Hitting Time and Place of Brownian Motion with Drift

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Abstract: We consider a \( d \)-dimensional Brownian motion in \( \mathbb{R}^d \) with drift. The explicit expressions are obtained for the joint density of the hitting time and place to a sphere, when the process starts either from the inside the sphere or from the out of sphere.

Keywords: Brownian motion with drift, hitting time, hitting place, Joint density.

1. INTRODUCTION

Let \( \{X(t), t \geq 0\} \) be a standard \( d \)-dimensional Brownian motion with drift \( c: X(t) = B(t) + ct, \ t \geq 0, \) where \( B(t) \) is the standard \( d \)-dimensional Brownian motion, \( c \in \mathbb{R}^d (d \geq 2) \) is a fixed vector. Let us denote by \( P_x(•) \) the probability measure on the path space of \( X \) corresponding to initial value \( X(0) = x \) and drift vector \( c, \ E_x(•) \) the corresponding expectation operator. For simplicity, we shall write \( P_i (•) \) and \( E_i (•) \) to refer the case \( c = 0. \) For \( r > 0, \) consider the sphere \( B_r = \{ x : x \in \mathbb{R}^d, \ |x| = r \}. \) The first hitting time of \( X \) through \( \partial B_r \) is defined as \( T_r = \inf \{ t > 0 : |X(t)| = r \}. \) As usual, we take \( \inf(\varnothing) = +\infty. \) The first hitting place is \( X(T_r). \) Because of the sample path continuity of the process, \( X(T_r) \) lies on \( \partial B_r. \)

A Laplace-Gegenbauer transform of the first hitting time and the first hitting place to a sphere centered at the origin was obtained by Hsu [5]; Betz and Gzyl [2, 3] gave another proof to Wendel's exterior problem. Yin [4] extended Wendel's results to the case of Brownian motion with constant drift. The joint density of the first hitting time and the first hitting place of a sphere by Brownian motion which starts at any point inside the sphere was obtained by Hsu [5]. The aim of this paper is to obtain the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion with or without drift which starts at any point in space.

The following notation can be found in [6]. Let \( J_\nu \) and \( N_\nu \) denote the first and second Bessel function of order \( \nu, \) respectively. Let \( I_\nu \) and \( K_\nu \) denote the first and second Bessel function of purely imaginary argument, respectively. Let \( C_m^n \) be the Gegenbauer polynomial of degree \( m \) and order \( \nu, \) which is defined via its generating function:

\[
(1 - 2 \beta t + \beta^2)^{-\nu} = \sum_{n=0}^{\infty} C_m^n(t) \beta^n.
\]

It is customary to take \( C_0^m = 1, C_0^m = 1, C_0^m = 2T_m/m, \) here \( T_m \) is the \( m \)th Tchebycheff polynomial: \( T_m(\cos \theta) = \cos m \theta. \) Set \( h = (d - 2)/2. \)

We use \( \{q_{m, n}, n \geq 1\} \) to denote the positive zeros of \( J_{m+h} \) in the ascending order.

2. LEMMAS

In this section we give several lemmas for latter use.

**Lemma 2.1.** ([6]) Let \( \sigma(dy) \) be the \( d-1 \) dimensional volume measure on \( \partial B_r \) \( (d \geq 2), \) then

\[
\int_{\partial B_r} C_m^k (\cos \theta) C_n^k (\cos \theta) \sigma(dy) = \begin{cases} 
\frac{2}{(m+h)!} \frac{\Gamma(h)}{\Gamma(h+m)} C_m^k(1), & m = k,\ d \geq 3, \\
\frac{2\pi r}{m} C_m^0(1), & m = k \neq 0, \ d = 2, \\
2\pi r, & m \neq k, \ d = 2, \\
0, & m \neq k, \ d \geq 2,
\end{cases}
\]

where \( \theta = \angle xy, \ x \in \mathbb{R}^d. \)

**Lemma 2.2.** ([5]) For \( |x| < r, \ \alpha > 0 \) and \( d \geq 2, \) then

\[
-2 \sum_{m=0}^{\infty} \frac{q_{m,n} J_{m+h}(\sqrt{2\alpha} |x|) q_{m,n}}{(2r^2 + \alpha) J_{m+h}(\sqrt{2\alpha r}) J_{m+h}(\sqrt{2\alpha r})} = I_{m+h}(\sqrt{2\alpha |x|}),
\]

where \( m \geq 0 \) is an integer.

**Lemma 2.3.** For \( |x| > r, \ \alpha > 0 \) and \( d \geq 2, \) then
\[ \int_0^\infty \lambda \left( J_{m+h}(\lambda |x|)N_{m+h}(\lambda r) - J_{m+h}(\lambda r)N_{m+h}(\lambda |x|) \right) d\lambda = -\frac{\pi}{2} \frac{K_{m+h}(\sqrt{2\alpha |x|})}{K_{m+h}(\sqrt{2\alpha r})}, \text{ where } m \geq 0 \text{ is an integer.} \]

**Proof.** Using the recurrence formulas (see [6]):

\[ \frac{d}{dt}(x^v K_v) = -K_v^{x^v K_v - 1} \frac{d}{dt}(x^v K_v) = -x^{v^2} K_{v+1}, \quad \frac{d}{dt}(x^v Z_v) = -x^{v^2} Z_{v-1}, \quad \frac{d}{dt}(x^v Z_v) = -x^{v^2} K_{v-1}, \]

where \( Z_v = J_v \) or \( N_v \), we get

\[ \int_0^\infty \pi \frac{K_{m+h}(\sqrt{2\alpha R})}{2} \frac{J_{m+h}(\lambda R)N_{m+h}(\lambda r) - J_{m+h}(\lambda r)N_{m+h}(\lambda R)}{K_{m+h}(\sqrt{2\alpha r})} R dR = -\frac{1}{2\alpha + \lambda^2}. \]

The result follows immediately from the Weber's inversion transform (see [7]). This ends the proof.

Letting \( k = 0 \) in Lemma 2.1, we have

**Lemma 2.4.** Let \( \sigma(dy) \) be the \( d-1 \) dimensional volume measure on \( \partial B_r(d \geq 2) \), then

\[ \int_{\partial B_r} C_m^h(\cos \theta)\sigma(dy) = \begin{cases} \frac{4}{\pi} \frac{h^{x^v K_v - 1}}{r^{x^v K_v}}, & m = 0, \\ 0, & m \geq 1, \end{cases} \]

where \( \theta = \angle x0y, \ x \in \mathbb{R}^d. \)

**Lemma 2.5.** Let \( x, \ c \in \mathbb{R}^d(d \geq 2) \), \( \sigma(dy) \) be the \( d-1 \) dimensional volume measure on \( \partial B_r \), then

\[ \int_{\partial B_r} e^{ixy} C_m^h(\cos \theta)\sigma(dy) = 2(\pi)^{d/2} h^{x^v K_v} I_{m+h}(|c| r) C_m^h(\cos \angle x0y). \]

where \( \theta = \angle x0y, \ x \in \mathbb{R}^d. \)

**Proof.** Using (1.5) in Yin [4] and Lemma 4 in [6, P.245].

**3. HITTING SPHERE FOR BROWNIAN MOTION**

In this section, we will give the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion. This is derived based on the Laplace-Gegenbauer transform obtained in Wendel [1]. The result in Theorem 3.1 is due to Hsu [5], which was obtained by solving the heat equation with Dirichlet boundary condition satisfied by the transition density function of a Brownian motion in a ball.

For the interior problem we have

**Theorem 3.1.** For \( x, y \in \mathbb{R}^d(d \geq 2), \ |x| < r, \ |y| = r \) and \( t > 0 \), then

(1) for \( d \geq 3 \), we have

\[ P \left( T_y \in dt, X(T_y) \in dy \right) d\sigma(dy) = \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{\Gamma(h)(m+h)C_m^h(\cos \theta)q_{m,n}J_{m+h}(|y| q_{m,n}) \cdot \frac{2x^v K_v}{\pi^2}}{2\pi^{h+1} r^{h+3} |x| h^{x^v K_v} J_{m+h}(|y| q_{m,n})} \cdot \frac{2x^v K_v}{\pi^2} \cdot \frac{2x^v K_v}{\pi^2} \cdot \frac{2x^v K_v}{\pi^2}, \tag{3.1} \]

(2) for \( d = 2 \), we have

\[ P \left( T_y \in dt, X(T_y) \in dy \right) d\sigma(dy) = \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{q_{m,n} J_0(|y| q_{m,n}) \cdot \frac{4x^v K_v}{\pi^2}}{2\pi r^2 J_0(|y| q_{m,n})} \cdot \frac{4x^v K_v}{\pi^2} - \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{q_{m,n} C_m^h(\cos \theta) J_{m+h}(|y| q_{m,n}) \cdot \frac{2x^v K_v}{\pi^2}}{\pi^2 r^2 J_m(|y| q_{m,n})} \cdot \frac{2x^v K_v}{\pi^2}, \tag{3.2} \]

where \( \theta = \angle x0y, \ \sigma \) is the \( d-1 \) dimensional volume measure on \( \partial B_r \).

**Proof.** Let us denote by \( H(t,y) \) the right hand side of (3.1). For \( \alpha > 0 \) and integer \( k \geq 0 \), using Lemmas 2.1 and 2.2 we have
On the other hand, from (3) in Wendel [1] we get

\[ \int_0^\infty \int_{0,1} e^{-\alpha t} C^k_i(\cos \theta) H(t, y) d\sigma(dy) = \frac{\Gamma(h)}{2\pi^{k+1} \rho^2} \sum_{m=0}^{\infty} (m+h) q_{m,h}(\frac{d}{dt} q_{m,h}) \frac{J_{m+h}(q_{m,h})}{J_{m+h}(q_{m,h})} \]

\times \int_0^\infty e^{-\frac{2\alpha}{\rho^2} t} \pi J_{m+h}(\cos \theta) C^k_m(\cos \theta) d\sigma(dy) = -\frac{2\alpha^2 C^k_i(1)}{|x|^k} \sum_{n=0}^{\infty} q_{n,n} J_{k+n}(\frac{d}{dt} q_{n,n}) \frac{J_{k+n}(q_{n,n})}{J_{k+n}(q_{n,n})}

\[ = \left( \frac{r}{|x|} \right)^h C^k_i(1) \frac{J_{k+n}(\sqrt{2\pi} |x|)}{J_{k+n}(\sqrt{2\pi})}. \]

It follows from (3.3) and (3.4) and the uniqueness that \( P_s(T_r \in dt, B(T_r) \in dy) = H(t, y) d\sigma(dy) \).

This proves (3.1). Eq. (3.2) can be proved along the same lines of (3.1) and thus the proof is omitted.

**Remark 3.1.** When \( r = 1 \), the result (3.1) coincides with (13) in Hsu [5].

**Corollary 3.1.** For \( x \in \mathbb{R}^d (d \geq 2), \ |x|<r \), and \( r>0 \), then

\[ P_s(T_r \in dt) / dt = \frac{q_{m,n}}{r^2 J_{m+n}(q_{m,n})} \left( \frac{|x|}{r} \right)^n J_{m+n}(q_{m,n}) e^{-\frac{2\alpha}{\rho^2} t}. \]

**Proof.** Integrating (3.1) or (3.2) with respect to \( y \in \partial B_r \), using Lemma 2.4 and \( J_{m+n}(q_{m,n}) = -J_{m+n}(q_{m,n}) \).

For the exterior problem we have

**Theorem 3.2.** For \( x, y \in \mathbb{R}^d (d \geq 2), \ |x|<r, \ |y|=r \) and \( r>0 \), then

(1) for \( d \geq 3 \), we have

\[ P_s(T_r \in dt, X(T_r) \in dy, T_r < \infty) / dt d\sigma(dy) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d+1}{2}} (r|x|)^{\frac{d}{2}}} \sum_{m=0}^{\infty} (m+h) C^k_i(\cos \theta) \]

\[ \times \int_0^\infty \frac{\lambda J_{m+k}(\lambda |x|) N_{m+k}(\lambda r) - J_{m+k}(\lambda r) N_{m+k}(\lambda |x|)}{J_{m+k}^2(\lambda r) + N_{m+k}^2(\lambda r)} e^{-\frac{2\alpha}{\rho^2} \lambda} d\lambda; \]

(3.6)

(2) for \( d = 2 \), we have

\[ P_s(T_r \in dt, X(T_r) \in dy, T_r < \infty) / dt d\sigma(dy) = \frac{\sum_{m=0}^{\infty} |x| D(m,|x|)}{\pi r} C^0_i(\cos \theta) \]

\[ \times \int_0^\infty \frac{\lambda J_{m}(\lambda |x|) N_{m}(\lambda r) - J_{m}(\lambda r) N_{m}(\lambda |x|)}{J_{m}^2(\lambda r) + N_{m}^2(\lambda r)} e^{-\frac{2\alpha}{\rho^2} \lambda} d\lambda, \]

(3.7)

Where \( \theta = \angle x, y \), \( \sigma \) is the \( d-1 \) dimensional volume measure on \( \partial B_r \) and \( D(m,|x|) = \frac{m}{2|m|}, \) if \( m \neq 0; \frac{1}{2|m|}, \) if \( m = 0 \).

**Proof.** Let us denote by \( G(t,y) \) the right hand side of (3.6). For \( \alpha > 0 \) and integer \( k \geq 0 \), using Lemmas 2.1 and 2.3 we have

\[ \int_0^\infty \int_{0,1} e^{-\alpha t} C^k_i(\cos \theta) G(t, y) d\sigma(dy) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d+1}{2}} (r|x|)^{\frac{d}{2}}} \sum_{m=0}^{\infty} (m+h) \int_{0,1} C^k_i(\cos \theta) C^k_m(\cos \theta) d\sigma(dy) \]

\[ \times \int_0^\infty \frac{\lambda J_{m+k}(\lambda |x|) N_{m+k}(\lambda r) - J_{m+k}(\lambda r) N_{m+k}(\lambda |x|)}{J_{m+k}^2(\lambda r) + N_{m+k}^2(\lambda r)} e^{-\frac{2\alpha}{\rho^2} \lambda} d\lambda dt = \left( \frac{r}{|x|} \right)^h \frac{K_{k+n}(\sqrt{2\pi} |x|)}{K_{k+n}(\sqrt{2\pi r})}. \]

(3.8)
\[ \int_0^\infty \int_{\partial B} e^{-\alpha|y|^2} C^k_n^m(\cos \theta) \mathbb{P}_x(T_r \in dt, X(T_r) \in dy, T_r < \infty) = \left( \frac{r}{|x|} \right)^3 \mathbb{C}_n^m(1) \frac{K_{k+h}(\sqrt{2\alpha |x|})}{K_{k+h}(\sqrt{2\alpha r})}. \] (3.9)

It follows from (3.8) and (3.9) and the uniqueness that \( \mathbb{P}_x(T_r \in dt, B(T_r) \in dy) = G(t, y)dt \sigma(dy) \).

This proves (3.6). (3.7) can be proved along the same lines as the case (3.6) and will be omitted.

The following corollary follows immediately from Theorem 3.2 and Lemma 2.4.

**Corollary 3.2.** For \( x \in \mathbb{R}^d (d \geq 2), \ |x| < r \) and \( t > 0 \), then

\[ P_x(T_r \in dt, T_r < \infty) = -\frac{1}{r} \int_0^\infty \lambda(J_n(\lambda |x|)N_n^*(\lambda r) - J_n(\lambda r)N_n^*(\lambda |x|)) \frac{1}{J_n^*(\lambda r) + N_n^*(\lambda r)} e^{-\frac{\lambda r}{2}} \frac{d\lambda}{\lambda}. \]

4. HITTING SPHERE FOR BROWNIAN MOTION WITH DRIFT

In this section, we will give the joint density of the first hitting time and the first hitting place of a sphere by Brownian motion with constant drift. The results can be proved, as in the last section, by inverting the Laplace-Gegenbauer transform for Brownian motion with drift obtained in Yin [4]. Or, using Girsanov’s change of measure theorem for Brownian motion. We give the results without proof.

For the interior problem we have

**Theorem 4.1.** For \( c, x, y \in \mathbb{R}^d (d \geq 2), \ |x| < r, \ |y| = r \) and \( t > 0 \), then

1. For \( d \geq 3 \), we have

\[ P_x(T_r \in dt, X(T_r) \in dy) = e^{-\alpha |c|^2} \int_0^\infty \Gamma(h)(m+h)C^h_m(\cos \theta) q_{m,a} J_{m}^{*}(q_{m,a}) e^{-\frac{\lambda r}{2}} \frac{d\lambda}{\lambda}; \]

2. For \( d = 2 \), we have

\[ P_x(T_r \in dt, X(T_r) \in dy) = e^{-\alpha |c|^2} \left\{ \sum_{m=0}^\infty \frac{d_m}{2\pi r^{3} J_m^{*} (q_{m,a})} e^{-\frac{\lambda r}{2}} + \sum_{m=1}^\infty \frac{q_{m,a} \cos(m \vartheta) J_m^{*}(q_{m,a})}{\pi r^{3} J_m^{*} (q_{m,a})} e^{-\frac{\lambda r}{2}} \right\}. \]

where \( \vartheta = \angle x y, \) \( \sigma \) is the \( d-1 \) dimensional volume measure on \( \partial B_r \).

**Corollary 4.1.** For \( c, x \in \mathbb{R}^d (d \geq 2), \ |x| < r \) and \( t > 0 \), then

1. For \( d \geq 3 \), we have

\[ P_x(T_r \in dt) = e^{-\alpha |c|^2} \frac{2^{h} \Gamma(h)}{r^{3} (|c|^{2} + |x|^{2})} \sum_{m=0}^\infty \frac{(m+h)C^h_m(\cos \angle c 0 x) q_{m,a} J_{m}^{*}(r |c|) J_{m}^{*}(q_{m,a})}{J_{m}^{*}(q_{m,a})} e^{-\frac{\lambda r}{2}} \frac{d\lambda}{\lambda}; \]

2. For \( d = 2 \), we have

\[ P_x(T_r \in dt) = e^{-\alpha |c|^2} \sum_{m=1}^\infty \frac{q_{m,a} J_{m}^{*}(q_{m,a})}{r^{2} J_0 (q_{m,a})} e^{-\frac{\lambda r}{2}} \frac{d\lambda}{\lambda} \]

\[ - e^{-\alpha |c|^2} \sum_{m=1}^\infty \frac{2 q_{m,a} \cos(m \angle c 0 x) J_{m}^{*}(r |c|) J_{m}^{*}(q_{m,a})}{r^{2} J_{m}^{*} (q_{m,a})} e^{-\frac{\lambda r}{2}} \frac{d\lambda}{\lambda}. \]

**Proof:** Integrating (3.1) or (3.2) with respect to \( y \in \sum_{i=1}^{d-1}(0, r) \) and using Lemma 2.5.

For the exterior problem we have
Theorem 4.2. For $c, x, y \in \mathbb{R}^d (d \geq 2), \ |x| > r, \ |y| = r$ and $t > 0$, then

(1) for $d \geq 3$, we have

$$P^c (T_c \in dt, B(T_c) \in dy, T_c < \infty) / dt \sigma(dy) = -\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(r | x |)^{\frac{d}{2}}} \sum_{m=0}^{\infty} (m + h)C_n^m (\cos \theta) \sqrt{J_m^2 (\lambda r) + N_m^2 (\lambda r)} \Delta x\Delta \lambda;$$

(2) for $d = 2$, we have

$$P^c (T_2 \in dt, B(T_2) \in dy, T_2 < \infty) / dt \sigma(dy) = -e^{-\frac{x^2}{2r^2}} \sum_{m=0}^{\infty} \frac{|x| D(m, | x |)}{\pi r} C_n^m (\cos \theta) \sqrt{J_m^2 (\lambda r) + N_m^2 (\lambda r)} \Delta x\Delta \lambda;$$

where $\theta = \angle x 0 y$, $\sigma$ is the $d-1$ dimensional volume measure on $\partial B_r$ and $D(m, | x |) = \frac{m}{2\pi | x |}$, if $m \neq 0$; $\frac{1}{2} \pi | x |$, if $m = 0$.

The following corollary follows immediately from Theorem 3.2 and Lemma 2.5.

Corollary 4.2. For $c, x \in \mathbb{R}^d (d \geq 2), \ |x| > r$ and $t > 0$, then

(1) for $d \geq 3$, we have

$$P^c (T_c \in dt, T_c < \infty) / dt = -e^{-\frac{x^2}{2r^2}} \frac{\Gamma(\frac{d}{2})}{\pi (|x| |y|)^{\frac{d}{2}}} \sum_{m=0}^{\infty} (m + h)\Delta x\Delta \lambda;$$

(2) for $d = 2$, we have

$$P^c (T_2 \in dt, T_2 < \infty) / dt = -e^{-\frac{x^2}{2r^2}} \left( \frac{1}{\pi} I_0 (r | y |) + \sum_{m=0}^{\infty} \frac{m}{\pi} I_m (r | y |) C_n^m (\cos \angle y 0 x) \right) \sqrt{J_m^2 (\lambda r) + N_m^2 (\lambda r)} \Delta x\Delta \lambda;$$

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