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RESEARCH ARTICLE

On The Distribution of Partial Sums of Randomly Weighted Powers of Uniform Spacings

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Abstract:
Objectives:
To study the asymptotic theory of the randomly weighted partial sum process of powers of k-spacings from the uniform distribution.

Methods:
Earlier results on the distribution of the uniform incremental randomly weighted sums.

Results and conclusions:
Our first contribution is to classify the multitude of earlier proofs in Section 3. The second contribution consists of a new class of proofs.

Keywords: Uniform spacings, Weak convergence, Gaussian process, Incremental asymptotic convergence, Random Sample, k spacings.

1. INTRODUCTION

Let \( 0 = U(0) \leq U(1) \leq U(2) \leq \cdots \leq U(n-1) \leq U(n) = 1 \) be the order statistics of a random sample of size \((n-1)\) from the U(0,1) distribution. Let \( k=1,2, \ldots \) be arbitrary but fixed and assume that \( n=mk \). The \( k \) (0,1) k-spacings are defined as

\[
R_{i,k} = U((i-1)k) - U((i-1)k), \quad i = 1, 2, \ldots, m. \tag{1}
\]

Let \( X_1, X_2, \ldots \) be iidrv with \( E(X) = \mu \), \( \text{Var}(X) = \sigma^2 < \infty \) and common distribution function \( F(.) \). Assume that the \( X_i \)'s are independent of the \( U_i \)'s. Define

\[
S_m(t, k, r, F) = \begin{cases} \sum_{i=1}^{[mt]} R_{i,k}^r X_i, & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 < t < \frac{1}{m} \end{cases}, \tag{2}
\]

where \([s]\) is the integer part of \( s \) and \( r>0 \) is fixed.

Looking at \( S_m(t, k, r, F) \) of (2) as a weighted partial sum of the \( X_i \)'s, Van Assche [1] obtained the exact distribution of \( S_1(1, 1, 1, F) \). Johnson and Kotz [2] studied some generalizations of Van Assche results. Soltani and Homei [3] considered the finite sample distribution of \( S_n(1, 1, 1, F) \). Soltani and Roozegar [4] considered the finite sample distribution of a case similar to \( S_m(1, k, 1, F) \) in which the spacings (1) are not equally spaced. It is interesting to note that \( S_m(t, k, r, F) \) of (2) is also a randomly weighted partial sum of powers of \( k \)-spacings from the U(0,1) distribution.

Here, we will obtain the asymptotic distribution of the stochastic process

\[
\alpha_m(t, k, r, F) = \begin{cases} \sum_{i=1}^{[mt]} (k^r m^{r-1} S_m(t, k, r, F) - t \mu_r F), & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 < t < \frac{1}{m} \end{cases}, \tag{3}
\]

where
\[ \mu_{lk} = \frac{\Gamma(k+l)}{\Gamma(k)}, k \geq 1, l > 0 \quad (4) \]

and \( \Gamma(.) \) is the gamma function.

The motivations and justifications of this work are given next. First, as noted by Johnson and Kotz [2], \( S_n (1, l, l, F) \) is a random mixture of distributions and as such it has numerous applications in Sociology and in Biology. Second, the asymptotic theory of \( S_n (t, k, r, F) \) is a generalization of important results of Kimball [5], Darling [6], LeCam [7], Sethuraman and Rao [8], Kozioł [9], Aly [10] and Aly [11] for sums of powers of spacings from the \( U(0, 1) \) distribution. Finally, we solve the open problem of proving the asymptotic normality of \( S_n (1, l, l, F) \) proposed by Soltani and Roozegar [4].

2. METHODS

2.1. The asymptotic distribution of \( \alpha_m (\cdot, k, r, F) \)

Let \( Y_i, Y_{2i}, \ldots \) be \( iidrV \) with the exponential distribution with mean 1 which are independent of the \( X_i \)'s. By Proposition 13.15 of Breiman [12] we have for each \( m \),

\[ \{ 0 = U(0), U(1), U(2), \ldots, U(n-1), U(n) = 1 \} \]

\[ = \left\{ 0, \frac{Y_1 + Y_2}{2 \sigma^2_{\alpha_1}}, \frac{Y_1 + Y_2 + Y_3 + Y_4}{4 \sigma^2_{\alpha_1}}, \ldots \right\} \quad (12) \]

Hence, for each \( m \),

\[ \{ R_{lk}, 1 \leq i \leq m \} \]

\[ = \left\{ \frac{Z_{lk}}{\sum_{i=1}^{m} Z_{lk}}, 1 \leq i \leq m \right\} \]

where for \( 1 \leq i \leq m \),

\[ Z_{lk} = Y_{(i-1)k+1} + \cdots + Y_{ik} \]

are \( iid \ Gamma (k, 1) \) random variables. Hence, for each \( m \)

\[ S_m (t, k, r, F) = \sum_{i=1}^{m} Z_{ik}X_i / \left( \sum_{i=1}^{m} Z_{ik} \right)^r, 0 \leq t \leq 1 \quad (5) \]

and

\[ \alpha_m (\cdot, k, r, F) = \left\{ \beta_m (\cdot, k, r, F) \right\} \quad (6) \]

where

\[ \beta_m (t, k, r, F) = \left\{ \begin{array}{ll} 0, & 0 \leq t < \frac{1}{m}, \\ m^2 \left\{ k^r \left( \sum_{i=1}^{m} Z_{ik}X_i \right)^{-r} \left( \sum_{i=1}^{m} Z_{ik} \right)^{r-1} \right\}, & \frac{1}{m} \leq t \leq 1 \end{array} \right. \quad (7) \]

Let \( \mu_{lk} \) be as in (4). Note that

\[ E(Z_{ik}^r) = \mu_{ik} \]

\[ E(Z_{ik}^r X_i) = \mu_{r, ik} \]

\[ \sigma^2_{r, ik} = \text{Var}(Z_{ik}^r X_i) = \sigma^2 \mu_{2r, ik} + \mu^2 \left\{ \mu_{2r, ik} - \mu^2_{r, ik} \right\} \quad (8) \]

and

\[ \text{Cov}(Z_{ik}^r X_i, Z_{lj}^s) = r \mu_{r, ik} \mu_{s, lj} \]

The following theorem will be needed in the sequel.

Theorem A. There exists a probability space on which a two-dimensional Wiener process \( \{ W(s) = (W_1(s), W_2(s)); s \geq 0 \} \) is defined such that

\[ \sup_{0 \leq t \leq 1} \left\{ \left( \sum_{j=1}^{m} Z_{ij}^r X_j - \mu_{r, ik} \right)^2 \right\}^{1/2} = \sigma_{r, ik} \]

\[ (Z_{jk} - k)^r - W^r_t \left( \frac{m^2}{m} \right) \quad (9) \]

where \( E(W(s)) = 0, \) and

\[ EW(s)W^r_t = \min(s, t) \left\{ \frac{\sigma_{r, ik}^2}{\mu_{r, ik} k} \right\} \]

(10)

Theorem A follows from the results of Einmahl [13], Zaitsev [14] and Götz and Zaitsev [15].

The main result of this paper is the following Theorem.

Theorem 1. On some probability space, there exists a sequence of mean zero Gaussian processes \( \Gamma_m (t, k, r, F), 0 \leq t \leq 1 \) such that

\[ \sup_{0 \leq t \leq 1} \left\{ \left( \sum_{j=1}^{m} Z_{ij}^r X_j - \mu_{r, ik} \right)^2 \right\}^{1/2} = \sigma_{r, ik} \quad (11) \]

where \( \Gamma_m (t, k, r, F) = \Gamma(t, k, r, F) \) for each \( m \), and

\[ E \{ \Gamma(t, k, r, F) \Gamma(s, k, r, F) \} = (t \wedge s) \sigma_{r, ik}^2 \]

(12)

Theorem 1 follows from (6) and the following Theorem.

Theorem 2. On the probability space of Theorem A,

\[ \sup_{0 \leq t \leq 1} \left\{ \beta_m (t, k, r, F) - \beta_m (t, k, r, F) \right\} = \frac{m^2}{m} \sigma_{r, ik}^2 \quad (13) \]

where \( W(.) \) is as in (9).

Proof of Theorem 2: We will only prove here the case when \( E(X) = \mu \neq 0 \). The case when \( \mu = 0 \) is straightforward and can be looked at as a special case of the case \( \mu \neq 0 \). Note that

\[ \beta_m (t, k, r, F) = \frac{m^2 \sigma_{r, ik}^2 \text{Var}(A_m (t))}{\left( \sum_{i=1}^{m} Z_{ik} \right)^r_{\mu} \text{Cov}(Z_{ik}^r X_i, Z_{lj}^s) \mu_{r, ik} \mu_{s, lj}} \quad (14) \]

where

\[ A_m (t) = \frac{1}{m} \sum_{i=1}^{m} Z_{ik}^r X_i - \mu_{r, ik} \frac{1}{k} \frac{1}{m} \sum_{i=1}^{m} Z_{ik}^r \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \left( Z_{ik}^r X_i - \mu_{r, ik} \right) + \mu_{r, ik} \frac{m(m-1)}{k} \]
\( \mu \mu_r k - t \mu \mu_r k \frac{1}{k!} \sum_{i=1}^{m} (Z_{i,k} - k + k)^r \). \quad (15) \\

It is clear that, uniformly in \( t, 0 \leq t \leq 1 \),
\[ \frac{|m^2 - m|}{m} \leq \frac{1}{m} \] \quad (16)

By (9), (15) and (16) we have, uniformly in \( t, 0 \leq t \leq 1 \),
\[ A_m(t) \overset{a.s.}{=} \frac{1}{m} W_1(mt) + G \left( \frac{1}{m} \sqrt{\log m} \right) + \mu \mu_r k t \]
\[ - t \mu \mu_r k \left( 1 + \frac{1}{mk} W_2(m) + o(m^{-\frac{3}{2}}) \right)^r + o(m^{-\frac{3}{2}}). \] \quad (19)

By Lemma 1.1.1 of Csörgő and Révész [17] we have, uniformly in \( t, 0 \leq t \leq 1 \),
\[ \frac{1}{m} |W_1([mt]) - W_1(mt)| \overset{a.s.}{=} O \left( \frac{1}{m} \sqrt{\log m} \right). \] \quad (18)

By (17) and (18) we have, uniformly in \( t, 0 \leq t \leq 1 \),
\[ A_m(t) \overset{a.s.}{=} \frac{1}{m} W_1(mt) + G \left( \frac{1}{m} \sqrt{\log m} \right) + \mu \mu_r k t \]
\[ - t \mu \mu_r k \left( 1 + \frac{1}{mk} W_2(m) + o(m^{-\frac{3}{2}}) \right)^r + o(m^{-\frac{3}{2}}). \]

By the LIL
\[ \left( \frac{1}{m} \sum_{i=1}^{m} Z_{i,k} \right)^r \overset{a.s.}{=} \left( k + O \left( m^{-\frac{3}{2}} \sqrt{\log m} \right) \right)^r \]
\[ \overset{a.s.}{=} r^r + O \left( m^{-\frac{3}{2}} \right). \] \quad (20)

By (14), (19) and (20) we have, uniformly in \( t, 0 \leq t \leq 1 \),
\[ \beta_m(t,k,r,F) \overset{a.s.}{=} \frac{k^r}{e^{r^2} o \left( m^{-\frac{3}{2}} \sqrt{\log m} \right)} \left( \frac{1}{m} W_1(mt) - \frac{r^r \mu \mu_r k}{m k} W_2(m) + o \left( m^{-\frac{3}{2}} \right) \right) \]
\[ \overset{a.s.}{=} \frac{1}{m} \left( W_1(mt) - \frac{r^r \mu \mu_r k}{k} W_2(m) + o \left( m^{-\frac{3}{2}} \right) \right). \]

This proves (13).

**Corollary 1.** By (4), (8) and (12),
\[ \Gamma(\cdot, k, r, F) \overset{D}{=} \lambda_{r,k} W(\cdot) + \frac{r^r \mu \Gamma(r+k) \sqrt{\Gamma(k)}}{\sqrt{k} \Gamma(k)} B(\cdot), \] \quad (21)

where
\[ \lambda_{r,k}^2 = \frac{\Gamma(2r+k)}{\Gamma(k)} \sigma^2 + \mu^2 \left( \frac{\Gamma(2r+k)}{\Gamma(k)} \right)^2 \left( \frac{r^r \mu \mu_r k}{k} \right)^2. \]

\( W(\cdot) \) is a Wiener process, \( B(\cdot) \) is a Brownian bridge and \( W(\cdot) \) and \( B(\cdot) \) are independent.

**Corollary 2.** By (11) and (21) we have, as \( m \rightarrow \infty \),
\[ \alpha_m(\cdot,k,r,F) \overset{D}{=} \Gamma(\cdot, k, r, F) \overset{D}{=} \lambda_{r,k} W(\cdot) + \frac{r^r \mu \Gamma(r+k) \sqrt{\Gamma(k)}}{\sqrt{k} \Gamma(k)} B(\cdot), \] \quad (22)

and, in particular,
\[ \alpha_m(1, k, r, F) \overset{D}{=} N \left( 0, \lambda_{r,k}^2 \right). \] \quad (23)

Some special cases of (22) and (23) are given. For \( r=1 \) and \( k \geq 1 \),
\[ \Gamma(\cdot, k, 1, F) \overset{D}{=} \sigma \sqrt{k(k+1)} W(\cdot) + \mu \sqrt{k} B(\cdot) \]
and
\[ \alpha_m(1, k, 1, F) \overset{D}{=} N(0, k(k+1) \sigma^2). \]

For \( r > 0 \) and \( k = 1 \),
\[ \Gamma(\cdot, 1, r, F) \overset{D}{=} \lambda_{r,1} W(\cdot) + r \mu \Gamma(r+1) B(\cdot) \]
and
\[ \alpha_m(1, 1, r, F) \overset{D}{=} N(0, \lambda_{r,1}^2), \]

where
\[ \lambda_{r,1}^2 = \sigma^2 \Gamma(2r+1) + \mu^2 \left( \Gamma(2r+1) - (1 + r^2) \Gamma^2(r+1) \right). \]

### 3. RESULTS

In this section, we will use the same notation of Section 1.

#### 3.1. The scaled sum case

Define
\[ T_{m,3}(t,k,r,F) = \frac{1}{\sum_{i=1}^{m} X_i} \sum_{i=1}^{[mt]} \frac{X_i}{m} \sqrt{\log \log m}, \quad 0 \leq t \leq 1 \]
and
\[ Y_{m,3}(t,k,r,F) = \frac{1}{\sum_{i=1}^{m} X_i} \sum_{i=1}^{[mt]} \frac{X_i}{m} \sqrt{\log \log m}, \quad 0 \leq t \leq \frac{1}{m}. \]

We can prove that
\[ y_{m,3}(t,k,r,F) \overset{D}{=} y_3(t,k,r,F), \]
where
\[ y_3(t,k,r,F) = \frac{1}{\mu \mu_r k} W_1(t) - \frac{1}{k} t W_3(1) - \frac{1}{\mu} t W_3(1). \]

\( (W_1(\cdot), W_2(\cdot), W_3(\cdot)) \) is a mean zero Gaussian vector with covariance \((\sigma \Lambda \sigma)^t\) and
\[ \Sigma_1 = \begin{pmatrix} \sigma_{\mu r k}^2 & \mu \mu_r k & \sigma_{\mu r k}^2 \\ \mu \mu_r k & k & 0 \\ \sigma_{\mu r k}^2 & 0 & \sigma^2 \end{pmatrix}. \]

Let
\[ \delta_{r,k} = \left( \frac{\mu \mu_r k}{\mu \mu_r k} - 1 \right) \left( \frac{\sigma^2}{\mu^2 k} + 1 \right) - \frac{r^2}{k} \]

We can show that
\[ \gamma_2(t, k, r, F) \overset{D}{=} \delta_{r,k}W(t) + \sqrt{\frac{r^2}{k} + \frac{\sigma^2}{\mu^2}}B(t), \]

where \( W(.) \) is a Brownian Motion and \( B(.) \) is a Brownian bridge and \( W(.) \) and \( B(.) \) are independent. Consequently,

\[ \gamma_{m,1}(1, k, r, F) \overset{D}{=} N(0, \delta_{r,k}^2). \]

When \( r=1, k \geq 1 \)

\[ \delta_{r,k}^2 = \frac{\sigma^2}{\mu^2}. \]

When \( r > 0, k = 1 \)

\[ \delta_{r,1}^2 = \left( \frac{\Gamma(2r+1)}{\Gamma^2(r+1)} \right) \left( \frac{\sigma^2}{\mu^2} + 1 \right) - r^2. \]

### 3.2. The Centered Sum Process

Let \( \bar{X} = \frac{1}{m} \sum_{j=1}^{m} X_j \) and define

\[ T_{m,2}(t, k, r, F) = \begin{cases} \frac{1}{m} \sum_{i=1}^{[mt]} R_{i,k}(X_i - \bar{X}), & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 \leq t < \frac{1}{m} \end{cases} \]

and

\[ \gamma_{m,2}(t, k, r, F) = \begin{cases} k^r m^{r-1} T_{m,2}(t, k, r, F), & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 \leq t < \frac{1}{m} \end{cases} \]

We can prove that

\[ \gamma_{m,2}(t, k, r, F) \overset{D}{=} \gamma_2(t, k, r, F), \]

where

\[ \gamma_2(t, k, r, F) = W_2(t) - \mu W_2(t) - \mu_{r,k} t W_3(1). \]

\((W(.)\text{, } W_2(.)\text{, } W_3(.)\)) is a mean zero Gaussian vector with covariance \((\Lambda \text{ s}) \Sigma_2\) and

\[ \Sigma_2 = \begin{bmatrix} \sigma_{r,k}^2 & \mu(\mu_{2r,k} - \mu_{r,k}^2) & \sigma^2 \mu_{r,k} \\ \mu(\mu_{2r,k} - \mu_{r,k}^2) & \mu_{r,k}^2 - \mu_{r,k}^2 & 0 \\ \sigma^2 \mu_{r,k} & 0 & \sigma^2 \end{bmatrix}. \]

We can show that

\[ \gamma_2(t, k, r, F) \overset{D}{=} \sigma \left( W_2(t) - \mu W_2(t) + \mu_{r,k} B(t) \right), \]

where \( W(.) \) is a Brownian Motion and \( B(.) \) is a Brownian bridge and \( W(.) \) and \( B(.) \) are independent. Consequently,

\[ \gamma_{m,2}(1, k, r, F) \overset{D}{=} N \left( 0, \sigma^2(\mu_{2r,k} - \mu_{r,k}^2) \right). \]

When \( r=1, k \geq 1 \)

\[ \gamma_{m,2}(1, k, r, F) \overset{D}{=} N(0, k \sigma^2). \]

When \( r > 0, k=1 \)

\[ \gamma_{m,2}(1, k, r, F) \overset{D}{=} N(0, \sigma^2(\Gamma(2r+1) - \Gamma^2(r+1))). \]

### 3.3. The Renewal Process

For simplicity, we will consider the case of \( r = 1 \). Define

\[ S_m^*(t) = \begin{cases} \frac{1}{m} \sum_{i=1}^{[mt]} R_{i,k}, & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 \leq t < \frac{1}{m} \end{cases} \]

\[ T_m^*(t) = \begin{cases} \frac{1}{m} \sum_{i=1}^{[mt]} Z_{i,k} X_i / (\sum_{i=1}^{m} Z_{i,k}), & \frac{1}{m} \leq t \leq 1 \\ 0, & 0 \leq t < \frac{1}{m} \end{cases} \]

\[ N_m(t) = \inf \{ u : S_m^*(u) > t \}, \]

\[ M_m(t) = \inf \{ u : T_m^*(u) > t \}, \]

\[ \alpha_m^*(t) = \left( m^2 k^2 \mu \right) (S_m^*(t) - t), \quad \frac{1}{m} \leq t \leq 1, \quad 0 \leq t < \frac{1}{m}, \]

\[ \beta_m^*(t) = \left( m^2 k^2 \mu \right) (T_m^*(t) - t), \quad \frac{1}{m} \leq t \leq 1, \quad 0 \leq t < \frac{1}{m}, \]

\[ \eta_m(t) = \frac{1}{m} \left( m^2 k^2 \mu - N_m(t) \right) \]

and

\[ \xi_m(t) = m^2 k^2 \mu (t - M_m(t)). \]

By (5), for each \( m \)

\[ \alpha_m^*(t) \overset{D}{=} \beta_m^*(t) \text{ and } \eta_m^*(t) \overset{D}{=} \xi_m^*(t). \] (24)

Note that (see (3))

\[ \alpha_m^*(t) = \alpha_m(\cdot, k, 1, F) \]

and hence, by Theorem 1

\[ \sup_{0 \leq t \leq 1} [\alpha_m^*(t) - \Gamma_m(t, k, 1, F)] \overset{P}{=} o \left( m^{-\frac{1}{2}} \right). \]

where \( \Gamma_m(\cdot, k, 1, F) \text{ is as in (11).} \)

**Theorem 3.** On the probability space of Theorem A,

\[ \sup_{0 \leq t \leq 1} [\eta_m(t) - \Gamma_m(t)] \overset{P}{=} o \left( m^{-\frac{1}{2}} \log m \log \log m \right)^{\frac{1}{2}}, \]

where

\[ \Gamma_m(t) = m^{-\frac{1}{2}} (W_2(mt) - \mu W_2(mt)) \]

and \( W(.) \) is as in (9).
Theorem 3 follows directly from (24) and the following Theorem.

**Theorem 4.** On the probability space of Theorem A,

\[ \sup_{0 \leq t \leq 1} |\xi_m(t) - \Gamma_m(t)| \overset{a.s.}{=} 0 \left( m^{-\frac{1}{2}} (\log m \log \log m)^{\frac{3}{2}} \right), \]

where \( \Gamma_m(t) \) is as in (25).

**Proof:** By (7),

\[ \beta^*_m(\cdot) = \beta_m(\cdot, 1, F). \]

Note that

\[ \xi_m(t) = \beta^*_m(M_m(t)) - m^2 k \mu \left( M_m(t) - t \right). \]

Hence

\[ \sup_{0 \leq t \leq 1} |\xi_m(t) - \Gamma_m(t)| \leq E_{m1} + E_{m2} + E_{m3}. \quad (26) \]

where

\[ E_{m1} = \sup_{0 \leq t \leq 1} |\beta^*_m(M_m(t)) - \Gamma_m(M_m(t))|, \]

\[ E_{m2} = m^2 k \mu \sup_{0 \leq t \leq 1} |M_m(t) - t| \]

and

\[ E_{m3} = \sup_{0 \leq t \leq 1} |\Gamma_m(M_m(t)) - \Gamma_m(t)|. \]

By Theorem 2 and the LIL for Wiener processes,

\[ E_{m1} \overset{a.s.}{=} o \left( m^{-\frac{1}{2}} \right). \quad (27) \]

and

\[ \sup_{0 \leq t \leq 1} |M_m(t) - t| \overset{a.s.}{=} O \left( \sqrt{m^{-1} \log \log m} \right) \]

By a Lemma of Horváth [18]

\[ \sup_{0 \leq t \leq 1} |M_m(t) - t| \overset{a.s.}{=} \sup_{0 \leq t \leq 1} |T_m(t) - t| \]

and hence

\[ \sup_{0 \leq t \leq 1} |M_m(t) - t| \overset{a.s.}{=} O \left( \sqrt{m^{-1} \log \log m} \right). \quad (28) \]

By the proof of Step 5 of Horváth [18] and Theorem 2 we can show that

\[ E_{m2} \overset{a.s.}{=} O \left( m^{-\frac{1}{2}} \log m \right). \quad (29) \]

As to \( E_{m3} \),

\[ E_{m3} \leq E_{m31} + E_{m32}. \quad (30) \]

where

\[ E_{m31} = \sup_{0 \leq t \leq 1} |W_1(M_m(t)) - W_1(t)| \]

and

\[ E_{m32} = m^2 k \mu \left[ W_2(m) \sup_{0 \leq t \leq 1} |M_m(t) - t| \right]. \]

By (28) and Lemma 1.1.1 of Csörgö and Révész [17] we have, uniformly in \( t, 0 \leq t \leq 1 \),

\[ E_{m31} = \sup_{0 \leq t \leq 1} |W_1(t + (M_m(t) - t)) - W_1(t)| \overset{a.s.}{=} O \left( m^{-\frac{1}{2}} \log \log m \right). \quad (31) \]

By (28) and the LIL for Wiener processes,

\[ E_{m32} \overset{a.s.}{=} O \left( m^{-\frac{1}{2}} \log \log m \right). \quad (32) \]

By (30)-(32),

\[ E_{m3} \overset{a.s.}{=} O \left( m^{-\frac{1}{2}} \log \log m \right). \quad (33) \]

By (26)-(33) we obtain Theorem 4.

4. THE RANDOM VECTOR CASE

Let \( X_1, X_2, \ldots \) be iid random vectors with

\[ E(X_i) = \mu = (\mu_1, \mu_2, \ldots, \mu_p)^T \] and \( \text{Var}(X_i) = \Sigma = [\sigma_{ij}] \).

Assume that the \( U_i \)s and the \( R_{i,k} \)s are same as in Section 1 and are independent of \( X_1, X_2, \ldots \). Define

\[ \zeta_m(t, k, r, F) = \begin{cases} \left[ \sum_{i=1}^{[mt]} R_{i,k} X_i \right] \frac{1}{m} \leq t < \frac{1}{m} \\ \begin{cases} \left[ \sum_{i=1}^{[mt]} R_{i,k} X_i \right] - t \mu_k \frac{1}{m} \leq t \leq \frac{1}{m} \end{cases} \end{cases} \]

and

\[ \xi_m(t, k, r, F) = \begin{cases} 0 \leq t < \frac{1}{m} \end{cases} \]

and

\[ \alpha_m(t, k, r, F) = \begin{cases} m \left[ k^r m^{-r-1} S_m(t, k, r, F) - t \mu_k \right] \frac{1}{m} \leq t \leq 1 \end{cases} \]

Theorem 5 is a generalization of Theorem 1.

**Theorem 5.** On some probability space, there exists a mean zero sequence of Gaussian processes \( \{ \Gamma_m(t, k, r, F), 0 \leq t \leq 1 \} \) such that
where, for each \( m \),

\[
\Gamma_m(c, k, r, F) \overset{D}{=} \Gamma(c, k, r, F),
\]

and

\[
E \Gamma(s, k, r, F) \Gamma'(t, k, r, F) = (t \wedge s) \sum_{r,k}^{(1)} \frac{r^2 t^2 (r+k)}{k t^2 (k)} t s \mu^t
\]

and

\[
\sum_{r,k}^{(1)} = \frac{\Gamma(2r+k)}{\Gamma(k)} \sum + \left\{ \frac{\Gamma(2r+k)}{\Gamma(k)} - \frac{k \Gamma^2(r+k)}{k t^2 (k)} \right\} \mu^t.
\]

**Corollary 1**. By (11) and (21) we have, as \( m \to \infty \),

\[
\alpha_m(c, k, r, F) \overset{D}{=} \Gamma(c, k, r, F)
\]

and, in particular,

\[
\alpha_m(1, k, r, F) \overset{D}{=} \mathcal{M}V\mathcal{N}(0, \Sigma^{(2)}_{r,k}),
\]

where

\[
\Sigma^{(2)}_{r,k} = \frac{\Gamma(2r+k)}{\Gamma(k)} \sum + \left( \frac{\Gamma(2r+k)}{\Gamma(k)} - \frac{k \Gamma^2(r+k)}{k t^2 (k)} \right) \mu^t.
\]

Particular cases of Corollary 1* are given next. For \( r = 1 \) and \( k \geq 1 \),

\[
E \Gamma(s, 1, r, F) \Gamma'(t, 1, r, F) = (t \wedge s) \sum_{r,1}^{(1)} - t s k \mu^t,
\]

\[
\sum_{r,1}^{(1)} = k (k+1) \sum + k \mu^t
\]

and

\[
\Gamma(1, 1, r, F) \overset{D}{=} \mathcal{M}V\mathcal{N}(0, k (k+1) \Sigma).
\]

For \( r > 0 \) and \( k = 1 \),

\[
E \Gamma(s, 1, 1, F) \Gamma'(t, 1, 1, F) = (t \wedge s) \sum_{r,1}^{(1)} - t s r^2 \Gamma^2 (r+1) \mu^t,
\]

\[
\sum_{r,1}^{(1)} = \Gamma(2r+1) \sum + \{ \Gamma(2r+1) - \Gamma^2 (r+1) \} \mu^t
\]

and

\[
\Gamma(1, 1, r, F) \overset{D}{=} \mathcal{M}V\mathcal{N}(0, \Sigma^*),
\]

where

\[
\Sigma^* = \Gamma(2r+1) \sum + \{ \Gamma(2r+1) - (1+r^2) \Gamma^2 (r+1) \} \mu^t.
\]

**CONCLUSION**

We proved the weak convergence of a stochastic process defined in terms of partial sums of randomly weighted powers of uniform spacings. The asymptotic results of several important generalizations and special cases are given.

**CONSENT FOR PUBLICATION**

Not applicable.

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Not applicable.

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