)(t) \in C^l(\Lambda, \mathbb{R}^{m\times m})$ and K, α, β are independent of λ.

Assumption **A3** implies that it is possible to separate solutions with different asymptotic rates and $\bar{P}^+_a(\lambda)$, $\bar{P}^+_b(\lambda)$, $\bar{P}^-_a(\lambda)$ and $\bar{P}^-_b(\lambda)$ are the projections on the corresponding linear subspaces. Note that for stationary points and periodic orbit **A2** and **A3** hold if **A1** is satisfied, see e. g. [9, 13].

Using a roughness theorem (see e. g. [25]) one can show that **A3** implies that the linear operator $L_+(\lambda) := \mathrm{d}/\mathrm{d}t - \partial f_+/\partial x\,(x_+(\cdot), \lambda)$, $\lambda \in \Lambda$ evaluated at solutions x_+ converging with asymptotic rates $\gamma_+ \in (\bar{\alpha}_+, \bar{\beta}_+)$ to $y_+(\lambda)$ have shifted exponential dichotomies with data $(K_+, \alpha_+, \beta_+, P^+_a(\lambda), P^+_b(\lambda))$, $\alpha_+ < 0$. In particular, the projections $P^+_a(\lambda)$, $P^+_b(\lambda)$ describe the linearisation along the solution $x_+(\cdot)$. Analogous result holds for "–".

We describe the set of points $(x_+(0), x_-(0), \lambda) \in \mathbb{R}^{2m+p}$ for which x_+ is γ_+–asymptotic with $y_+(\lambda)(0)$ and x_- is γ_-–asymptotic with $y_-(\lambda)(0)$ by

$$\mathbf{M}^0_\gamma := \{(x^0_+, x^0_-, \lambda) \mid (x^0_+, \lambda) \in \mathbf{M}^{+0}_{\gamma_+},\ (x^0_-, \lambda) \in \mathbf{M}^{-0}_{\gamma_-}, \lambda \in \Lambda\}$$

with $\gamma = (\gamma_+, \gamma_-)$ as index. Since $\mathbf{M}^{+0}_{\gamma_+}$ and $\mathbf{M}^{-0}_{\gamma_-}$ are C^1–manifolds of dimensions $m^+_a + p$ and $m^-_a + p$ the set $\mathbf{M}^0_\gamma$ is an ($m^+_a + m^-_a$ + p)–dimensional C^1–manifold. To define non-degeneracy we assume in addition to **A1**–**A3**

A4. **W** is an ($m^+_b + m^-_b$)–dimensional manifold in $\mathbb{R}^{2m+p}$ which is described by the set

$$\mathbf{W} := \{(x^0_+, x^0_-, \mu) \in \mathbb{R}^{2m+p} \mid g(x^0_+, x^0_-, \mu) = 0\}$$

with a function $g \in C^1(\mathbb{R}^{2m+p}, \mathbb{R}^{m_a^+ + m_a^- + p})$ so that $g'(x_+^0, x_-^0, \mu)$ has full rank for all $(x_+^0, x_-^0, \mu) \in \mathbf{W}$.

A5. $T_{(\bar{x}(0),\bar{\lambda})}\mathbf{W} + T_{(\bar{x}(0),\bar{\lambda})}\mathbf{M}_\gamma^0 = \mathbb{R}^{2m+p}$.

Assumption **A4** implies by $(m_a^+ + m_a^- + p) + (m_b^+ + m_b^-) = 2m + p$ that **A5** is equivalent to

$$T_{(\bar{x}(0),\bar{\lambda})}\mathbf{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathbf{M}_\gamma^0 = \{0\}.$$

Choices for the number of parameters p and the manifold **W** in different applications are provided in Section II.

Definition 3. 2 (Generalised Connecting Orbit)

Let **A1–A4** hold. We call $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ a **generalised connecting orbit** from **W** to y_+ and y_- of type (γ_+, γ_-) if $\bar{x}_+$ and $y_+(\bar{\lambda})$ are γ_+–asymptotic at $\bar{\lambda}$, $\bar{x}_-$ and $y_-(\bar{\lambda})$ are γ_-–asymptotic at $\bar{\lambda}$ and $(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \in \mathbf{W}$. If in addition the transversality condition **A5** holds, then it is called a **nondegenerate generalised connecting orbit** from **W** to y_+ and y_- of type (γ_+, γ_-).

We relate non-degeneracy to non-singularity of a linear operator in Lemma 3.3 below. The derivation of this non-singularity condition also introduces some notions used in the proof of our main Theorem 4.1.

Let $D^0 : \mathbb{R}^{2m+p} \to \mathbb{R}^{m_a^+ + m_a^- + p}$ be the linear operator defined by the Jacobian matrix

$$\left(\frac{\partial g}{\partial x_+}, \frac{\partial g}{\partial x_-}, \frac{\partial g}{\partial \lambda}\right)(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}), \tag{5}$$

of g evaluated at $(\bar{x}_+(0), \bar{x}_-(0), \lambda)$. Then

$$T_{(\bar{x}(0),\bar{\lambda})}\mathbf{W} = \ker(D^0) =: \mathbf{N}(D^0).$$

As seen in [10, Prop. 3, Corr. 1], there exists an open neighbourhood $Y \times \Omega$ of $(0, 0, \lambda) \subset \mathbf{R}(P_a^+(0)) \times \mathbf{R}(P_a^-(0)) \times \mathbb{R}^p$ in which the manifold $\mathbf{M}_\gamma^0$ is locally parametrised by a function

$$b(\xi_+, \xi_-, \lambda) := (x_+(\xi_+, \lambda)(0), x_-(\xi_-, \lambda)(0), \lambda).$$

By this definition $b(0, 0, \bar{\lambda}) = (\bar{x}_+(0), \bar{x}_-(0), \lambda)$ and b maps to

$$W := \{(x_+(\xi_+, \lambda)(0), x_-(\xi_-, \lambda)(0), \lambda) | (\xi_+, \xi_-, \lambda) \in Y \times \Omega\}\ \mathbf{M}_{\gamma_-}^0.$$

Therefore the tangent map $B(\xi_+, \xi_-, \lambda)(\eta_+, \eta_-, \mu)$ of b is defined by

$$\begin{pmatrix} \frac{\partial}{\partial \xi_+} x_+(\xi_+, \lambda)(0)\eta_+ + \frac{\partial}{\partial \lambda} x_+(\xi_+, \lambda)(0)\mu \\ \frac{\partial}{\partial \xi_-} x_-(\xi_-, \lambda)(0)\eta_- + \frac{\partial}{\partial \lambda} x_-(\xi_-, \lambda)(0)\mu \\ \mu \end{pmatrix}.$$

With the same arguments as in [10, Prop. 3] we observe

$$\frac{\partial}{\partial \xi} x_+(0, \bar{\lambda})(0) = P_a^+(0) \text{ and } \frac{\partial}{\partial \xi} x_-(0, \bar{\lambda})(0) = P_a^-(0).$$

Thus $\eta_+ \in \mathbf{R}(P_a^+(0))$ and $\eta_- \in \mathbf{R}(P_a^-(0))$ imply that the tangent map $B^0 := B(0, 0, \bar{\lambda})$ of b at the point $(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda})$ is

$$B^0(\eta_+, \eta_-, \mu) = \begin{pmatrix} \eta_+ + \frac{\partial}{\partial \lambda} x_+(0, \bar{\lambda})(0)\mu \\ \eta_- + \frac{\partial}{\partial \lambda} x_-(0, \bar{\lambda})(0)\mu \\ \mu \end{pmatrix}. \tag{6}$$

For $(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \in W$ the tangent space of $\mathbf{M}_\gamma^0$ is $T_{(\bar{x}(0),\bar{\lambda})}\mathbf{M}_\gamma^0 = \mathbf{R}(B^0)$ and

$$T_{(\bar{x}(0),\bar{\lambda})}\mathbf{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathbf{M}_\gamma^0 = \mathbf{N}(D^0) \cap \mathbf{R}(B^0).$$

Using this result and the definitions of D^0 and B^0 we obtain the following lemma.

Lemma 3.3. Suppose that **A1–A4** hold and let $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ be a generalised connecting orbit from **W** to y_+ and y_- of type (γ_+, γ_-). Then $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ is nondegenerate if and only if the linear operator

$$D^0 \circ B^0 = D(\bar{x}_+(0), \bar{x}_-(0), \bar{\lambda}) \circ B(0, 0, \bar{\lambda})$$

is nonsingular.

IV. THE APPROXIMATION OF GENERALISED CONNECTING ORBITS

The numerical method for approximating generalised connecting orbits on finite intervals is introduced and analysed in IV.1 In particular, the error is shown to decay exponentially with the length of the intervals. The detailed proof of the main theorem is provided in IV.2. In IV.3 we present the boundary corrector method and in IV.4 we comment on restrictions to the length of intervals.

IV.1. The Approximation Theorem

In this section we set up an approximation theory for generalised connecting orbits. We truncate $[0, \infty)$ to finite intervals $J_+ = [0, T_+]$ and $J_- = [0, T_-]$ and approximate both parts $\bar{x}_+$ and $\bar{x}_-$ of a generalised connecting orbit on J_+ and J_-, respectively. At T_+ and T_- we use asymptotic boundary conditions.

Assume **A1–A4**. Let $(\bar{x}, \bar{\lambda}) = (\bar{x}_+, \bar{x}_-, \bar{\lambda})$ be a non-degenerate generalised connecting orbit from **W** to y_+ and y_- of type (γ_+, γ_-) with $\bar{\alpha}_+ < \gamma_+ < \min(0, \bar{\beta}_+)$ and $\bar{\alpha}_- < \gamma_- < \min(0, \bar{\beta}_-)$. Moreover, suppose that the operators $L_+ := d/dt - \partial f_+/\partial x(\bar{x}_+, \bar{\lambda})$ and $L_- := d/dt - \partial f_-/\partial x(\bar{x}_+, \bar{\lambda})$ have shifted exponential dichotomies with dichotomy data $(K_+, \alpha_+, \beta_+, P_a^+, P_b^+)$ and $(K_-, \alpha_-, \beta_-, P_a^-, P_b^-)$ so that α_+, β_+ satisfy $\bar{\alpha}_+ < \alpha_+ < \gamma_+ < \beta_+ < \bar{\beta}_+$ and α_-, β_- satisfy $\bar{\alpha}_- < \alpha_- < \gamma_- < \beta_- <$

$\bar{\beta}_-$. We assume that the boundary conditions are regular in the following sense:

A6. The functions $M_+(T_+, \cdot) \in C^1(\Lambda, \mathbb{R}^{m_b^+ \times m})$ and $M_-(T_-, \cdot) \in C^1(\Lambda, \mathbb{R}^{m_b^- \times m})$ satisfy

$$\mathbf{N}(M_+(T_+, \bar{\lambda})) \cap \mathbf{R}(\bar{P}_b^+(T_+)\bar{\lambda}) = \{0\},$$

$$\mathbf{N}(M_-(T_-, \bar{\lambda})) \cap \mathbf{R}(\bar{P}_b^-(T_-)\bar{\lambda}) = \{0\}$$

for all $T_+, T_- \in \mathbb{R}_+$ sufficiently large. Moreover, let $M_+(T_+, \lambda)$, $M_-(T_-, \lambda)$ and the derivatives $\frac{\partial}{\partial\lambda} M_+(T_+, \lambda)$, $\frac{\partial}{\partial\lambda} M_-(T_-, \lambda)$ be uniformly bounded by constants M_{M_+}, M_{M_-}, $M^1_{M_-}$ and $M^1_{M_+}$ and Lipschitz continuous with constants L_{M_+}, L_{M_-}, $L^1_{M_+}$ and $L^1_{M_-}$. The inverse functions $N_+ := (M_+(T_+, \bar{\lambda})|\bar{\mathbf{R}}_b^+)^{-1}$ and $N_- := (M_-(T_-, \bar{\lambda})\bar{\mathbf{R}}_b^-)^{-1}$ are uniformly bounded by $M_{\bar{N}_+} > 0$ and $M_{\bar{N}_-} > 0$.

We abbreviate the subspaces $\bar{\mathbf{R}}_b^+ := \mathbf{R}(\bar{P}_b^+(T_+))(\bar{\lambda})$, $\bar{\mathbf{R}}_b^- = \mathbf{R}(\bar{P}_b^-(T_-))(\bar{\lambda})$, $\mathbf{R}_b^+ := \mathbf{R}(P_b^+(T_+))(\bar{\lambda})$, $\mathbf{R}_b^- = \mathbf{R}(P_b^-(T_-)(\bar{\lambda})$. The function $\mathbf{R}_b^+ \to \mathbb{R}^{m_b^+} : x \to M_+(T_+, \bar{\lambda})x$ is denoted by $M_+(T_+, \bar{\lambda})|\mathbf{R}_b^+$. As seen in [26] the assumption **A6** implies that $M_+(T_+, \bar{\lambda})|\mathbf{R}_b^+$ and $M_-(T_-, \bar{\lambda})|\mathbf{R}_b^-$ are nonsingular and that their inverse functions $N_+ := (M_+(T_+, \bar{\lambda})|\mathbf{R}_b^+)^{-1}$ and $N_- := (M_-(T_-, \bar{\lambda})|\mathbf{R}_b^-)^{-1}$ are uniformly bounded by $M_{N_+} := 4KM_{N_+} > 0$ and $M_{N_-} := 4KM_{N_-} > 0$.

Remark: If the assumption of uniformly bounded functions is too strong, assumption **A6** may be weakened so that uniformity has to be satisfied only on subsets $\mathbf{I}_+, \mathbf{I}_- \subset \mathbb{R}_+$ so that there exist sequences $\{T_+^i\}_{i\in N} \subset \mathbf{I}_+$, $\{T_-^i\}_{i\in N} \subset \mathrm{I}_-$ with $\lim_{i\to\infty} T_+^i = \infty$ and $\lim_{i\to\infty} T_-^i = \infty$. In the case of equilibria we always choose $[0, \infty)$ and in the case of 1–periodic orbits (scaled system) we may choose $\cup_{N\in\mathbb{N}}[N - \tau_-, N + \tau_+]$, $\tau_-, \tau_+ \in [0, 1]$. For details see [26].

$\mathbf{N}(M_+(T_+, \bar{\lambda})) \cap \mathbf{R}(\bar{P}_b^+(T_+))(\bar{\lambda}) = \{0\}$ is not a very restrictive condition since such boundary condition only excludes endpoints in the γ_+–unstable subspace. Matrices $M_+(T_+, \bar{\lambda})$, $M_-(T_-, \bar{\lambda})$ satisfying the following more restrictive assumption **A6*** allow for endpoints in the γ_+–stable subspace and in the γ_-–stable subspace only. In this case we get quadratic error estimates

$$\begin{aligned} &|M_+(T_+, \bar{\lambda})(\bar{x}(T_+) - y(\bar{\lambda})(T_+))| \\ &\quad \le C_M^+ |\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)|^2 \\ &|M_-(T_-, \bar{\lambda})(\bar{x}(T_-) - y(\bar{\lambda})(T_-))| \\ &\quad \le C_M^- |\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)|^2 \end{aligned} \tag{7}$$

for constants $C_M^+ > 0$, $C_M^- > 0$ and sufficiently large T_+, T_-. Such boundary conditions imply that we do not need an additional gap in the structure of the exponents and they induce a better performance of the approximation method. However, it needs additional effort to compute such boundary matrices. For equilibria and periodic solutions such asymptotic boundary conditions can be computed by solving an eigenvalue problem (equilibrium) or the adjoint variational equation (periodic solution) as shown in [9, 13] and implemented in [5-7].

A6* The functions $M_+(T_+,\cdot)$, $M_-(T_-,\cdot)$ satisfy as sumption **A6** and in addition

$$\mathbf{N}(M_+(T_+, \bar{\lambda})) = \mathbf{N}(\bar{P}_b^+(T_+))(\bar{\lambda}) \text{ and}$$

$$\mathbf{N}(M_-(T_-, \bar{\lambda})) = \mathbf{N}(\bar{P}_b^-(T_-)(\bar{\lambda})).$$

Within the proof of Theorem 4.1 we use the assumption that the estimate

$$e^{\Delta_+ T_+} + e^{\Delta_- T_-} \le \frac{\varepsilon}{r(\beta_+, T_+) + r(\beta_-, T_-)}$$

is satisfied for T_+, T_-, $\varepsilon > 0$, $\beta_+, \beta_- \in \mathbb{R}$, $\Delta_+ < -|\beta_+|$, $\Delta_- < -|\beta_-|$ and with

$$r(\beta, T) := \begin{cases} e^{|\beta| T} + \dfrac{1}{|\beta|}(e^{|\beta| T} - 1) & : \beta \ne 0, \\ 1 + T & : \beta = 0. \end{cases}$$

Therefore we define the set of pairs (T_+, T_-) satisfying this condition by

$$\mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-) := \left\{ (T_+, T_-) \in \mathbb{R}_+^2 \,\middle|\, e^{\Delta_+ T_+} + e^{\Delta_- T_-} \le \frac{\varepsilon}{r(\beta_+, T_+) + r(\beta_-, T_-)} \right\}.$$

More details about this set and sufficient conditions for (T_+, T_-) to be in this set are provided later on in IV.3. To argue that this set is not empty lets consider one of the results of IV. 3 in advance. Let (T_+, T_-) be of the form $(\alpha T, T)$ and $\alpha \in (|\beta_-|/|\Delta_+|, |\Delta_-|/|\beta_+|)$. Then there exists some T with $(\alpha T, T) \in \mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ for all $T \ge \bar{T}$. Such an α always exists since $\Delta_+ < -|\beta_+|$ and $\Delta_- < -|\beta_-|$ imply $|\beta_+|\cdot|\beta_-| < |\Delta_+|\cdot|\Delta_-|$.

Remark: In the following theorem we define $\Delta_+ = d_+\gamma_+ - \beta_+$ and $\Delta_- = d_-\gamma_- - \beta_-$ with $d_+ = d_- = 1$ or $d_+ = d_- = 2$. Thus, $\Delta_+ < -|\beta_+|$ is equivalent to $d_+\gamma_+ < \min(0, 2\beta_+)$ which implies an additional restriction $\gamma_+ < 2\beta_+$ only if $d_+ = 1$ (“simple” boundary conditions **A6**) and $\beta_+ < 0$ (strongly stable case). The same holds for “–”.

We approximate a generalised connecting orbit $(\bar{x}_+, \bar{x}_-, \bar{\lambda})$ by $(z_J^+, z_J^-, v_J) \in \mathbf{Y} := C^1(J_+, \mathbb{R}^m) \times C^1(J_-, \mathbb{R}^m) \times \mathbb{R}^p$ and define the norm on the Banach spaces $\mathbf{Y}$ by

$$\begin{aligned} &\| x_+, x_-, \lambda \|_{\mathbf{Y}} = \| x_+ \|_{\mathbf{Y}}^+ + \| x_- \|_{\mathbf{Y}}^- + |\lambda|, \\ &\| x_+ \|_{\mathbf{Y}}^+ = \sup_{t\in J_+}(|x_+(t)|\, q_+(t)), \quad q_+(t) = \frac{2}{1+e^{\beta_+ t}}, \\ &\| x_+ \|_{\mathbf{Y}}^- = \sup_{t\in J_-}(|x_-(t)|\, q_-(t)), \quad q_-(t) = \frac{2}{1+e^{\beta_- t}}. \end{aligned}$$

With these notations we formulate the central theorem of this paper. The very technical proof is provided in a separate section IV. 2.

Theorem 4.1. Suppose that **A1–A5** and **A6*** hold. Let $d_+ = d_- = 2$ and let $(\bar{x}, \bar{\lambda})$ be a nondegenerate generalised connecting orbit from **W** to y_+ and y_- of type (γ_-, γ_+). Let $L_+ := d/dt - \partial f_+/\partial x(\bar{x}_+, \bar{\lambda})$, $L_- := d/dt - \partial f_-/\partial x(\bar{x}_-, \bar{\lambda})$ have shifted exponential dichotomies with data $(K_+, \alpha_+, \beta_+, P_a^+, P_b^+)$, $(K_-, \alpha_-, \beta_-, P_a^-, P_a^-)$ and γ_+, γ_- satisfying $\alpha_+ < \gamma_+ < \min(0, \beta_+)$ and $\alpha_- < \gamma_- < \min(0, \beta_-)$.

Then there exist $\delta > 0$, $C_+ > 0$, $C_- > 0$, $\varepsilon > 0$ and $\bar{T}_+$, $\bar{T}_-$ sufficiently large, so that for all $(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, d_+\gamma_+ - \beta_+, \beta_-, d_-\gamma_- - \beta_-)$ with $T_+ > \bar{T}_+$, $T_- \geq \bar{T}_-$ the operator equation $H_J(z_+, z_-, v) = 0$ defined by

$$\begin{pmatrix} \dot{z}_+ - f_+(z_+, v) \\ \dot{z}_- - f_-(z_-, v) \\ g(z_+(0), z_-(0), v) \\ M_+(T_+, v)(z_+(T_+) - y_+(v)(T_+)) \\ M_-(T_-, v)(z_-(T_-) - y_-(v)(T_-)) \end{pmatrix} = 0 \tag{8}$$

has a unique solution (z_J^+, z_J^-, v_J) in a ball in **Y** with radius $\bar{\delta} = \delta/[r(\beta_+, T_+) + r(\beta_-, T_-)]$, denoted by $\mathbf{B}(\bar{\delta})(\bar{x}_+|J_+, \bar{x}_-|J_-, \bar{\lambda})$. With $\Delta_+ := 2\gamma_+ - \beta_+ < 0$ and $\Delta_- := 2\gamma_- - \beta_- < 0$ the pointwise error estimates of z_J^+, z_J^- and v_J are

$$|v_J - \bar{\lambda}| \leq C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-} \tag{9}$$

$$|z_J^+(t) - \bar{x}_+ \mid J_+(t)|$$

$$\leq (C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}) \begin{cases} e^{\beta_+ t} & : \beta_+ > 0 \\ 1 & : \beta_+ \leq 0 \end{cases} \tag{10}$$

$$|z_J^-(t) - \bar{x}_- \mid J_-(t)|$$

$$\leq (C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}) \begin{cases} e^{\beta_- t} & : \beta_- > 0 \\ 1 & : \beta_- \leq 0 \end{cases} \tag{11}$$

If $M_+(T_+, \lambda)$ and $M_-(T_-, \lambda)$ only satisfy **A6** but the exponents satisfy the additional gap–conditions $\alpha_+ < \gamma_+ < \min(0, 2\beta_+)$ and $\alpha_- < \gamma_- < \min(0, 2\beta_-)$, then the existence result holds for $d_+ = d_- = 1$ and the error estimates hold with $\Delta_+ := \gamma_+ - \beta_+ < -|\beta_+| \leq 0$ and $\Delta_- := \gamma_- - \beta_- < -|\beta_-| \leq 0$.

In the following corollaries we apply Theorem 4.1 to different combinations of connecting orbits as they are typical in the literature. For these corollaries we always assume **A6*** and therefore that the boundary matrices $M_+(T_+,\cdot)$, $M_-(T_-,\cdot)$ are computed parametrically and solve the eigenvalue problem (equilibrium) or the adjoint variational equation (periodic solution).

Theorem 4.1 applied to connecting orbits with hyperbolic equilibria or periodic orbits ($\bar{\beta}_+ \geq 0$, $\bar{\beta}_- \geq 0$) induces the following corollary.

Corollary 4.2. Suppose that **A1–A5** and **A6*** hold. Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate point–to–periodic or periodic–to–periodic connecting orbit with $\bar{\beta}_+ \geq 0$ and $\bar{\beta}_- \geq 0$. Let γ_+, γ_- be given with $\alpha_+ < \gamma_+ < 0$ and $\alpha_- < \gamma_- < 0$. Then there exist $\delta > 0$, $C^+ > 0$, $C^- > 0$, $\varepsilon > 0$ and $\bar{T}_+$, $\bar{T}_-$ sufficiently large, so that for all $(T_+, T_-) \in \mathbf{D}(\varepsilon, 0, 2\gamma_+, 0, 2\gamma_-)$ with $T_+ \geq \bar{T}_+$, $T_- \geq \bar{T}_-$ the operator equation (8) has a unique solution $(z_J^+, z_J^-, v_J) \in \mathbf{B}$ and the following estimates hold

$$\begin{aligned} &|v_J - \bar{\lambda}| \leq C^+ e^{2\gamma_+ T_+} + C^- e^{2\gamma_- T_-}, \\ &\| z_J^+ - \bar{x}_+ \mid J_+ \|_\infty \leq C^+ e^{2\gamma_+ T_+} + C^- e^{2\gamma_- T_-}, \\ &\| z_J^- - \bar{x}_- \mid J_- \|_\infty \leq C^+ e^{2\gamma_+ T_+} + C^- e^{2\gamma_- T_-}. \end{aligned}$$

We get this result applying Theorem 4.1 with $\beta_+ = \beta_- = 0$. Since the typical error estimates are of the form $C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}$, it is numerically convenient to choose $(T_+, T_-) = (T|\Delta_-|/|\Delta_+|, T)$ so that $\Delta_+ T_+ = \Delta_- T_-$. For sufficiently large T the condition $(T|\Delta_-|/|\Delta_+|, T) \in \mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ is always satisfied and therefore no restriction. In case of a connecting orbit from a hyperbolic equilibrium with $\bar{\beta}_- > 0$ to a hyperbolic periodic orbit we concentrate on interval length with $2\gamma_+ T_+ = (2\gamma_- - \beta_-)T_-$ and get a super–convergence result due to the multiplicative factor $e^{\beta_- T_-}$ in (11).

Corollary 4.3. Suppose that **A1–A5** and **A6*** hold. Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate point–to–periodic connecting orbit with $\bar{\beta}_- > 0$. Let γ_+, γ_- be given with $\alpha_+ < \gamma_+ < 0$ and $\alpha_- < \gamma_- < 0$. Then there exists $\bar{T} > 0$ and suitable constants, so that the operator equation (8) has a unique solution $(z_J^+, z_J^-, v_J) \in \mathbf{B}$ for all $T \geq \bar{T}$ and $(T_+, T_-) = (T(2\gamma_- - \beta_-)/(2\gamma_+), T)$ and the following estimates hold

$$\begin{aligned} &|v_J - \bar{\lambda}| \leq C e^{(2\gamma_- - \beta_-)T}, \\ &\| z_J^+ - \bar{x}_+ \mid J_+ \|_\infty \leq C e^{(2\gamma_- - \beta_-)T}, \\ &\| z_J^- - \bar{x}_- \mid J_- \|_\infty \leq C e^{2\gamma_- T}. \end{aligned}$$

The following corollary applies to strongly stable manifolds and centre stable manifolds in case of periodic orbits.

Corollary 4.4. Suppose that **A1–A5** and **A6*** hold. Let $(\bar{x}, \bar{\lambda})$ be a nondegenerate connecting orbit with convergence in the strongly stable manifold or in the centre stable manifold so that $\bar{\beta}_+ \leq 0$, $\bar{\beta}_- \leq 0$. Let γ_+, γ_- be given with $\alpha_+ < \gamma_+ < \beta_+ \leq \bar{\beta}_+$ and $\alpha_- < \gamma_- < \beta_- \leq \bar{\beta}_-$. Then there exist $\delta > 0$, $C^+ > 0$, $C^- > 0$, $\varepsilon > 0$ and $\bar{T}_+$, $\bar{T}_-$ sufficiently large, so that for all $(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, 2\gamma_+ - \beta_+, \beta_-, 2\gamma_- - \beta_-)$ with $T_+ \geq \bar{T}_+$, $T_- \geq \bar{T}_-$ the operator equation (8) has a unique solution $(z_J^+, z_J^-, v_J) \in \mathbf{B}$ and the following estimates hold

$$|v_J - \bar{\lambda}| \le C^+ e^{(2\gamma_+ - \beta_+)T_+} + C^- e^{(2\gamma_- - \beta_-)T_-},$$
$$\| z_J^+ - \bar{x}_+ \,|\, J_+ \|_\infty \le C^+ e^{(2\gamma_+ - \beta_+)T_+} + C^- e^{(2\gamma_- - \beta_-)T_-},$$
$$\| z_J^- - \bar{x}_- \,|\, J_- \|_\infty \le C^+ e^{(2\gamma_+ - \beta_+)T_+} + C^- e^{(2\gamma_- - \beta_-)T_-}.$$

For hyperbolic point–to–point connecting orbits we get super-convergence in the parameter as in [13, 17, 27, 28] for $\beta_+ > 0$ and $\beta_- > 0$ if we restrict to pairs (T_+, T_-) satisfying $\beta_+ T_+ = \beta_- T_-$. For these pairs we obtain

$$|v_J - \bar{\lambda}| \le C^+ e^{(2\gamma_+ - \beta_+)T_+} + C^- e^{(2\gamma_- - \beta_-)T_-},$$
$$\| z_J^+ - \bar{x}_+ \,|\, J_+ \|_\infty \le C^+ e^{2\gamma_+ T_+} + C^- e^{2\gamma_- T_-},$$
$$\| z_J^- - \bar{x}_- \,|\, J_- \|_\infty \le C^+ e^{2\gamma_+ T_+} + C^- e^{2\gamma_- T_-}.$$

As seen in the different arguments it is numerically convenient to choose the length of both intervals with an appropriate rate. The error estimates above may help to choose them adequate.

IV.2. Proof of the Main Theorem 4.1

Assumptions **A1**–**A5** imply the existence of a γ_+ – stable manifold of y_+ characterised by a differentiable function $x_+(\xi^+, \lambda)$ parameterised by $\xi^+ \in \mathbf{R}_a^+$ and $\lambda \in \Omega$ and a γ_-– stable manifold of y_- characterised by a differentiable function $x_-(\xi^-, \lambda)$ parameterised by $\xi^- \in \mathbf{R}_a^-$ and $\lambda \in \Omega$, see [10, Prop. 3] or [26, Prop. 2.2] for details.

In order to describe the linearisation along the generalised connection orbit $(\bar{x}, \bar{\lambda})$ let $S^\pm(\cdot,\cdot)$ be the solution operator[4] of $L_\pm x = 0$ on $J_\pm = [0, T_\pm]$ (or $J_\pm = \mathbb{R}_+$, meaning "$T_\pm = \infty$"), which implies $L_\pm S^\pm(\cdot, s) = 0$ and $S^\pm(s, s) = \mathrm{Id}_{\mathbb{R}^m}$ for all $s \in J_\pm$. Then,

$$s_J^\pm(w)(t) := \int_0^t S^\pm(t,s) P_a^\pm(s) w(s) \mathrm{d}s - \int_t^{T_+} S^\pm(t,s) P_b^\pm(s) w(s) \mathrm{d}s \tag{12}$$

solves $L_\pm x = w$ on $J_\pm$. As show in detail in [10, Prop. 3] or [26, Prop. 2.2] the partial derivatives of $x_\pm$ at $(0, \lambda)$ are

$$\frac{\partial}{\partial \xi} x_\pm(0, \bar{\lambda})(\cdot) = S^\pm(\cdot, 0) P_a^\pm(0) \tag{13}$$

$$\frac{\partial}{\partial \lambda} x_\pm(0, \bar{\lambda})(\cdot) = s_{(0,\infty)}^\pm(\Psi_\pm(\cdot)) - S^\pm(\cdot, 0) P_a^\pm(0) \frac{\partial}{\partial \lambda} y_\pm(\bar{\lambda})(0) + \frac{\partial}{\partial \lambda} y_\pm(\bar{\lambda})(\cdot) \tag{14}$$

with

$$\Psi_\pm(s) := \left(\frac{\partial}{\partial \lambda} f(\bar{x}_\pm(s), \bar{\lambda}) - \frac{\partial}{\partial \lambda} f(\bar{y}_\pm(s), \bar{\lambda}) \right) + \left(\frac{\partial}{\partial x} f(\bar{x}_\pm(s), \bar{\lambda}) - \frac{\partial}{\partial x} f(\bar{y}_\pm(s), \bar{\lambda}) \right) \frac{\partial}{\partial \lambda} y_\pm(\bar{\lambda})(s). \tag{15}$$

[4] We index the first two parts of (x_+, x_-, λ) by "+" and "–". Definitions and assertions which hold for both parts are often indexed by "±", so that "$(x_\pm, \lambda)$ solves $\dot{x}_+ = f_\pm(x_\pm, \lambda)$" means that (x_+, λ) solves $\dot{x}_+ = f_+(x_+, \lambda)$ and (x_-, λ) solves $\dot{x}_- = f_-(x_-, \lambda)$.

In addition to $\mathbf{Y}$ let $\mathbf{Z} = C^0(J_+, \mathbb{R}^m) \times C^0(J_-, \mathbb{R}^m) \times \mathbb{R}^{m_a^+ + m_a^- + p} \times \mathbb{R}^{m_b^+} \times \mathbb{R}^{m_b^-}$ be a Banach space with norm

$$\|(v_+, v_-, r_0, r_+, r_-)\|_{\mathbf{Z}} = C_{\mathbf{Z}}^{J_+} \| v_+ \|_{\mathbf{Z}}^+ + C_{\mathbf{Z}}^{J_-} \| v_- \|_{\mathbf{Z}}^- + |r_0| + C_{\mathbf{Z}}^{J_+} |r_+| e^{-\beta_+ T_+} + C_{\mathbf{Z}}^{J_-} |r_-| e^{-\beta_- T_-}$$

using

$$\| v_+ \|_{\mathbf{Z}}^+ = \| v_+ \|_{\beta_+}^+ + \int_0^{T_+} |v_+(s)| e^{-\beta_+ s} \mathrm{d}s \quad \text{with } \| v_+ \|_{\beta_+}^+ = \sup_{t \in J_+} (|v_+(t)| e^{-\beta_+ t}) \text{ and}$$

$$\| v_- \|_{\mathbf{Z}}^- = \| v_- \|_{\beta_-}^- + \int_0^{T_-} |v_-(s)| e^{-\beta_- s} \mathrm{d}s \quad \text{with } \| v_- \|_{\beta_-}^- = \sup_{t \in J_-} (|v_-(t)| e^{-\beta_- t}).$$

The constants are defined by

$$C_{\mathbf{Z}}^{J_+} = (K_+ + M_{M_+} M_{N_+} K_+^2) \| R_0^+ \|,$$
$$C_{\mathbf{Z}}^{J_-} = (K_- + M_{M_-} M_{N_-} K_-^2) \| R_0^- \|,$$
$$C_{\mathbf{Z}}^{J_+} = M_{N_+} K_+ \| R_0^+ \|,$$
$$C_{\mathbf{Z}}^{J_-} = M_{N_-} K_- \| R_0^- \|.$$

Here R_0^+, R_0^- and Q_0 are the partial derivatives of g such that the matrix $(R_0^+ \; R_0^- \; Q_0)$ describes D^0 as defined in (5). B^0 is defined as in (6) so that we obtain

$$D^0 \mathrm{o} B^0(\eta_+, \eta_-, \mu) = R_0^+ \left[\eta_+ + \frac{\partial}{\partial \lambda} x_+(0, \bar{\lambda})(0) \mu \right] + R_0^- \left[\eta_- + \frac{\partial}{\partial \lambda} x_-(0, \bar{\lambda})(0) \mu \right] + Q_0 \mu \tag{16}$$

is nonsingular (see Lemma 3.3).

Therefore, $D^0 \circ B^0(\eta_+, \eta_-, 0) = R_0^+ \eta_+ + R_0^- \eta_-$ implies rank $(R_0^+) > 0$, rank $(R_0^-) > 0$ and $\| R_0^+ \| > 0$, $\| R_0^- \| > 0$. To prove Theorem 4.1 we make use of the following lemma (see also [13, 29]).

Lemma 4.5 (Perturbation Lemma) Let F: $\mathbf{B}(\delta)(y_0) \to \mathbf{Z}$ be a C^1–function from a δ-ball in $\mathbf{Y}$ into $\mathbf{Z}$ (Banach spaces). Assume that $F'(y_0)$ is a homeomorphism and there exist constants κ and σ so that

$$\|F'(y) - F'(y_0)\| \le \kappa < \sigma \le \|F'(y_0)^{-1}\|^{-1},$$
$$\|F(y_0)\| \le (\sigma - \kappa)\delta$$

holds for all $y \in \mathbf{B}(\delta)(y_0)$. Then F has a unique zero $\bar{y}$ in $\mathbf{B}(\delta)(y_0)$ and

$$\| \bar{y} - y_0 \| \le (\sigma - \kappa)^{-1} \|F(y_0)\|,$$
$$\|y_1 - y_2\| \le (\sigma - \kappa)^{-1} \|F(y_1) - F(y_2)\|$$

for all $y_1, y_2 \in \mathbf{B}(\delta)(y_0)$.

Sketch of the Proof of Theorem 4.1

We apply the **Perturbation Lemma 4.5** for fixed $(T_+, T_-) \in \mathbf{D}(\gamma, \beta_+, d_+\gamma_+ - \beta_+, \beta_-, d_-\gamma_- - \beta_-)$ with $F = H_J$ and $y_0 =$

$(\bar{x}_+|J_+,\ \bar{x}_-|J_-,\ \lambda) =: (\bar{x}|J,\ \bar{\lambda})$ and abbreviate $(\omega_+, \omega_-, r_0, r_+, r_-)$ by (ω, r_0, r). We prove the assumptions of Lemma 4.5 by the following steps:

Step 1. We show that there exists a $C_{lin} > 0$ with $||(\upsilon, \mu)||_{\mathbf{Y}} \le C_{lin}||(\omega, r_0, r)||_{\mathbf{Z}}$ for each solution (υ, μ) of $H_J'(\bar{x}|J, \bar{\lambda})(\upsilon, \mu) = (\omega, r_0, r)$.

Step 2. The derivative $H_J'(\bar{x}|J, \bar{\lambda})$ is a homeomorphism. Thus we observe

$$\begin{aligned}&\| H_J'(\bar{x}\mid J,\bar{\lambda})^{-1}(\omega,r_0,r)\|_{\mathbf{Y}}\\ &\quad\le\ C_{lin}\| H_J'(\bar{x}\mid J,\bar{\lambda})H_J'(\bar{x}\mid J,\bar{\lambda})^{-1}(\omega,r_0,r)\|_{\mathbf{Z}}\\ &\quad=\ C_{lin}\|(\omega,r_0,r)\|_{\mathbf{Z}}\end{aligned}$$

and define $\sigma := 1/C_{lin} \le || H_J'(\bar{x}|J, \bar{\lambda})^{-1}||^{-1}_{\mathbf{YZ}}$.

Hence "$\sigma \le ||F'(y_0)^{-1}||^{-1}$" in Lemma 4.5.

Step 3. We show that there exist $C_{Lip} > 0$, $\delta := \sigma/(2C_{Lip}) > 0$ with $|| H_J'(z, \nu) - H_J'(\bar{x}|J, \bar{\lambda})||_{\mathbf{YZ}} \le \kappa := \sigma/2$ for all $(z, \nu) \in \mathbf{B}(\tilde{\delta})(\bar{x}|J, \bar{\lambda})$ with $\tilde{\delta} := \delta/[r(\beta_+, T_+) + r(\beta_-, T_-)]$. Notice that the ball $\mathbf{B}(\tilde{\delta})(\bar{x}|J, \bar{\lambda})$ has to shrink for larger T_+, T_-. This implies "$||F'(y) - F'(y_0)|| \le \kappa < \sigma$" in Lemma 4.5 for fixed T_+, T_-.

Step 4. There exist $\varepsilon > 0$, $\bar{C}_\pm > 0$ and $\bar{T}_\pm$ with

$$\begin{aligned}\| H_J(\bar{x}\mid J,\bar{\lambda})\|_{\mathbf{Z}} &\le \tilde{C}^+e^{\Delta_+T_+}+\tilde{C}^-e^{\Delta_-T_-}\\ &\le(\sigma-\kappa)\frac{\delta}{r(\beta_+,T_+)+r(\beta_-,T_-)}\end{aligned}$$

for all $T = (T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ and $T_+ \ge \bar{T}_+$, $T_- \ge \bar{T}_-$. For the second inequality the assumption $(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ is essential.

This implies "$||F(y_0)|| \le (\sigma - \kappa)\tilde{\delta}$" in Lemma 4.5 for fixed T_+, T_- and a radius $\tilde{\delta} = \delta/[r(\beta_+, T_+) + r(\beta_-, T_-)]$ which depends on the length of the intervals.

Step 5. Lemma 4.5 implies that H_J has a unique solution (z_J, ν_J) in a ball $\mathbf{B}(\tilde{\delta})(\bar{x}|J, \bar{\lambda})$ satisfying

$$\| z_J, \nu_J) - (\bar{x}|J, \bar{\lambda})\|_{\mathbf{Y}} \le 2C_{lin}\|H_J(\bar{x}|J, \bar{\lambda})\|_{\mathbf{Z}}.$$

With constants $C^+ := 2C_{lin}\tilde{C}^+$, $C^- := 2C_{lin}\tilde{C}^-$ and by step 4 we obtain

$$\| z_J, \nu_J) - (\bar{x}|J, \bar{\lambda})\|_{\mathbf{Y}} \le C^+e^{\Delta_+T_+}+C^-e^{\Delta_-T_-}.$$

Proof: Now we derive the details of the proof.

Step 1. Let $(\omega_+, \omega_-, r_0, r_+, r_-) \in \mathbf{Z}$ be arbitrary and let $(\upsilon_+, \upsilon_-, \mu) \in \mathbf{Y}$ be a solution of the inhomogeneous equation

$$H_J'(\bar{x}_+\mid J_+,\bar{x}_-\mid J_-,\bar{\lambda})(\upsilon_+,\upsilon_-,\mu)=(\omega_+,\omega_-,r_0,r_+,r_-). \tag{17}$$

This equivalent is equivalent to the variational equation

$$\omega_+ = \dot{\upsilon}_+ - A_+(\cdot)\upsilon_+ - V_+(\cdot)\mu, \tag{18}$$

$$\omega_- = \dot{\upsilon}_- - A_-(\cdot)\upsilon_- - V_-(\cdot)\mu, \tag{19}$$

$$r_0 = R_0^+\upsilon_+(0) + R_0^-\upsilon_-(0) + Q_0\mu, \tag{20}$$

$$r_+ = R_+\upsilon_+(T_+) + Q_+\mu, \tag{21}$$

$$r_- = R_-\upsilon_-(T_-) + Q_-\mu, \tag{22}$$

with

$$\begin{aligned}A_\pm(\cdot) &= \frac{\partial}{\partial x} f_\pm(\bar{x}_\pm(\cdot),\bar{\lambda}),\\ V_\pm(\cdot) &= \frac{\partial}{\partial\lambda} f_\pm(\bar{x}_\pm(\cdot),\bar{\lambda}),\\ R_0^\pm &= \frac{\partial}{\partial x_\pm} g(\bar{x}_+(0),\bar{x}_-(0),\bar{\lambda}),\\ Q_0 &= \frac{\partial}{\partial\lambda} g(\bar{x}_+(0),\bar{x}_-(0),\bar{\lambda}),\\ R_\pm &= M_\pm(T_\pm,\bar{\lambda}),\\ Q_\pm &= \frac{\partial}{\partial\lambda} M_\pm(T_\pm,\bar{\lambda})(\bar{x}_\pm(T_\pm)-y_\pm(\bar{\lambda})(T_\pm)),\\ &\quad M_\pm(T_\pm,\bar{\lambda})\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm).\end{aligned}$$

By assumption **A4** and **A5** this system is a linear boundary value problem of dimension $m_a^+ + m_a^- + p + m_b^+ + m_b^+ = 2m + p$ for which the Fredholm alternative holds and a unique solution $(\upsilon_+, \upsilon_-, \mu) \in \mathbf{Y}$ exists.

Defining $\xi_a^\pm = P_a^\pm(0)\upsilon_\pm(0) \in \mathbf{R}(P_a^\pm(0))$ and $\xi_b^\pm = P_b^\pm(T_+)(\upsilon_\pm(T_\pm) - \frac{\partial}{\partial\lambda}(\bar{\lambda})(T_\pm)\mu) \in \mathbf{R}(P_b^\pm(T_+))$ the unique solutions υ_+, υ_- of

$$L_\pm\upsilon_\pm = (V_\pm(\cdot)\mu + \omega_\pm),\ t \in J_\pm,$$

$$P_a^\pm(0)\upsilon_\pm(0) = \xi_a^\pm,$$

$$P_b^\pm(T_\pm)\upsilon_\pm(T_\pm) = \xi_b^\pm + P_b^\pm(T_\pm)\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu$$

solve (18), (19) and are of the form

$$\begin{aligned}\upsilon_\pm(t) &= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_j^\pm(\omega_\pm)(t)\\ &\quad + s_j^\pm(V_\pm(\cdot)\mu)(t)\\ &\quad + S_\pm(t,T_\pm)P_b^\pm(T_\pm)\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu\end{aligned} \tag{23}$$

$$\begin{aligned}&= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(\omega_\pm)(t)\\ &\quad + \frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(T_\pm)\mu + s_J^\pm(\Psi_\pm(\cdot))(t)\mu\\ &\quad - S_\pm(t,0)P_a^\pm(0)\frac{\partial}{\partial\lambda} y_\pm(\bar{\lambda})(0)\mu\end{aligned} \tag{24}$$

$$\begin{aligned}&= S_\pm(t,0)\xi_a^\pm + S_\pm(t,T_\pm)\xi_b^\pm + s_J^\pm(\omega_\pm)(t)\\ &\quad + \frac{\partial}{\partial\lambda} x_\pm(0,\bar{\lambda})(t)\mu\\ &\quad + S_\pm(t,T_\pm)\int_{T_\pm}^\infty S_\pm(T_\pm,s)P_b^\pm(s)x \in \Psi_\pm(s)\mu \mathrm{d}s\end{aligned} \tag{25}$$

(see also [10, Eqs. (51), (54), (55)]). Here $S_\pm$ and $s_J^\pm$ are defined as in (12) and the derivatives of $x_\pm$ are defined by (13)-(15). To obtain pointwise estimates for υ_+, υ_- we first use (20)–(22) to get estimates for $\xi_a^\pm$, $\xi_b^\pm$ and μ.

Using the definitions of D^0 and B^0 and equations (16), (20) and (25) we obtain

$$\begin{aligned} &D^0 \mathrm{o} B^0(\xi_a^+,\xi_a^-,\mu) + R_0^+ S_+(0,T_+)\xi_b^+ + R_0^- S_-(0,T_-)\xi_b^- \\ &= r_0 + R_a^+(T_+) + R_a^-(T_-) \end{aligned}$$

with

$$\begin{aligned} R_a^\pm(T_\pm) &:= R_0^\pm \int_0^{T_\pm} S_\pm(0,s) P_b^\pm(s)\omega_\pm(s)\,\mathrm{d}s \\ &- R_0^\pm S_\pm(0,T_\pm)\int_{T_\pm}^\infty S(T_\pm,s)P_b^\pm(s)\Psi_\pm(s)\mu\,\mathrm{d}s \end{aligned}$$

From $s_J^\pm(\Psi_\pm(\cdot))(T_\pm) = P_a^\pm(T_\pm)\, s_J^\pm(\Psi_\pm(\cdot))(T_\pm)$, (14), (21), (22) and (24) we get

$$\begin{aligned} &M_\pm(T_\pm,\bar\lambda)\xi_b^\pm \\ &= r_\pm - \frac{\partial}{\partial\lambda} M_\pm(T_\pm,\bar\lambda)\mu(x(T_\pm) - y_\pm(\bar\lambda)(T_\pm)) \\ &\quad + M_\pm(T_\pm,\bar\lambda)P_a^\pm(T_\pm)\left(\frac{\partial}{\partial\lambda} y_\pm(\bar\lambda)(T_\pm) - \frac{\partial}{\partial\lambda} x_\pm(0,\bar\lambda)(T_\pm)\right)\mu \\ &\quad - M_\pm(T_\pm,\bar\lambda)S_\pm(T_\pm,0)\xi_a^\pm - M_\pm(T_\pm,\bar\lambda)s_J^\pm(\omega_\pm)(T_\pm) \\ &= R_b^\pm(T_\pm). \end{aligned}$$

Therefore, defining $\hat\xi_b^\pm := \xi_b^\pm\, \varepsilon$ we get the linear system

$$\begin{aligned} D^0 \mathrm{o} B^0(\xi_a^+,\xi_a^-,\mu) + R_0^+ S_+(0,T_+)\hat\xi_b^+ e^{\beta_+ T_+} & \\ + R_0^- S_-(0,T_-)\hat\xi_b^- e^{\beta_- T_-} & \\ = r_0 + R_a^+(T_+) + R_a^-(T_-) & \\ M_+(T_+,\bar\lambda)\hat\xi_b^+ = R_a^+(T_+)e^{-\beta_+ T_+} & \\ M_-(T_-,\bar\lambda)\hat\xi_b^- = R_a^-(T_-)e^{-\beta_- T_-}. & \end{aligned}$$

We estimate $|R_a^\pm(T_\pm)|$ and $|R_b^\pm(T_\pm)|\, e^{-\beta T_\pm}$ by

$$\begin{aligned} &|R_a^\pm(T_\pm)| \\ &\quad \le \|(\tilde\omega_\pm,0,0)\|_{\mathbf{Z}} + \|R_0^\pm\| K_\pm c_{\Delta,\beta}^\pm e^{(\gamma_\pm-\beta_\pm)T_\pm}|\mu| \\ &|R_b^\pm(T_\pm)|e^{-\beta_\pm T_\pm} \\ &\quad \le \frac{\|(\tilde\omega_\pm,0,\tilde r_\pm)\|_{\mathbf{Z}} + \dfrac{\tilde c_\pm}{C_{\xi,\mu}} e^{(\gamma_\pm-\beta_\pm)T_\pm}(|\xi_a^\pm| + |\mu|)}{K_\pm M_{N_\pm}\|R_0^\pm\|} \end{aligned}$$

with abbreviations $\tilde\omega_+ := (\omega_+, 0)$, $\tilde r_+ := (r_+, 0)$, $\tilde\omega_- := (0, \omega_-)$, $\tilde r_- := (0, r_-)$, $C_{\xi,\mu} := \|(D^0 \circ B^0)^{-1}\|$ and some constants $c_{\Delta,\beta}^\pm > 0$ and $\tilde c^\pm > 0$.

To estimate $|(\xi_a^+, \xi_a^-, \mu)|$ we notice that μ is involved in both parts of the problem, thus we have to choose $\bar T_\pm$ as follows: Let $a > 1$ and $T_\pm \ge \bar T_\pm := [\ln(\tilde c_+) - \ln(1 - 1/a) + \ln(2)]/(\beta_\pm - \gamma_\pm)$ so that $\tilde c_\pm e^{(\gamma_\pm-\beta_\pm)T_\pm} \le \frac{1}{2}(1 - 1/a)$ holds. Now we estimate

$$\begin{aligned} |(\xi_a^+,\xi_a^-,\mu)| &\le \|D^0 \mathrm{o} B^0)^{-1}\|\ \|(\omega_+,\omega_-,r_0,r_+,r_-)\|_{\mathbf{Z}} \\ &\quad + \tilde c_+(|\xi_a^+| + |\mu|)e^{(\gamma_+-\beta_+)T_+} \\ &\quad + \tilde c_-(|\xi_a^-| + |\mu|)e^{(\gamma_--\beta_-)T_-} \\ &\le a\cdot C_{\xi,\mu}\|(\omega_+,\omega_-,r_0,r_+,r_-)\|_{\mathbf{Z}} \end{aligned}$$

$$\begin{aligned} |\hat\xi_b^\pm| &\le \frac{1}{K_\pm\|R_0^\pm\|}\|(\tilde\omega_\pm,0,\tilde r_\pm)\|_{\mathbf{Z}} \\ &\quad + \frac{\tilde c_\pm}{C_{\xi,\mu}K_\pm\|R_0^\pm\|}(|\xi_b^+| + |\mu|)e^{(\gamma_\pm-\beta_\pm)T_\pm} \\ &\le \frac{1+a}{2}\frac{1}{K_\pm\|R_0^\pm\|}\|(\omega_+,\omega_-,r_0,r_+,r_-)\|_{\mathbf{Z}} \\ &\le \frac{a}{K_\pm\|R_0^\pm\|}\|(\omega_+,\omega_-,r_0,r_+,r_-)\|_{\mathbf{Z}}\,. \end{aligned}$$

for all $T_+ \ge \bar T_+$ and $T_- \ge \bar T_-$. These estimates for ξ_a^+, ξ_a^-, $\hat\xi_b^+$, $\hat\xi_b^-$, μ are used to get an estimate for the solution (υ_+, υ_-, v) of (18)–(22) and hence of (17).

$$|\upsilon_\pm(t)| \le \frac{1+e^{\beta_\pm}t}{2} C_\upsilon^\pm a\|(\omega_+,\omega_-,r_0,r_+,r_-)\|_{\mathbf{Z}}$$

with some constant $C_\upsilon^\pm > 0$ (For details on these comprehensive, but straightforward calculations see [26] or [10]). Therefore we observe that $\|\upsilon_\pm\|_{\mathbf{Y}}^\pm = \sup_{t\in J_\pm}(|\upsilon_\pm(t)|q_\pm(t)) \le a\ (C^\pm\, \|(\omega_+, \omega_-, r_0, r_+, r_-)\|_{\mathbf{Z}}$ and with $C_{lin} := a(C_{\xi,\mu} + C_\upsilon^+ + C_\upsilon^-)$

$$\|(\upsilon_+, \upsilon_-, \mu)\|_{\mathbf{Y}} \le C_{lin}\|(\omega_+, \omega_-, r_0, r_+, r_-)\|_{\mathbf{Z}}.$$

Step 2. By **Step 1** we observe for a solution of (17) that $H_J'(\bar x|J, \bar\lambda)^{-1}(\omega_+, \omega_-, r_0, r_+, r_-) = (\upsilon_+, \upsilon_-, \mu)$ and get the estimate

$$\begin{aligned} &\|H_J'(\bar x|J, \bar\lambda)^{-1}(\omega_+, \omega_-, r_0, r_+, r_-)\|_{\mathbf{Y}} \\ &\quad = \|(\upsilon_+, \upsilon_-, \mu)\|_{\mathbf{Y}} \le C_{lin}\|(\omega_+, \omega_-, r_0, r_+, r_-)\|_{\mathbf{Z}} \end{aligned}$$

Thus $\sigma := 1/C_{lin} \le \|H_J'(\bar x|J, \bar\lambda)^{-1}\|_{\mathbf{YZ}}^{-1}$.

Step 3. For any (z, v), $(\upsilon, \mu) \in \mathbf{Y}$ we estimate by comprehensive, but straightforward calculations (for details see [26])

$$\begin{aligned} &\left\| H_J'(z,v)\begin{pmatrix}\upsilon\\ \mu\end{pmatrix} - H_J'(\bar x|J,\bar\lambda)\begin{pmatrix}\upsilon\\ \mu\end{pmatrix}\right\|_{\mathbf{Z}} \\ &\le C_{Lip}^+\|(z_+ - \bar x_+|J_+)\|_{\mathbf{Y}}^+(\|\upsilon_+\|_{\mathbf{Y}}^+ + |\mu|\, r(\beta_+,T_+)) \\ &\quad + C_{Lip}^-\|(z_- - \bar x_-|J_-)\|_{\mathbf{Y}}^-(\|\upsilon_-\|_{\mathbf{Y}}^- + |\mu|\, r(\beta_-,T_-)) \\ &\quad + C_{Lip}^-\|(\upsilon_+,\upsilon_-,\mu)\|_{\mathbf{Y}} \\ &\qquad |v - \bar\lambda|(r(\beta_+,T_+) + r(\beta_-,T_-)) \\ &\le C_{Lip}\|(z - \bar x|J, v - \bar\lambda)\|_{\mathbf{Y}} \\ &\qquad \|(\upsilon,\mu)\|_{\mathbf{Y}}\,(r(\beta_+,T_+) + r(\beta_-,T_-)) \end{aligned}$$

with some constant $C_{Lip} > 0$. Defining $\kappa := \dfrac{\sigma}{2}$ and $\delta := \dfrac{\sigma}{2C_{Lip}}$ we obtain

$$\| H_J'(z,v) - H_J'(\bar{x} \mid J, \bar{\lambda}) \|_{\mathbf{YZ}} \le \frac{\sigma}{2} = \kappa < \sigma \le \| H_J'(\bar{x} \mid J, \bar{\lambda})^{-1} \|_{\mathbf{YZ}}^{-1}$$

for any $(z, v) \in \mathbf{B}\left(\frac{\delta}{r(\beta_+, T_+) + r(\beta_-, T_-)}\right)(\bar{x} \mid J, \bar{\lambda})$.

Step 4. The truncation error is

$$\|H_J(\bar{x} \mid J, \bar{\lambda})\|_{\mathbf{Z}}$$

$$= \left\| \begin{pmatrix} 0 \\ 0 \\ 0 \\ M_+(T_+)(\bar{\lambda})(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)) \\ M_-(T_-)(\bar{\lambda})(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)) \end{pmatrix} \right\|_{\mathbf{Z}}$$

$$= C_{\mathbf{Z}}^+ \mid M_+(T_+, \bar{\lambda})(\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)) \mid e^{-\beta_+ T_+} + C_{\mathbf{Z}}^- \mid M_-(T_-, \bar{\lambda})(\bar{x}_-(T_-) - y_-(\bar{\lambda})(T_-)) \mid e^{-\beta_- T_-}$$

$$\le C_{\mathbf{Z}}^+ \tilde{C}_{tr}^+ \| (\bar{x}_+(T_+) - y_+(\bar{\lambda})(T_+)) \|^{d_+} e^{-\beta_+ T_+} + C_{\mathbf{Z}}^- \tilde{C}_{tr}^- \| (\bar{x}_-(T_-) - y_+(\bar{\lambda})(T_-)) \|^{d_-} e^{-\beta_- T -}$$

$$\le \bar{C}(e^{(d_+\gamma_+ - \beta_+)T_+} + e^{(d_-\gamma_- - \beta_-)T_-})$$

with $C_{\mathbf{Z}}^{\pm} = M_{N_\pm} K_\pm \| R_0^\pm \|, C_{\bar{x}}^{\pm}$ so that

$\| \bar{x}_\pm(T_\pm) - y_\pm(\bar{\lambda})(T_\pm) \| \le C_{\bar{x}}^{\pm} e^{\gamma_\pm T_\pm}$ holds,

$$\tilde{C}_{tr}^{\pm} := \begin{cases} M_{M_\pm} & : d_\pm = 1, \\ C_{M_-}^{\pm} & : d_\pm = 2(C_M^{\pm} \text{ so that (7) holds}) \end{cases}$$

and $\bar{C} = \max\left(C_{\mathbf{Z}}^+ \tilde{C}_{tr}^+ (C_{\bar{x}}^+)^{d_+}, C_{\mathbf{Z}}^- \tilde{C}_{tr}^- (C_{\bar{x}}^-)^{d_-}\right)$.

For $(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, d_+\ \gamma_+ - \beta_+, \beta_-, d_-\ \gamma_- - \beta_-)$ with $\varepsilon := (\sigma\delta)/2\tilde{C}$ and $T_\pm \ge \bar{T}_\pm$ the inequality

$$\bar{C}(e^{(d_+\gamma_+ - \beta_+)T_+} + e^{(d_-\gamma_- - \beta_-)T_-}) \le \frac{\delta}{(r(\beta_+, T_+) + r(\beta_-, T_-)} \frac{\sigma}{2}$$

is satisfed. This implies the estimate

$$\| H_J(\bar{x} \mid J, \bar{\lambda} \|_{\mathbf{Z}} \le \frac{\delta}{(r(\beta_+, T_+) + r(\beta_-, T_-)} \frac{\sigma}{2}.$$

$(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, d_+\ \gamma_+ - \beta_+, \beta_-, d_-\ \gamma_- - \beta_-)$ is important to guarantee this estimate. Essentially, it becomes necessary to control $e^{(d_+\gamma_+ - \beta_+)T_+ + |\beta_-|T_-}$ and $e^{(d_-\gamma_- - \beta_-)T_- + |\beta_+|T_+}$ for large T_+ and T_-, which is not possible for all pairs (T_+, T_-). A discussion of this problem is given in IV.4.

Step 5. By Lemma 4.5 there exists a unique solution (z_J, v_J) in the ball $\mathbf{B}(\tilde{\delta})(\bar{x} \mid J, \bar{\lambda})$ with radius $\tilde{\delta} = \delta/[r(\beta_+, T_+) + r(\beta_-, T_-)]$. With $C^{\pm} = 2C_{\text{lin}} \tilde{C}^{\pm}$ we get

$$\| (z_J, \upsilon_J) - (\bar{x} \mid J, \bar{\lambda}) \|_{\mathbf{Y}} \le C^+ e^{(d_+\gamma_+ - \beta_+)T_+} + C^- e^{(d_-\gamma_- - \beta_-)T_-}.$$

Moreover, we get the estimates (9)-(11) since

$$| v_J - \bar{\lambda} | \le C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}$$

$$\| z_J^{\pm}(t) - \bar{x}_\pm \mid J_\pm(t) \| \le (C^+ e^{\Delta_+ T_+} + C^- e^{\Delta_- T_-}) \frac{1 + e^{\beta_\pm t}}{2}.$$

IV.3. The Boundary Corrector Method

To avoid the parameter-dependent computation of the asymptotic boundary matrices $M_+(T_+,v)$, $M_-(T_-,v)$ we develop an iterative method with at most three iterations called **boundary corrector method** for generalised connecting orbits. In addition to **A1–A5** we assume

A7. Let $\bar{\alpha} < 2\bar{\beta}$. Let $\Lambda_0 \subset \Lambda$ satisfy $\bar{\lambda} \in \Lambda_0$ and $M_+(T_+, \cdot) \in C^1(\Lambda_0, \mathbb{R}^{m_b^+ \times m})$, $M_-(T_-, \cdot) \in C^1(\Lambda_0, \mathbb{R}^{m_b^- \times m})$ so that **A6*** holds for all $\lambda \in \Lambda_0$.

The idea of the **boundary corrector method** is to start with an initial parameter μ_0 and compute $M_+(T_+,\mu_0)$, $M_-(T_-,\mu_0)$. Generically, these matrices satisfy $\mathbf{N}(M_+(T_+, \bar{\lambda})) \cap \mathbf{R}(\bar{P}_b^+(T_+)(\bar{\lambda})) = \{0\}$ and $\mathbf{N}(M_-(T_-, \bar{\lambda})) \cap \mathbf{R}(\bar{P}_b^-(T_-)(\bar{\lambda})) = \{0\}$. Applying the theorem with these matrices gives us a solution with error estimates $e^{(\gamma_+ - \beta_+)T_+} + e^{(\gamma_- - \beta_-)T_-}$, for $|v_0 - \bar{\lambda}|$ in particular. Starting the second iterate with $\mu_1 = v_0$ we have a sufficiently good initial guess so that applying the theorem with updated matrices $M_+(T_+, \mu_1)$, $M_-(T_-, \mu_1)$ gives us a solution with error estimates $e^{(2\gamma_+ - \beta_+)T_+} + e^{(2\gamma_- - \beta_-)T_-}$, if $\beta_+, \beta_- > 0$ and $e^{2(\gamma_+ - \beta_+)T_+} + e^{2(\gamma_- - \beta_-)T_-}$ and $e^{3(\gamma_+ - \beta_+)T_+} + e^{3(\gamma_- - \beta_-)T_-}$ after a third step with $\mu_2 = v_1$ if $\beta_+ < 0$ and $\beta_- < 0$. With $\gamma_+ < 2\beta_+$ and $\gamma_- < 2\beta_-$ we obtain again error estimates of the form $e^{(2\gamma_+ - \beta_+)T_+} + e^{(2\gamma_- - \beta_-)T_-}$. The boundary corrector method for generalised connecting orbits is defined as follows:

1. Start with some $\mu_0 \in \Lambda_0$, $i = 0$
2. Compute $M_+(T_+, \mu_i)$ and $M_-(T_-, \mu_i)$
3. Calculate $(z_{i+1}^+, z_{i+1}^-, \mu_{i+1})$ as the solution of

$$\begin{pmatrix} \dot{z}_+ - f_+(z_+, \upsilon) \\ \dot{z}_- - f_-(z_-, \upsilon) \\ g(z_+(0), z_-(0), \upsilon) \\ M_+(T_+, \mu_i)(z_+(T_+) - y_+(\upsilon)(T_+)) \\ M_-(T_-, \mu_i)(z_-(T_-) - y_-(\upsilon)(T_-)) \end{pmatrix} = 0$$

4. Repeat with **2**. with ("$i = i + 1$") and updated parameter once if $\beta_+ \ge 0$ and $\beta_- \ge 0$ or twice if $\beta_+ < 0$ or $\beta_- < 0$.

Propositon 4.6. Suppose that the assumptions of Theorem 4.1 and **A7** hold. Moreover, let $\bar{T}_+$, $\bar{T}_-$ be sufficiently large. Then there exists some $C_{bcm} > 0$ so that for all $T_+ \ge \bar{T}_+$, $T_- \ge \bar{T}_-$ with $(T_+, T_-) \in \mathbf{D}(\varepsilon, \beta_+, d_+\gamma_+ - \beta_+, \beta_-, d_-\gamma_- - \beta_-)$ for $d_+ = d_- = 1$ and $d_+ = d_- = 2$ the result (z_J, v_J) of the boundary corrector method can be estimated by

$$\|(z_J, v_J) - (\bar{x} \mid J, \bar{\lambda})\|_{\mathbf{Y}} \le C_{bem}\left(e^{(2\gamma_+ - \beta_+)T_+} + e^{(2\gamma_- - \beta_-)T_-}\right).$$

Proof: The proof is similar the one in [10, Prop. 4] since

$$e^{(\gamma_+ - \beta_+)T_+} e^{(\gamma_- - \beta_-)T_-} \le \max\left(e^{2(\gamma_+ - \beta_+)T_+}, e^{2(\gamma_- - \beta_-)T_-}\right)$$

Therefore, the boundary corrector method has the same exponential rate $e^{(2\gamma_+-\beta_+)T_+}$ + $e^{(2\gamma_--\beta_-)T_-}$ as the solution of Theorem 4.1 with $d_+ = d_- = 2$.

IV.4. Length of Approximation Intervals

Following the proof of Theorem 4.1 it becomes necessary that the pair (T_+, T_-) satisfies the estimate

$$e^{\Delta_+T_+}+e^{\Delta_-T_-}\leq\frac{\varepsilon}{(r(\beta_+,T_+)+r(\beta_-,T_-)}. \tag{26}$$

Now we derive sufficient conditions for (26). Let $\varepsilon > 0$, β_+, $\beta_- \in \mathbb{R}$, $\Delta_+ < -|\beta_+| \leq 0$ and $\Delta_- < -|\beta_-| \leq 0$ be given and define $\upsilon(0, T) = 1 + T$ and $\upsilon(\beta, T) = \min(1 + 1/|\beta|, 1 + T)$ for $\beta \neq 0$. Then we see that $r(\beta, T) \leq e^{|\beta|T}\, \upsilon(\beta, T)$ and that (26) holds for each pair (T_+, T_-) which satisfies

$$(e^{\Delta_+T_+}+e^{\Delta_-T_-})(e^{|\beta_+|T_+}\upsilon(\beta_+,T_+)+e^{|\beta_-|T_-}\upsilon(\beta_-,T_-))\leq\varepsilon. \tag{27}$$

The estimate (27) is satisfied if

$$\Delta_+T_+ + |\beta_-|T_- + \ln(\upsilon(\beta_-,T_-))\leq\left(\frac{\varepsilon}{4}\right), \tag{28}$$

$$\Delta_-T_- + |\beta_+|T_+ + \ln(\upsilon(\beta_+,T_+))\leq\left(\frac{\varepsilon}{4}\right), \tag{29}$$

$$(\Delta_+ + |\beta_+|)T_+ + \ln(\upsilon(\beta_+,T_+))\leq\left(\frac{\varepsilon}{4}\right), \tag{30}$$

$$(\Delta_- + |\beta_-|)T_- + \ln(\upsilon(\beta_-,T_-))\leq\left(\frac{\varepsilon}{4}\right), \tag{31}$$

hold. First we choose minimal $\hat{T}_+$, $\hat{T}_- \in \mathbb{R}_+$ so that (30) and (31) hold for all $T_+ \geq \hat{T}_+$ and $T_- \geq \hat{T}_-$. Then we define

$$Q_+(T_-)=\frac{|\beta_-|T_- + \ln(\upsilon(\beta_-,T_-)) - \ln\left(\frac{\varepsilon}{4}\right)}{-\Delta_+},$$

$$Q_-(T_+)=\frac{|\beta_+|T_+ + \ln(\upsilon(\beta_+,T_+)) - \ln\left(\frac{\varepsilon}{4}\right)}{-\Delta_-}$$

and obtain that (28) and (29) hold in the domain (see gray area in Fig. (**1**))

$$\begin{aligned}&\tilde{\mathbf{D}}(\varepsilon,\beta_+,\Delta_+,\beta_-,\Delta_-)\\ &\quad= \{(T_-,T_+)\,|\,T_+\geq\max(\hat{T}_+,Q_+(T_-))\\ &\qquad\qquad \text{and } T_-\geq\max(\hat{T}_-,Q_-(T_+))\}.\end{aligned}$$

If $\beta_+ \neq 0$, $\beta_- \neq 0$ and $T_+ \geq 1/|\beta_+|$, $T_- \geq 1/|\beta_-|$, the functions Q_+ and Q_- are linear with slopes $|\beta_-|/|\Delta_+|$ and $|\beta_+|/|\Delta_-|$. Therefore $\Delta_+ < -|\beta_+|$ and $\Delta_- < -|\beta_-|$ implies that $|\beta_+| \cdot |\beta_-| < \Delta_+\, \Delta_-$ and hence $|\beta_+|/|\Delta_-|$ (the slope of Q_-) is less than $\frac{|\Delta_+|}{|\beta_-|}$ (the slope of Q_+^{-1}). Thus Q_- and Q_+^{-1} intersect and $\tilde{\mathbf{D}}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-) \neq \emptyset$. Therefore $\tilde{\mathbf{D}}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-) \subset$

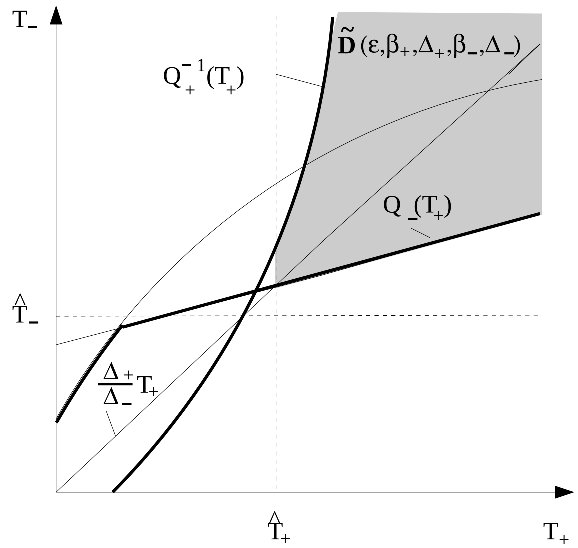

Fig. (1). Typical diagram for $\tilde{\mathbf{D}}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ with $\beta_- = 0$, $\beta_+ \neq 0$.

$\mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-) \neq \emptyset$. In particular, for $a \in (|\beta_-|/|\Delta_+|, |\Delta_-|/|\beta_+|)$ there exists some $\bar{T}$ with $(aT, T) \in \mathbf{D}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-)$ for all $T \geq \bar{T}$. Similar results hold for $\beta_+ = 0$ or $\beta_- = 0$ by using definition of Q_+ and Q_-. If $\beta_+ = 0$, then $Q_-(T_+) = [\ln(1 + T_+) - \ln(\varepsilon/4)]/|\Delta_-|$ and for T_+ large also $T_- \geq Q_-(T_+)$ have to be large. Nevertheless, in this case one might choose $a \in (|\beta_-|/|\Delta_+|, \infty)$. Analogously on may choose $a \in (0, |\Delta_-|/|\beta_+|)$ if $\beta_- = 0$. Summarising, we obtain the following Lemma:

Lemma 4.7. Let $\varepsilon > 0$, β_+, $\beta_- \in \mathbb{R}$ and $\Delta_+ < -|\beta_+|$, $\Delta_- < -|\beta_-|$. Then (26) is satisfied for all $(T_+, T_-) \in \tilde{\mathbf{D}}(\varepsilon, \beta_+, \Delta_+, \beta_-, \Delta_-) \neq \emptyset$.

Remark: If $\mathbf{I}_+ \times \mathbf{I}_-$ is defined as in the remark on page 6 then the set $\tilde{\mathbf{D}}$ (ε, β_+, Δ_+, β_-, Δ_-) has a sufficiently large intersection with $\mathbf{I}_+ \times \mathbf{I}_-$. Since the error is of the form $C^+e^{\Delta_+T_+}$ + $C^-e^{\Delta_-T_-}$ choosing $(T_+, T_-) = (T\Delta_-/\Delta_+, T)$ implies $\Delta_+T_+ = \Delta_-T_-$ and $|\Delta_-|/|\Delta_+| \in (|\beta_-|/|\Delta_+|, |\Delta_-|/|\beta_+|)$ is always satisfied. By this observation, it is not convenient to fix one interval length an enlarge the other one.

V. THE LORENZ SYSTEM

In this section we apply the method and our theoretical results to the Lorenz system

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1),\\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3,\\ \dot{x}_3 &= x_1x_2 - bx_3.\end{aligned}$$

We detect a point to periodic connecting orbit plotted in Fig. (**2**) and Fig. (**3**) and continue this connecting orbit by varying the parameter σ to get Fig. (**4**). To apply the results of this paper we solve the boundary value problem.[5]

[5]We use the collocation boundary value solver Colcon and its version for parameter continuation Colpar from Bader and Kunkel [30].

$$\begin{pmatrix} \dot{x} - \bar{T} f(x, r; \sigma, b) \\ \dot{y} - \bar{T} f(y, r; \sigma, b) \\ \dot{z} + \tau f(z, r; \sigma, b) \\ x(0) - z(0) \\ \chi(y(2), \phi) \\ y(2) - y(3) \\ V_+(x(3) - y(3)) \\ V_- z(3) \end{pmatrix} = 0.$$

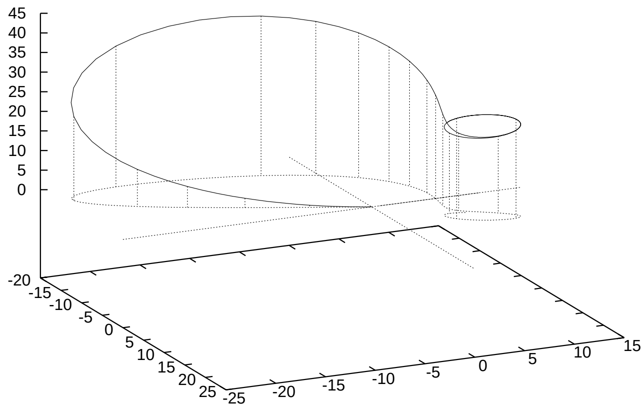

Fig. (2). Approximation of a point to periodic connecting orbit in the Lorenz system with parameters σ = 10 and b = 8/3.

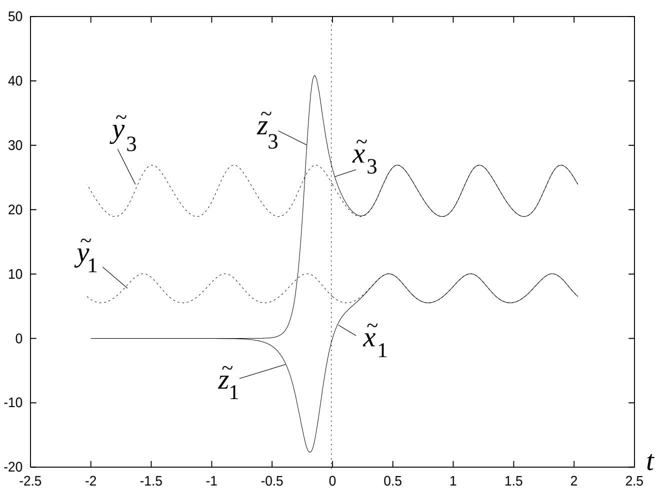

Fig. (3). The first and third component of the connecting orbit (—) and the periodic orbit (--) in the rescaled version.

We truncate the scaled solution on the interval [0,3], such that we compute three periods of the periodic orbit. The periodic orbit is fixed at $t = 2$ and $t = 3$ by $y(2) = y(3)$ and a simple phase fixing function $\chi\,(y(2), \phi) = y_1(2) - \phi$.

The boundary value solver restricts us to compute on the interval [0,3] even for the first part z of the connecting orbit. Thus we also scale the first part by a constant τ. At $r = 24.0$ the "unstable eigenvalue" of 0 is 10.1365 and the "stable Floquet multiplier" of the periodic orbit is $0.93 \cdot 10^{-5}$. This yields for the systems scaled by $\tau = 2/3$ and the period $\bar{T} \simeq 0.677167$ roughly similar exponents $\alpha_+ \simeq -7.845$ and $\alpha_- \simeq -6.76$ for this scaled system.

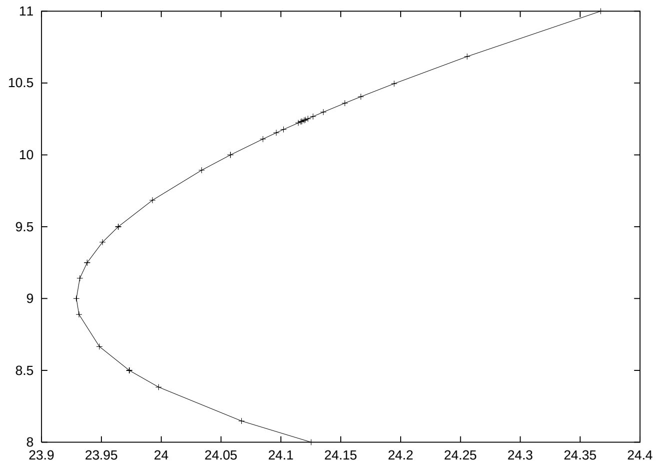

Fig. (4). Parameter-space r–σ. Branch of point to periodic connecting orbits.

Remark: For the boundary value solver it is essential to have an appropriate initial approximation for the solution. By applying the strategy in [10] we get such an initial approximation of a point to periodic connecting orbit. We approximate the "first part" (a solution in the unstable manifold of 0 which intersects the hyperplane $\{(0, x_2, x_3) | x_2, x_3 \in \mathbb{R}\}$ at $t = 0$) for different parameter values r and define an approximation of the intersection points which is linear in r. For the "second part" (a solution in the stable manifold of the periodic orbit) we compute a solution which has its initial value on the linearisation (mentioned above) and which is asymptotic to the periodic orbit. This is a solution with a relatively large error at $t = 0$ and it does not allow for continuation, but it is a sufficiently accurate first guess for a solution of (32). Even if the orbits look alike, (figure 6 in [10]) is this initial guess whereas Fig. (**2**) is the result of our method.

The value $\phi = 6.5043$ as well as the matrices $V_+, V_- \in \mathbb{R}^{2,3}$ for the asymptotic boundary conditions are defined by the initial approximation and the linearisation of the periodic orbit.

We apply the method with fixed matrices $V_+, V_- \in \mathbb{R}^{2,3}$ so that we apply Theorem 4.1 to this scaled system with $\gamma_+ = \gamma_-$ close to $\max(\alpha_+, \alpha_-)$, $\beta_+ = \beta_- = 0$ (as in Corollary **1**) and with $d_+ = d_- = 1$. Thus we get error estimates with exponents $\gamma_+ T_+ = \gamma_- T_-$ and with $T_+ = T_-$ due to the scaling of the system. An implementation of the Lorenz system with simultaneous parameter dependent computation of matrices $V_+, V_- \in \mathbb{R}^{2,3}$ satisfying **A6*** and therefore with exponent $2\gamma_+ T_+$ and $2\gamma_- T_-$ for the error estimates is provided e. g. in [5, 7, 11].

As result (32) we get an approximation for the unknown parameter $r \simeq 24.05803$, the unknown period $\bar{T} \simeq 0.677167$ and initial solutions for the "first part" of the connecting orbit $z(t) \in \mathbb{R}^3$, for the "second part" $x(t) \in \mathbb{R}^3$ and for the periodic orbit $y(t) \in \mathbb{R}^3$.

The phase portrait is plotted in Fig. (**2**) and in Fig. (**3**) we plot the first and third component of the connecting orbit (–) and the periodic orbit (--) in the rescaled version, such that

$\tilde{x} = x(\cdot/\bar{T})$, $\tilde{z} = z(-\cdot/\tau)$ and $\tilde{y} = y(\cdot/\bar{T})$ for $t \geq 0$ and $\tilde{y} = y(\cdot/\bar{T} + 3)$ for $t < 0$. Parameter continuation with respect to σ yields a branch of point to periodic connecting orbits. In Fig. (**4**) we plot the pairs of parameters (r, σ) corresponding to these point to periodic connecting orbits.

CONCLUDING REMARKS

We have developed error estimates of a computational method for a general type of a connecting orbit which includes most of the common connecting orbits and we provided a detailed numerical analysis framework for different connecting orbits. Practical implementations of several examples are provided in [5-7, 11]. These implementations include a smooth parametric computation of the limiting orbits and the boundary matrices.

Connecting orbits which converge, but not with an exponential rate (in our notion this means $\alpha = 0$) are not covered by our theory. The case of a homoclinic connecting orbit of a semi-hyperbolic equilibrium is analysed in [28, 31].

It seems straightforward to transfer the results of this paper from the case with two subproblems to the case with a finite number of subproblems.

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