Statistical Properties of the Periodogram for Stable Random Field

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Abstract: Let \( X(t) \) \((t \in \mathbb{Z})\) be a discrete stable random field. The problem of estimating the spectral density field based on \( X(t) \) is considered. Moments and the asymptotic moments of the spectral sample, the periodogram, based on \( X(t) \) are calculated.

Keywords: Stable distributions, stationary stable processes, spectral representation, symmetric stable distribution, stable random field, periodogram.

1. INTRODUCTION

Paul Levy in the 1920s began the study of general stable distributions. He was interested in stable distributions because they are precisely the limit distributions that can occur in the Generalized Central Limit Theorem. A lot of works on stable distributions and related topics have been done for last two decades, see [1] and the references therein. Besides to the uses of stable distributions in probability and statistics, they have a wide applications in many fields Zolotarev [2], Ghazal [3-5], Combanis [6] and Hosoya [7].

Spectral analysis is an important technique in the statistical analysis. So, spectral representation of symmetric stable processes have been considered by Hardine [8]. Moreover, Masry [9] is concerned with the estimation of the spectral density for stationary stable processes. Therefore, we consider an estimation for the spectral density field of homogeneous symmetric complex \((n; 1, \alpha)\) - stable random field.

The paper is organized as follows: Section 2 is devoted to introduce some basic definitions and Lemmas which will be used later. In section 3, we are going to investigate the first and second moments of a periodogram for stable random fields. In section 4, the asymptotic moments of a periodogram for stable random fields will be given.

2. PRELIMINARIES

Definition (2.1): A complex random variable \( Z = Z_1 + iZ_2 \) where \( Z_1, Z_2 \in \mathbb{R} \), has a symmetric stable distribution with \( \alpha \), \( 0 < \alpha \leq 2 \), if \( Z_1 \) and \( Z_2 \) have the same distribution and symmetric stable with \( \alpha \).

Definition (2.2): A random field \( \xi(\lambda), \lambda \in \mathbb{R}^n \), is called \((n; 1, \alpha)\) continuous stable random field if the linear combination \( \sum_{k=1}^{n} Z_k \xi(\lambda) \) has a symmetric complex stable random field with \( Z_k \in \mathbb{C} \) (the set of complex numbers); \( \xi(\lambda) \in \mathbb{R}^n \) and \( 0 < \alpha < 2 \).

Definition (2.3): An \((n; 1, \alpha)\) - stable random field \( X(t) \) will be called discrete if \( t \in \mathbb{Z}^n \).

Definition (2.4): A random \((n; 1, \alpha)\) - stable random field, \( \xi(\lambda), \lambda \in \mathbb{R}^n \), will be called homogeneous if all limiting distribution, \( \tilde{\xi}(\lambda) \overset{d}{=} \xi(\lambda + \tau), \tau \in \mathbb{Z}^n \) where \( =d \) (homogeneity of random field is defined by the translation-invariance of the finite-dimensional distribution.

Definition (2.5): Let \( \xi(\lambda), \lambda \in \mathbb{R}^n \) be a continuous homogeneous symmetric complex \((n; 1, \alpha)\) - stable random field with independent \((n; 1, \alpha)\) - stable increment. A complex \((n; 1, \alpha)\) - stable random field \( \eta(v), v \in \mathbb{R}^n \), called harmonic homogeneous stable field if it has the spectral representation

\[
\eta(v) = \int \exp(i\langle v, \lambda \rangle) d\xi(\lambda).
\]

The spectral representation for discrete harmonic \((n; 1, \alpha)\) - stable random field \( X(t) \) can be written in the form

\[
X(t) = \int_{\mathbb{R}^n} \exp(i\langle v, \lambda \rangle) d\xi(\lambda)
\tag{2.1}
\]

where, \( \xi(\lambda), \lambda \in \mathbb{R}^n \), is an \((n; 1, \alpha)\) - stable random field with independent increments satisfy

\[
[E |d\xi(\lambda)|^\alpha]^F = C(P, \alpha) \Phi(\lambda) d\lambda
\]

where \( C(P, \alpha) \) depends on \( P \) and \( \alpha \) and \( \Phi(\lambda) \) is a nonnegative integrable function called the spectral density field of \( X(t), t \in \mathbb{Z}^n \).

We will construct an estimation for the nonnegative integrable function \( \Phi(\lambda), \lambda \in \mathbb{R}^n \), on observations \( X(t), t \in \mathbb{Z}^n \). For construction, we will use the periodogram as in Masry [9] which estimated the spectral density function for stable random process. Let \( h(t) = h(t_1, t_2, \ldots, t_j) \) be a bounded even function. Let \( T = (T_1, T_2, \ldots, T_n) \) where \( T_j = 2\tau_j + 1; j = 1, n \) and \( t_j = (t_1, t_2, \cdots, t_j) \). We define a finite Fourier transform of the function \( h(t) \) by

\[
\hat{h}(\lambda) = \int_{\mathbb{R}^n} h(t) \exp(-i\langle t, \lambda \rangle) dt.
\]
The function $H^T(\lambda)$ satisfy

$$H^T(\lambda) = \sum_{t \in \mathbb{Z}} h(t) \exp(-i(t, \lambda)), \text{ where } \lambda \in \mathbb{R}^n.$$ \hspace{1cm} (2.2)

The function $H^T(\lambda)$ satisfy

$$B^T_\alpha = \int_{\mathbb{R}^n} \left| H^T(\lambda) \right|^\alpha d\lambda < \infty \text{ with } 0 < \alpha < 2.$$ \hspace{1cm} (2.3)

Let

$$H_f(\lambda) = A_f H^T(\lambda)$$ \hspace{1cm} (2.4)

where $A_f = \left[ \frac{1}{B^T_\alpha} \right]^{\frac{1}{\alpha}} \in \mathbb{R}^n$ and

$$\int |H_f(\lambda)|^\alpha d\lambda = 1.$$ \hspace{1cm} (2.5)

We consider the statistic

$$d_f(\lambda) = A_f \Re \left[ \sum_{t \in \mathbb{Z}} \exp(-i(t, \lambda)) h(t) X(t) \right]$$ \hspace{1cm} (2.6)

to estimate the spectral density $\Phi(\lambda)$.

**Definition (2.6):** Let $0 < P < \alpha < 2$, $\lambda \in \mathbb{R}^n$. The statistic $I_f(\lambda)$ will be called a periodogram $(n; 1, \alpha)$ - stable random field $X(t)$, $t \in T^n$, where

$$I_f(\lambda) = K(P, \alpha) |d_f(\lambda)|^\alpha,$$ \hspace{1cm} (2.7)

$$K(P, \alpha) = \frac{D(P)}{F(P, \alpha) C_\alpha^{(P, \alpha)}}$$ \hspace{1cm} (2.8)

where $P = n \alpha$ and $C_\alpha = \frac{1}{\pi} \int_0^\pi |\cos \theta|^\alpha d\theta$.

The following Lemma can be proved as in [9] and will be used in Section 3.

**Lemma (2.1):** Let $L^\alpha (0 < \alpha < 2)$ be the set of all measurable function on $\mathbb{R}$, for which

$$\int |g(\lambda)|^\alpha d\lambda < \infty, \text{ where } \int \mu \in \mathbb{R}^n.$$ \hspace{1cm} (2.9)

$G(\lambda)$ is non-negative bounded on $\mathbb{R}^n$ with $G(\Pi, \Pi, \Pi, \ldots, \Pi) > 0$.

Then for homogeneous symmetric complex $(n; 1, \alpha)$ - stable field $\xi(\lambda), \lambda \in \mathbb{R}^n$, we have

$$E \exp[i \Re \left[ \int g(\lambda) d(\lambda) \right]] = \int |g(\lambda)|^\alpha \Phi(\mu) d\lambda$$ \hspace{1cm} (3.1)

where

$$\gamma_f^{(\alpha)}(\lambda) = \int |H_f(\lambda - \mu)|^\alpha \Phi(\mu) d\lambda.$$ \hspace{1cm} (3.2)

**Theorem (3.1):** Let $\lambda \in \mathbb{R}$.

(i) $E I_f(\lambda) = \frac{\left[ \gamma_f^{(\alpha)}(\lambda) \right]_P^P}{P \in (0, \alpha)}$ \hspace{1cm} (3.3)

(ii) $D I_f(\lambda) = \left[ \frac{K^2(P, \alpha)}{K(2P, \alpha)} - 1 \right] \left[ \gamma_f^{(\alpha)}(\lambda) \right]_P^P, P \in (0, \alpha).$ \hspace{1cm} (3.4)

**Proof:** From (2.9), we have

$$|d_f(\lambda)|^\alpha = D^{-1}(P) \Re \left[ \int \frac{1 - \exp(iu d_f(\lambda))}{|u|^{\alpha P}} du \right].$$ \hspace{1cm} (3.5)

From (2.5), (3.2) and (3.6) we can obtain

$$E I_f(\lambda) = \frac{1}{F(P, \alpha) C_\alpha^{(P, \alpha)}} \int \frac{1 - \exp(C_\alpha a^\alpha \gamma_f^{(\alpha)}(\lambda))}{|u|^{\alpha P}} du.$$ \hspace{1cm} (3.6)

By putting $u = a^\alpha \gamma_f^{(\alpha)}(\lambda)$ and using (2.7) we have

$$E I_f(\lambda) = \frac{1}{F(P, \alpha)} \int \frac{1 - \exp(-|x|^P)}{|x|^{\alpha P}} dx [\gamma_f^{(\alpha)}(\lambda)]^P.$$ \hspace{1cm} (3.7)
Part (ii) can be proved by a similar way.

**Theorem (3.2):**
\[
\text{Cov}(I_f(\lambda^{(1)}), IT(\lambda^{(2)})) = \int \left\{ \frac{1}{F(P, \alpha)C_{\alpha}^{\alpha}} \right\} \int \left\{ \exp{C_T}(u_1, u_2) \right\} \frac{du_1 du_2}{|u_1 u_2|^{\alpha}}
\]
where
\[
C_T(u_1, u_2) = -C_\alpha \int \left[ \mu H_f(\lambda^{(1)} - \mu) + u_2 H_f(\lambda^{(2)} - \mu)^\alpha \Phi(\mu) d\mu \right]
\]
\[
C_T(u_1, u_2) = -C_\alpha \int \left[ \mu H_f(\lambda^{(1)} - \mu) + u_2 H_f(\lambda^{(2)} - \mu)^\alpha \Phi(\mu) d\mu \right]
\]

**Proof:** From (2.5), (2.6) and (2.9) we have
\[
I_f(\lambda) = \int \frac{1}{F(P, \alpha)C_{\alpha}^{\alpha}} \int \left\{ \cos(u d_f(\lambda)) \right\} \frac{du}{|u|^{\alpha}}
\]
From (3.6) we obtain
\[
\int \left\{ \cos(u d_f(\lambda)) \right\} \frac{du}{|u|^{\alpha}} = \int \left\{ \cos(u d_f(\lambda)) \right\} \frac{du}{|u|^{\alpha}}
\]
By the definition of the covariance function, we can obtain
\[
\text{Cov}(I_f(\lambda^{(1)}), I_f(\lambda^{(2)})) = \int \left\{ \frac{1}{F(P, \alpha)C_{\alpha}^{\alpha}} \right\} \int \left\{ \exp{C_T}(u_1, u_2) \right\} \frac{du_1 du_2}{|u_1 u_2|^{\alpha}}
\]
Since,
\[
E\left\{ \sum_{j=1}^{\alpha} \exp(-C_\alpha | u_j |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(1)})) \right\}
\]
\[
= E\left\{ \cos(u d_f(\lambda^{(1)})) \cos(u d_f(\lambda^{(2)})) \right\}
\]
\[
- \exp(-C_\alpha | u_1 |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(2)})) E\left\{ \cos(u d_f(\lambda^{(1)})) \right\}
\]
\[
- \exp(-C_\alpha | u_2 |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(1)})) E\left\{ \cos(u d_f(\lambda^{(2)})) \right\}
\]
\[
+ \exp(-C_\alpha | u_1 |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(1)}) + | u_2 |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(2)}))
\]
From Lemma (3.2), we get
\[
E\left\{ \cos(u d_f(\lambda^{(1)})) \cos(u d_f(\lambda^{(2)})) \right\}
\]
\[
= \frac{1}{2} \exp(-C_\alpha | u_1 |^\alpha H_f(\lambda^{(1)} - \mu) + u_2 H_f(\lambda^{(2)} - \mu)^\alpha \Phi(\mu) d\mu |)
\]
Furthermore, by using (3.2) we get
\[
E\left\{ \cos(u d_f(\lambda^{(1)})) \right\} = \exp(-C_\alpha | u_j |^\alpha \Gamma_f^{(\alpha)}(\lambda^{(1)}))
\]
By substituting from (3.11) and (3.12) in (3.10), then we have
\[
\text{Cov}(I_f(\lambda^{(1)}), I_f(\lambda^{(2)}))
\]
\[
= \frac{1}{2} \exp(-C_\alpha | u_1 |^\alpha H_f(\lambda^{(1)} - \mu) + u_2 H_f(\lambda^{(2)} - \mu)^\alpha \Phi(\mu) d\mu |)
\]
\[
+ \frac{1}{2} \exp(-C_\alpha | u_1 |^\alpha H_f(\lambda^{(1)} - \mu) - u_2 H_f(\lambda^{(2)} - \mu)^\alpha \Phi(\mu) d\mu |)
\]
Finally, if we replace \(u_i\) by \((-u_i)\) and using (3.3), (3.8), then the proof can be completed.

4. **ASYMPTOTIC BEHAVIOUR OF THE MEAN, DESPESION AND COVARIANCE FOR THE PERIODOGRAM**

In this section we obtain the asymptotic properties of the periodogram \(I_f(\lambda)\).

**Definition (4.1):** A positive kernel \(F_t(\lambda)\), \(\lambda \in \Pi^n\), is said to be an \(n - measure sequence function\) if for all \(\lambda \in \Pi\) and \(\delta > 0\), the following conditions are satisfied:

(i) \(F_t(\lambda) \geq 0\);

(ii) \(\int \int F_t(\lambda) d\lambda = 1\);

(iii) \(\lim_{T \to \infty} \int \int F_t(\lambda) d\lambda = 0\);

where \(S_\delta = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), ||\lambda|| < \delta, \delta > 0\}\)

The following lemma will be used later in the proof of Theorem (4.1).

**Lemma (4.1):** Let \(F_t(\lambda), \lambda \in \Pi^n,\) be a positive kernel. If \(g(\lambda)\) is bounded on \(\Pi\) and continuous at a point \(\lambda^*, \lambda^* \in \Pi\), then
\[
\lim_{T \to \infty} \int \int F_t(\lambda + \lambda^*) d\lambda = g(\lambda^*).
\]

**Proof:** Since \(\int \int F_t(\lambda + \lambda^*) d\lambda = 1\), then
\[
\int \int F_t(\lambda + \lambda^*) d\lambda - g(\lambda^*) \leq 0
\]
Since \(g(\lambda)\) continuous at \(\lambda^*\) then for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(|g(\lambda + \delta^*) - g(\lambda^*)| \leq \varepsilon\) for all \(\lambda \in S_\delta\). So,
\[
\int \int F_t(\lambda) g(\lambda + \lambda^*) d\lambda - g(\lambda^*) \leq \varepsilon
\]

Finally, if we replace \(u_i\) by \((-u_i)\) and using (3.3), (3.8), then the proof can be completed.
Since $g(\lambda)$ is bounded on $\prod^{\infty}_{n=1} \lambda^n$, then there is a real number $L < \infty$ such that $\lambda^n \prod^{\infty}_{n=1} \lambda^n$.

Therefore,
\[
\int F_\lambda(\lambda) g(\lambda + \lambda') - g(\lambda') \, d\lambda \leq 2L \int \lambda^n \prod^{\infty}_{n=1} \lambda^n \lambda^n \lambda' \lambda'
\]

According to definition (4.1) and $|g(\lambda + \lambda') - g(\lambda')|$ may be made arbitrarily small by choice of $\varepsilon$ as $g(\lambda)$ continuous at a point, we can conclude that
\[
\lim_{\varepsilon \to 0} \int F_\lambda(\lambda) g(\lambda + \lambda') - g(\lambda') \, d\lambda = 0.
\]

**Theorem (4.1):** Let $\lambda \in \prod^n$. Then
\[
(i) \lim_{\varepsilon \to 0} E I_\varepsilon(\lambda) = [\Phi(\lambda)]^{2P}, P \in (\alpha, \alpha) \tag{4.1}
\]
\[
(ii) \lim_{\varepsilon \to 0} D I_\varepsilon(\lambda) = \left[ \frac{K^2(P, \alpha)}{K(2P, \alpha)} - 1 \right][\Phi(\lambda)]^{2\alpha P} P = (0, \frac{\alpha}{2}) \tag{4.2}
\]

**Proof:** The proof comes directly by substituting $-\lambda + \mu = \nu$ in (3.3) and using Theorem (3.1) with Lemma (4.1).

**Theorem (4.2):** Suppose $0 \leq \alpha \leq \frac{\alpha}{2} < 2$ with $\lambda^{(1)} \in \prod^n$, $\lambda^{(2)} \in \prod^n$ and $\lambda^{(1)} \neq \lambda^{(2)}$, i.e. let $\Phi(\lambda)$, $\lambda \in \prod^n$, be continuous at $\lambda^{(1)}$, $\lambda^{(2)}$ and $\Phi(\lambda^{(1)}) \neq 0$, $\Phi(\lambda^{(2)}) \neq 0$. If
\[
\lim_{\varepsilon \to 0} \frac{B_\alpha^{(1)}(\lambda^{(1)}), \lambda^{(2)}}{B_\alpha^{(2)}(\lambda^{(1)}), \lambda^{(2)}} = 0 \tag{4.3}
\]

where
\[
B_\alpha^{(1)}(\lambda^{(1)}), \lambda^{(2)} = \int \Pi^{\beta}\left| H^{(1)}(\lambda^{(1)} - \lambda)H^{(2)}(\lambda^{(2)} - \lambda) \right|^\alpha \, d\lambda
\]

Then
\[
\lim_{\varepsilon \to 0} \text{Cov}[I_\varepsilon(\lambda^{(1)}), I_\varepsilon(\lambda^{(2)})] = 0
\]

**Proof:** From Theorem (3.2), we get
\[
|\text{Cov}[I_\varepsilon(\lambda^{(1)}), I_\varepsilon(\lambda^{(2)})]| \leq \frac{1}{F(\lambda, \alpha) C_\alpha^{(1)}} \times \text{exp}\left\{\sum_{j=1}^{P} \left[ C^{(1)}(u_j, u_j) - C^{(2)}(u_j, u_j) \right] \right\} \frac{du_j}{u_j^{2P}}
\]

From (3.8) and Lemma (2.3) we have
\[
|C^{(1)}(u_j, u_j) - C^{(2)}(u_j, u_j)| \leq 2C_\alpha |u_j|^{2P} \frac{\Phi(\mu)}{\prod^{\infty}_{n=1} \lambda^n} \tag{4.4}
\]

Hence,
\[
|\text{Cov}[I_\varepsilon(\lambda^{(1)}), I_\varepsilon(\lambda^{(2)})]| \leq \frac{1}{F(\lambda, \alpha) C_\alpha^{(1)}} \times \text{exp}\left\{\sum_{j=1}^{P} \left[ C^{(1)}(u_j, u_j) - C^{(2)}(u_j, u_j) \right] \right\} \frac{du_j}{u_j^{2P}}
\]

\[
\text{exp}\left\{\sum_{j=1}^{P} \left[ C^{(1)}(u_j, u_j) - C^{(2)}(u_j, u_j) \right] \right\} \frac{du_j}{u_j^{2P}} \leq \frac{1}{C_\alpha^{(1)}}
\]

Finally, we can conclude that
\[
\lim_{\varepsilon \to 0} \text{exp}\left\{\sum_{j=1}^{P} \left[ C^{(1)}(u_j, u_j) - C^{(2)}(u_j, u_j) \right] \right\} \frac{du_j}{u_j^{2P}} < \infty
\]

and this completes our proof.
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